Investigation of implicit time stepping for grid-induced stiffness in discontinuous Galerkin time-domain methods on unstructured triangular meshes

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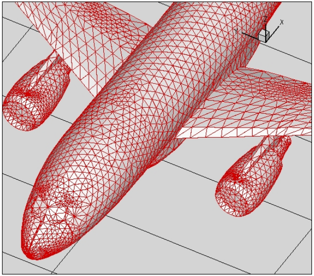
Outline

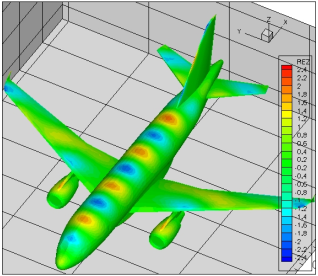
Context and motivations

- Time-domain electromagnetic wave propagation
- Irregularly shaped geometries, heterogeneous media
  - Unstructured, locally refined, triangular (2D)/tetrahedral (3D) meshes
- Numerical ingredients (starting point to this study)
  - Discontinuous Galerkin time-domain (DGTD) methods
  - Nodal (Lagrange type) polynomial interpolation
  - Explicit time integration

Context and motivations

- Scattering of a plane wave by an aircraft, F=1 GHz
  - Mesh: # vertices = 153,821 , # tetrahedra = 883,374
  - $L_{\text{min}} = 0.000601$ m , $L_{\text{max}} = 0.121290$ m $(\approx \dfrac{\lambda}{2.5})$, $L_{\text{avg}} = 0.039892$ m
  - $\Delta t_{\text{min}} =$ 0.24 picosec and $\Delta t_{\text{max}} = 40.50$ picosec

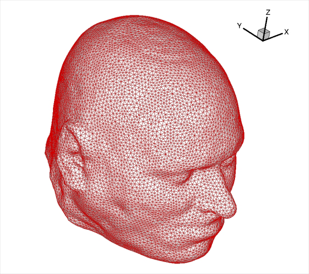


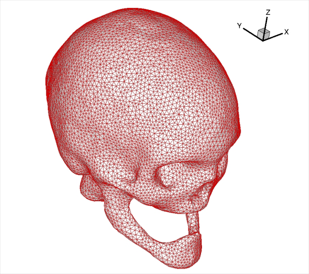


Context and motivations

- Propagation in heterogeneous media, F=1.8 GHz
  - Mesh: # vertices = 311,259 , # tetrahedra = 1,862,136
  - $L_{\mathrm{min}} = 0.650$ mm , $L_{\mathrm{max}} = 8.055$ mm , $L_{\mathrm{avg}} = 4.064$ mm
  - $\Delta t_{\mathrm{min}} = 0.019$ picosec and $\Delta t_{\mathrm{max}} = 0.203$ picosec

Tissue	L_{min} (mm)	L_{max} (mm)	L_{avg} (mm)	λ (mm)
Skin	1.339	8.055	4.070	26.73
Skull	1.613	7.786	4.069	42.25
CSF	0.650	7.232	4.059	20.33
Brain	0.650	7.993	4.009	25.26





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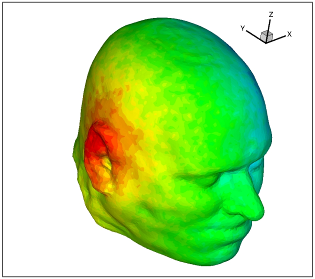

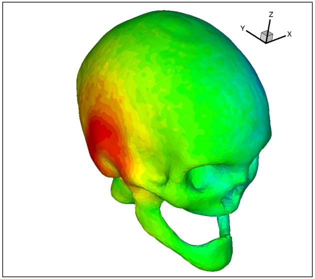

Context and motivations

- Possible routes to overcome grid-induced stiffness
  - Local time-step strategies with explicit time integration
  - Locally implicit (hybrid explicit/implicit) time integration
- Objective of this study
  - Investigate strengthes and weaknesses of implicit time integration
    - Numerical aspects (stability, dispersion error)
    - Computational aspects

- Initially introduced to solve neutron transport problems (W. Reed and T. Hill, 1973)
- Became popular as a framework for solving hyperbolic or mixed hyperbolic/parabolic problems
- Recently developed for elliptic problems
- Somewhere between a finite element and a finite volume method, gathering many good features of both
- Main properties
  - Can easily deal with discontinuous coefficients and solutions
  - Can handle unstructured, non-conforming meshes
  - Yield local finite element mass matrices
  - High-order accurate methods with compact stencils
  - Naturally lead to discretization and interpolation order adaptivity
  - Amenable to efficient parallelization

- F. Bourdel, P.A. Mazet and P. Helluy
  Proc. 10th Inter. Conf. on Comp. Meth. in Appl. Sc. and Eng., 1992
  - Triangular meshes, first-order upwind DG method (i.e FV method)
  - Time-domain and time-harmonic Maxwell equations
- M. Remaki and L. Fezoui, INRIA RR-3501, 1998
  - Time-domain Maxwell equations
  - Triangular meshes, P1 interpolation, Runke-Kutta time integration (RKDG)
- J.S. Hesthaven and T. Warburton (J. Comput. Phys., Vol. 181, 2002)
  - Tetrahedral meshes, high order Lagrange polynomials, upwind flux
  - Runge-Kutta time integration
- B. Cockburn, F. Li and C.-W. Shu (J. Comput. Phys., Vol. 194, 2004)
  - Locally divergence-free RKDG formulation
- L. Fezoui, S. Lanteri, S. Lohrengel and S. Piperno (M2AN, Vol. 39, No. 6, 2005)
  - Tetrahedral meshes, high order Lagrange polynomials
  - Leap-frog time integration scheme, centered flux
- G. Cohen, X. Ferrieres and S. Pernet (J. Comput. Phys., Vol. 217, 2006)
  - Hexahedral meshes, high order Lagrange polynomials, penalized formulation
  - Leap-frog time integration scheme

Content

Outline

Implicit DGTD-$\mathbb{P}_p$ method

- Time-domain Maxwell's equations

$$\begin{cases} \varepsilon \partial_t \mathbf{E} - \text{curl}(\mathbf{H}) = 0 \\ \mu \partial_t \mathbf{H} + \text{curl}(\mathbf{E}) = 0 \end{cases}$$

$$\mathbf{E} = (E_x, E_y, E_z)^t \quad \text{and} \quad \mathbf{H} = (H_x, H_y, H_z)^t$$

- Boundary conditions: $\partial\Omega = \Gamma_a \cup \Gamma_m$

$$\begin{cases} \mathbf{n} \times \mathbf{E} = 0 \text{ on } \Gamma_m \\ \mathbf{n} \times \mathbf{E} - \sqrt{\dfrac{\mu}{\varepsilon}}\mathbf{n} \times (\mathbf{H} \times \mathbf{n}) = \mathbf{n} \times \mathbf{E}_{\text{inc}} - \sqrt{\dfrac{\mu}{\varepsilon}}\mathbf{n} \times (\mathbf{H}_{\text{inc}} \times \mathbf{n}) \text{ on } \Gamma_a \end{cases}$$

Implicit DGTD-$\mathbb{P}_p$ method

- System formulation: $\mathbf{W} = (\mathbf{E}, \mathbf{H})^t$

$$\begin{cases} G_0 \partial_t \mathbf{W} + G_x \partial_x \mathbf{W} + G_y \partial_y \mathbf{W} + G_z \partial_z \mathbf{W} = 0 \text{ in } \Omega \\ (M_{\Gamma_m} - G_{\mathbf{n}})\mathbf{W} = 0 \text{ on } \Gamma_m \\ (M_{\Gamma_a} - G_{\mathbf{n}})\mathbf{W} = (M_{\Gamma_a} - G_{\mathbf{n}})\mathbf{W}_{\text{inc}} \text{ on } \Gamma_a \end{cases}$$

- Flux matrices
  - $G_l \mathbf{W} = (\mathbf{H} \times \mathbf{e}_l, -\mathbf{E} \times \mathbf{e}_l)^t$, $\{\mathbf{e}_l\}_{x,y,z}$ is the canonical base of $\mathbb{R}^3$
  - $G_{\mathbf{n}} \mathbf{W} = (\mathbf{H} \times \mathbf{n}, -\mathbf{E} \times \mathbf{n})^t$

$$G_0 = \begin{pmatrix} \varepsilon \,\mathrm{Id}_3 & 0_{3\times 3} \\ 0_{3\times 3} & \mu \,\mathrm{Id}_3 \end{pmatrix}$$

$$G_{\mathbf{n}} = \begin{pmatrix} 0_{3\times 3} & N_{\mathbf{n}} \\ N_{\mathbf{n}}^t & 0_{3\times 3} \end{pmatrix} \text{ with } N_{\mathbf{n}} = \begin{pmatrix} 0 & \mathbf{n}_z & -\mathbf{n}_y \\ -\mathbf{n}_z & 0 & \mathbf{n}_x \\ \mathbf{n}_y & -\mathbf{n}_x & 0 \end{pmatrix}$$

Implicit DGTD-$\mathbb{P}_p$ method

- Diagonalization: $G_{\mathbf{n}} = T\Lambda T^{-1}$ and $G_{\mathbf{n}}^{\pm} = T\Lambda^{\pm} T^{-1}$
- Boundary flux matrices

$$\left\{\begin{array}{l} M_{\Gamma_m} = \begin{pmatrix} 0_{3\times 3} & N_{\mathbf{n}} \\ -N_{\mathbf{n}}^t & 0_{3\times 3} \end{pmatrix} \\ M_{\Gamma_a} = |G_{\mathbf{n}}| \end{array}\right.$$

- First-order absorbing condition: $(M_{\Gamma_a} - G_{\mathbf{n}})\mathbf{W} = 0 \Leftrightarrow G_{\mathbf{n}}^{-}\mathbf{W} = 0$

Implicit DGTD-$\mathbb{P}_p$ method

- Semi-discretization in time
  - Crank-Nicolson scheme
  - $\mathbf{W}^n$: approximation of $\mathbf{W}$ at time $t_n = n\Delta t$

$$G_0 \left(\frac{\mathbf{W}^{n+1} - \mathbf{W}^n}{\Delta t} \right) + (G_x \partial_x + G_y \partial_y + G_z \partial_z) \left(\frac{\mathbf{W}^{n+1} + \mathbf{W}^n}{2} \right) = 0$$

- Boundary value problem: for each t_n,

$$\begin{cases} \beta G_0 \mathbf{W} + (G_x \partial_x + G_y \partial_y + G_z \partial_z)\mathbf{W} = \mathbf{F} \text{ in } \Omega \\ (M_{\Gamma_m} - G_{\mathbf{n}})\mathbf{W} = 0 \text{ on } \Gamma_m \\ (M_{\Gamma_a} - G_{\mathbf{n}})\mathbf{W} = (M_{\Gamma_a} - G_{\mathbf{n}})\mathbf{W}_{\text{inc}} \text{ on } \Gamma_a \end{cases}$$

  - $\beta = \dfrac{2}{\Delta t}$ and $\mathbf{W} = \mathbf{W}^{n+1}$
  - $\mathbf{F} = \beta G_0 \mathbf{W}^n - (G_x \partial_x + G_y \partial_y + G_z \partial_z)\mathbf{W}^n$

Implicit DGTD-$\mathbb{P}_p$ method

- Semi-discrete electromagnetic energy

Let $\mathcal{E}^n = \frac{1}{2}\int_\Omega (\mathbf{W}^n)^t (G_0 \mathbf{W}^n) dx = \frac{1}{2}\int_\Omega \varepsilon \parallel \mathbf{E}^n \parallel^2 dx + \frac{1}{2}\int_\Omega \mu \parallel \mathbf{H}^n \parallel^2 dx$

Then $\mathcal{E}^n$ is exactly conserved if $\Gamma_a = \emptyset$ or $\mathcal{E}^{n+1} \leq \mathcal{E}^n$ if $\Gamma_a \neq \emptyset$

$$\frac{\mathcal{E}^{n+1} - \mathcal{E}^n}{\Delta t} = -\int_\Omega \left(\frac{\mathbf{W}^{n+1} + \mathbf{W}^n}{2}\right)^t \mathcal{G}_{xyz} \left(\frac{\mathbf{W}^{n+1} + \mathbf{W}^n}{2}\right) dx$$

with $\mathcal{G}_{xyz} \equiv (G_x\partial_x + G_y\partial_y + G_z\partial_z)$ and,

$$\int_\Omega \mathbf{W}^t \mathcal{G}_{xyz} \mathbf{W} = \frac{1}{2}\int_{\Gamma_m} \mathbf{W}^t (M_{\Gamma_m}\mathbf{W}) ds + \frac{1}{2}\int_{\Gamma_a} \mathbf{W}^t (M_{\Gamma_a}\mathbf{W}) ds$$

but M_{Γ_m} skew-symmetric and $|G_\mathbf{n}|$ positive then,

$$\int_\Omega \mathbf{W}^t \mathcal{G}_{xyz} \mathbf{W} = \frac{1}{2}\int_{\Gamma_a} \mathbf{W}^t (|G_\mathbf{n}|\mathbf{W}) ds \geq 0$$

Implicit DGTD-$\mathbb{P}_p$ method

- Discretization in space
  - Triangulation of Ω: $\overline{\Omega_h} \equiv \mathcal{T}_h = \bigcup_{K \in \mathcal{T}_h} \overline{K}$
    - $\mathcal{F}_0$: set of purely internal faces
    - $\mathcal{F}_m$ and $\mathcal{F}_a$: sets of faces on the boundaries Γ_m and Γ_a
  - $V_h = \left\{ \mathbf{V} \in [L^2(\Omega)]^3 \times [L^2(\Omega)]^3 \,|\, \forall K \in \mathcal{T}_h \;\;,\;\; \mathbf{V}_{|K} \in \mathbb{P}_p(K) \right\}$
  - Variational formulation

$$\int_K (G_0 \partial_t \mathbf{W}_h)^t \, \mathbf{V} dx - \int_K \mathbf{W}_h^t \left(\sum_{l \in \{x,y,z\}} G_l \partial_l \mathbf{V} \right) dx$$
$$+ \sum_{F \in \partial K} \int_F (\Phi_F(\mathbf{W}_h))^t \, \mathbf{V} ds = 0 \;\;,\;\; \forall \mathbf{V} \in V_h$$

  - Numerical flux

$$\Phi_F(\mathbf{W}_h) = \begin{cases} I_{F,K} G_{\mathbf{n}_F} \{\mathbf{W}_h\} \text{ if } F \in \mathcal{F}_0 \\ \frac{1}{2}(M_{F,K} + I_{F,K} G_{\mathbf{n}_F})\mathbf{W}_h \text{ if } F \in (\mathcal{F}_m \cup \mathcal{F}_a) \end{cases}$$

- Discretization in space
  - Bounadry flux matrices: $M_{F,K} = \begin{cases} I_{F,K} \begin{pmatrix} 0_{3\times 3} & N_{\mathbf{n}_F} \\ -N^t_{\mathbf{n}_F} & 0_{3\times 3} \end{pmatrix} \text{ if } F \in \mathcal{F}_m \\ |G_{\mathbf{n}_F}| \text{ if } F \in \mathcal{F}_a \end{cases}$
  - For $F = K \cap \tilde{K}$ ($\tilde{K}$ neighbor of K): $\begin{cases} [\![\mathbf{V}_h]\!]_F = I_{F,K}\mathbf{V}_{h|K} + I_{F\tilde{K}}\mathbf{V}_{h|\tilde{K}} \\ \{\mathbf{V}_h\}_F = \dfrac{1}{2}(\mathbf{V}_{h|K} + \mathbf{V}_{h|\tilde{K}}) \end{cases}$

$$
\begin{aligned}
\int_{\Omega_h} (G_0 \partial_t \mathbf{W}_h)^t \, \mathbf{V} dx - \sum_{K \in \mathcal{T}_h} \int_K \mathbf{W}_h^t \left(\sum_{l \in \{x,y,z\}} G_l \partial_l \mathbf{V} \right) dx \\
+ \sum_{F \in \mathcal{F}_m \cup \mathcal{F}_a} \int_F \left(\frac{1}{2} (M_{F,K} + I_{F,K} G_{\mathbf{n}_F}) \mathbf{W}_h \right)^t \mathbf{V} ds \\
+ \sum_{F \in \mathcal{F}_0} \int_F (G_{\mathbf{n}_F} \{\mathbf{W}_h\}_F)^t \, [\![\mathbf{V}]\!]_F ds = 0
\end{aligned}
$$

Outline

Properties of the fully discrete scheme

- Equivalent, face oriented, formulation

  Since (with $\tilde{K}$ a neighboring element of K),

$$\sum_{F\in\mathcal{F}_0}\int_F \left(G_{\mathbf{n}_F}\{\mathbf{W}_h\}_F\right)^t [\![\mathbf{V}]\!]_F ds = \sum_{K\in\mathcal{T}_h}\sum_{\substack{F\in\partial K\\ F\in\mathcal{F}_0}}\int_F \frac{1}{2}\left(G_{\mathbf{n}_F}\mathbf{W}_{h|K}\right)^t \left(I_{F,K}\mathbf{V}_{|K}\right) ds + \sum_{K\in\mathcal{T}_h}\sum_{\substack{F\in\partial K\\ F\in\mathcal{F}_0}}\int_F \frac{1}{2}\left(G_{\mathbf{n}_F}\mathbf{W}_{h|\tilde{K}}\right)^t \left(I_{F,K}\mathbf{V}_{|K}\right) ds$$

  Then,

$$2\int_{\Omega_h}\left(G_0\partial_t\mathbf{W}_h\right)^t\mathbf{V}dx - \sum_{K\in\mathcal{T}_h}\int_K \mathbf{W}_h^t\left(\sum_{l\in\{x,y,z\}} G_l\partial_l\mathbf{V}\right)dx + \sum_{K\in\mathcal{T}_h}\int_K\left(\sum_{l\in\{x,y,z\}} G_l\partial_l\mathbf{W}_h\right)^t\mathbf{V}dx + \sum_{F\in\mathcal{F}_m\cup\mathcal{F}_a}\int_F\left(M_{F,K}\mathbf{W}_h\right)^t\mathbf{V}ds + \sum_{F\in\mathcal{F}_0}\int_F\left[\left(G_{\mathbf{n}_F}\mathbf{W}_{h|K}\right)^t\left(I_{F,\tilde{K}}\mathbf{V}_{|\tilde{K}}\right) + \left(G_{\mathbf{n}_F}\mathbf{W}_{h|\tilde{K}}\right)^t\left(I_{F,K}\mathbf{V}_{|K}\right)\right]ds = 0$$

Properties of the fully discrete scheme

- Bilinear forms

Find $\mathbf{W}_h \in V_h$ such that $\forall \mathbf{V} \in V_h$

$$a(\partial_t \mathbf{W}, \mathbf{V}) + b(\mathbf{W}, \mathbf{V}) + c(\mathbf{W}, \mathbf{V}) = 0, \ \forall \mathbf{V} \in V_h$$

$$\left\{ \begin{array}{rcl} a(\mathbf{W}, \mathbf{V}) & = & 2 \displaystyle\int_{\Omega_h} (G_0 \mathbf{W})^t \, \mathbf{V} dx \\ b(\mathbf{W}, \mathbf{V}) & = & - \displaystyle\int_{\Omega_h} \mathbf{W}^t \left(\sum_{l \in \{x,y,z\}} G_l \partial_l \mathbf{V} \right) dx + \int_{\Omega_h} \left(\sum_{l \in \{x,y,z\}} G_l \partial_l \mathbf{W} \right)^t \mathbf{V} dx \\ c(\mathbf{W}, \mathbf{V}) & = & \displaystyle\sum_{F \in \mathcal{F}_0} \int_F \left[\left(G_{\mathbf{n}_F} \mathbf{W}_{|K} \right)^t \left(I_{F,\tilde{K}} \mathbf{V}_{|\tilde{K}} \right) + \left(G_{\mathbf{n}_F} \mathbf{W}_{|\tilde{K}} \right)^t \left(I_{F,K} \mathbf{V}_{|K} \right) \right] ds \\ & + & \displaystyle\sum_{F \in \mathcal{F}_m \cup \mathcal{F}_a} \int_F (M_{F,K} \mathbf{W})^t \, \mathbf{V} ds \end{array} \right.$$

Properties of the fully discrete scheme

- Properties of the bilinear forms
  - $b(\mathbf{W}, \mathbf{V})$ is skew-symmetric
  - Since $G_{\mathbf{n}_F}$ is symmetric and $I_{F,\tilde{K}} = -I_{F,K}$ then,

$$c(\mathbf{W}_h, \mathbf{W}_h) = \sum_{F \in \mathcal{F}_m \cup \mathcal{F}_a} \int_F \left(M_{F,K} \mathbf{W}_h\right)^t \mathbf{W}_h ds$$

but M_{Γ_m} is skew-symmetric thus,

$$c(\mathbf{W}_h, \mathbf{W}_h) = \sum_{F \in \mathcal{F}_a} \int_F \left(|G_{\mathbf{n}_F}| \mathbf{W}_h\right)^t \mathbf{W}_h ds$$

- Fully discrete formulation based on Crank-Nicolson scheme

Find $\mathbf{W}_h^{n+1} \in V_h$ such that $\forall \mathbf{V} \in V_h$

$$\begin{aligned} \beta a(\mathbf{W}_h^{n+1}, \mathbf{V}) &+ b(\mathbf{W}_h^{n+1}, \mathbf{V}) + c(\mathbf{W}_h^{n+1}, \mathbf{V}) \\ &= \beta a(\mathbf{W}_h^{n}, \mathbf{V}) - b(\mathbf{W}_h^{n}, \mathbf{V}) - c(\mathbf{W}_h^{n}, \mathbf{V}) \end{aligned}$$

with $\beta = \dfrac{2}{\Delta t}$

Properties of the fully discrete scheme

- Well-posedness of the discrete problem
  The homogeneous discrete problem,

$$\text{Find } \mathbf{W}_h \in V_h \text{ such that}$$
$$\beta a(\mathbf{W}_h, \mathbf{V}) + b(\mathbf{W}_h, \mathbf{V}) + c(\mathbf{W}_h, \mathbf{V}) = 0, \ \forall \mathbf{V} \in V_h$$

  possesses only the trivial solution

- Elements of the proof
  - $b(\mathbf{W}, \mathbf{V})$ is skew-symmetric
  - G_0 is symmetric positive definite
  - $|G_{\mathbf{n}_F}|$ is positive

$$\begin{aligned} \beta a(\mathbf{W}_h, \mathbf{W}_h) \quad + \quad & b(\mathbf{W}_h, \mathbf{W}_h) \\ + \quad & c(\mathbf{W}_h, \mathbf{W}_h) = 2\beta \int_{\Omega_h} (G_0 \mathbf{W}_h)^t \, \mathbf{W}_h dx \\ & + \sum_{F \in \mathcal{F}_a} \int_F \mathbf{W}_h^t |G_{\mathbf{n}_F}| \mathbf{W}_h ds = 0 \end{aligned}$$

Properties of the fully discrete scheme

- Fully discrete energy: $\mathcal{E}_h^n = \dfrac{1}{4} a(\mathbf{W}_h^n, \mathbf{W}_h^n)$

  Then $\mathcal{E}^n$ is exactly conserved if $\Gamma_a = \emptyset$ or $\mathcal{E}_h^{n+1} \leq \mathcal{E}_h^n$ if $\Gamma_a \neq \emptyset$

- Elements of the proof
  - Choose as a test function $\mathbf{V} = \dfrac{\mathbf{W}_h^{n+1} + \mathbf{W}_h^n}{2}$
  - $b(\mathbf{W}, \mathbf{V})$ is a skew-symmetric
  - $G_{\mathbf{n}_F}$ and $|G_{\mathbf{n}_F}|$ are symmetric

- Convergence of the fully discrete scheme
  - L. Fezoui, S. Lanteri, S. Lohrengel and S. Piperno
    M2AN, Vol. 39, No. 6, 2005

$$\mathcal{O}(Th^{\min(s,p)}) + \mathcal{O}(\Delta t^2)$$

for the total error in $C^0([0,T]; L^2(\Omega))$ with $s > \dfrac{1}{2}$ a regularity parameter

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Outline

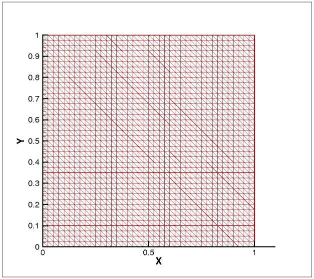
Numerical results

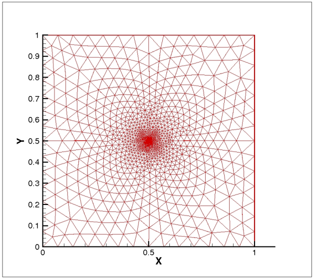
- Two-dimensional Maxwell's equation (TM)

$$\begin{cases} \mu \dfrac{\partial H_x}{\partial t} + \dfrac{\partial E_z}{\partial y} = 0 \\ \mu \dfrac{\partial H_y}{\partial t} - \dfrac{\partial E_z}{\partial x} = 0 \\ \varepsilon \dfrac{\partial E_z}{\partial t} - \dfrac{\partial H_y}{\partial x} + \dfrac{\partial H_x}{\partial y} = 0 \end{cases}$$

  - Implicit DGTD-$\mathbb{P}_p$ method
    - Triangular mesh
    - Sparse block matrix, $3n_p \times 3n_p$ (with $n_p = ((p+1)(p+2))/2$)
    - MUMPS multifrontal sparse matrix solver
      (P.R. Amestoy, I.S. Duff and J.-Y. L'Excellent, CMAME, Vol. 184, 2000)
    - LU factors computed before entering the time stepping loop

Numerical results
Eigenmode in a metallic cavity



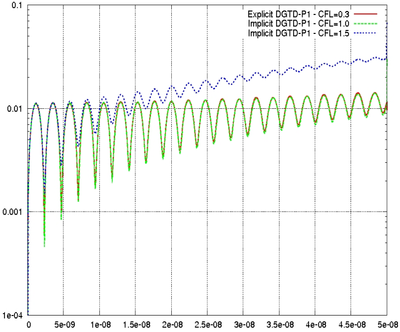


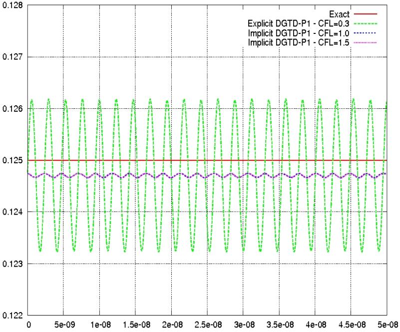
$\#$ vertices $= 1{,}681$
$\#$ elements $= 3{,}200$

$(\Delta t)_u = 58.92$ picosec

$\#$ vertices $= 1{,}400$
$\#$ elements $= 2{,}742$

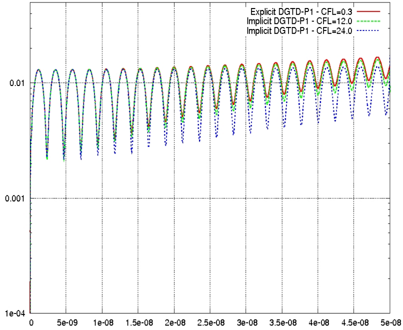
$(\Delta t)_m = 1.44$ picosec
$(\Delta t)_M = 235.54$ picosec

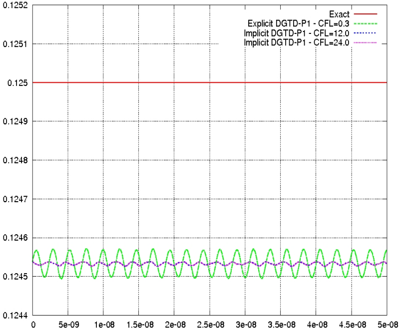




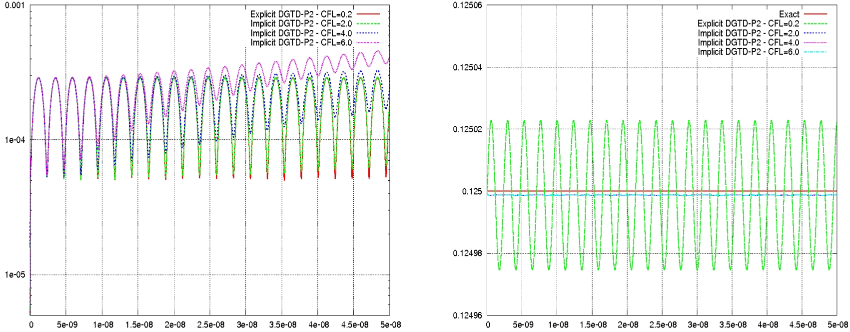
DGTD-$\mathbb{P}_1$ method: time evolutions of the L2 error (left) and discrete energy (right)
Uniform mesh

Numerical results
Eigenmode in a metallic cavity

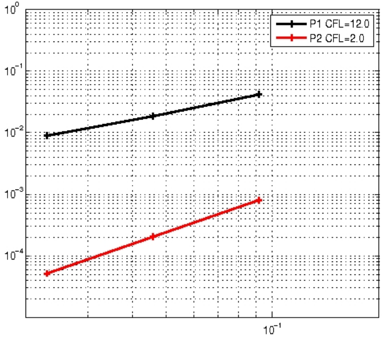




DGTD-$\mathbb{P}_1$ method: time evolutions of the L2 error (left) and discrete energy (right)
Non-uniform mesh



DGTD-$\mathbb{P}_2$ method: time evolutions of the L2 error (left) and discrete energy (right)
Non-uniform mesh



Numerical convergence of the implicit DGTD-$\mathbb{P}_1$ and DGTD-$\mathbb{P}_2$ methods
Non-uniform mesh
Global (space and time) L2 error versus maximal edge length (logarithmic scales)

Slopes: 1.11 (DGTD-$\mathbb{P}_1$ method) and 1.98 (DGTD-$\mathbb{P}_2$ method)

Computing times (AMD Opteron 2 GHz based workstation)

Time integration	Method	CFL-$\mathbb{P}_p$	CPU time
Explicit	DGTD-$\mathbb{P}_1$	0.3	15 sec
Implicit	-	1.0	44 sec
-	-	1.5	30 sec

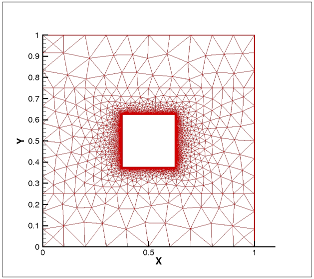
Uniform mesh

Time integration	Method	CFL-$\mathbb{P}_p$	CPU time
Explicit	DGTD-$\mathbb{P}_1$	0.3	443 sec
Implicit	-	12.0	133 sec
-	-	24.0	67 sec
Explicit	DGTD-$\mathbb{P}_2$	0.2	2057 sec
Implicit	-	2.0	1923 sec
-	-	4.0	938 sec
-	-	6.0	620 sec

Non-uniform mesh

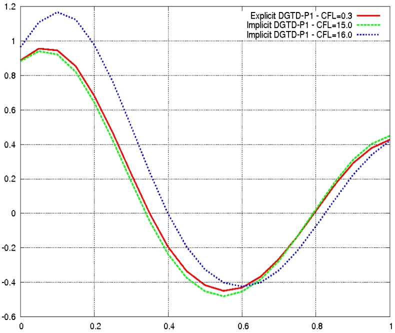
Factorization: CPU time, RAM size (LU/total)
$<$ 1 sec, 13 MB/26 MB (DGTD-$\mathbb{P}_1$) and 2 sec, 34 MB/64 MB (DGTD-$\mathbb{P}_2$)

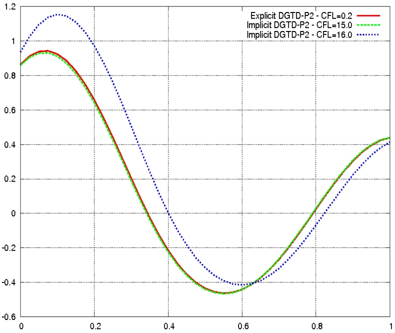
Numerical results
Scattering of a plane wave by a PEC square



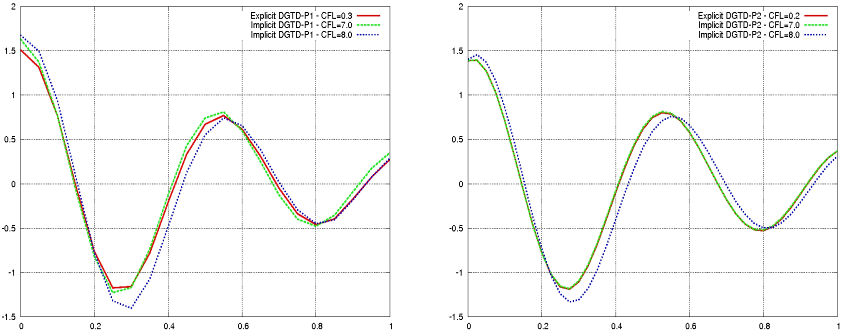
$\#$ vertices $= 6{,}018$
$\#$ elements $= 10{,}792$

$(\Delta t)_m = 0.95$ picosec
$(\Delta t)_M = 328.60$ picosec

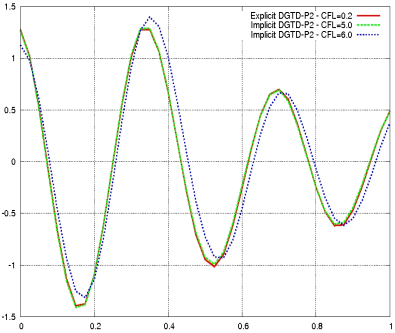


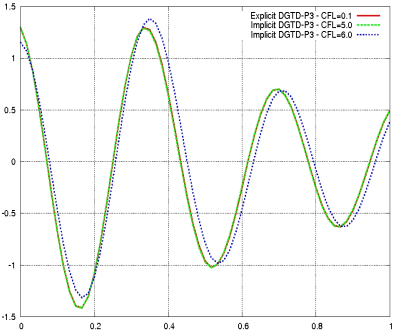


F=300 MHz: 1D distribution of DFT(E_z), $y = 0.75$ m
Left: DGTD-$\mathbb{P}_1$ method - Right: DGTD-$\mathbb{P}_2$ method



F=600 MHz: 1D distribution of DFT(E_z), $y = 0.75$ m
Left: DGTD-$\mathbb{P}_1$ method - Right: DGTD-$\mathbb{P}_2$ method





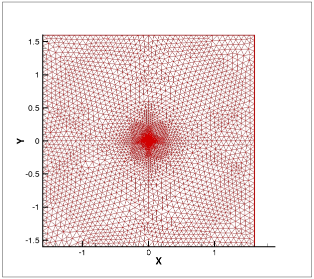
F=900 MHz: 1D distribution of DFT(E_z), $y = 0.75$ m
Left: DGTD-$\mathbb{P}_2$ method - Right: DGTD-$\mathbb{P}_3$ method

Computing times (AMD Opteron 2 GHz based workstation)

Frequency	Time integration	Method	CFL-$\mathbb{P}_p$	CPU time
300 MHz	Explicit	DGTD-$\mathbb{P}_1$	0.3	1602 sec
-	Implicit	-	15.0	370 sec
-	Explicit	DGTD-$\mathbb{P}_2$	0.2	5677 sec
-	Implicit	-	15.0	762 sec
600 MHz	Explicit	DGTD-$\mathbb{P}_1$	0.3	758 sec
-	Implicit	-	7.0	383 sec
-	Explicit	DGTD-$\mathbb{P}_2$	0.2	3074 sec
-	Implicit	-	7.0	767 sec
900 MHz	Explicit	DGTD-$\mathbb{P}_2$	0.2	2191 sec
-	Implicit	-	5.0	746 sec
-	Explicit	DGTD-$\mathbb{P}_3$	0.1	8771 sec
-	Implicit	-	5.0	1591 sec

Computing times (AMD Opteron 2 GHz based workstation)
Factorization phase

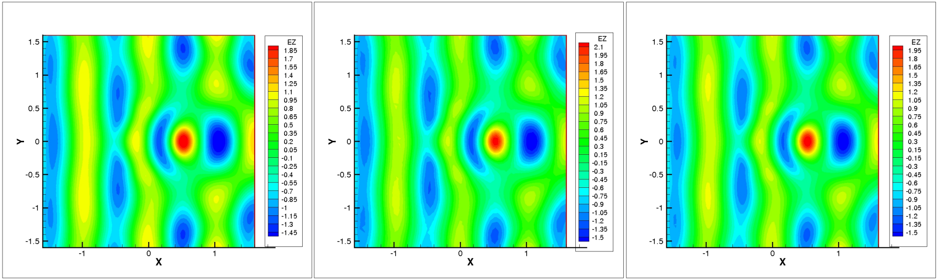
Frequency	Method	CFL-$\mathbb{P}_p$	CPU time	RAM size (LU/total)
300 MHz	DGTD-$\mathbb{P}_1$	15.0	3 sec	38 MB/ 73 MB
-	DGTD-$\mathbb{P}_2$	15.0	8 sec	106 MB/194 MB
600 MHz	DGTD-$\mathbb{P}_1$	7.0	3 sec	38 MB/ 73 MB
-	DGTD-$\mathbb{P}_2$	7.0	8 sec	106 MB/194 MB
900 MHz	DGTD-$\mathbb{P}_2$	5.0	8 sec	106 MB/194 MB
-	DGTD-$\mathbb{P}_3$	5.0	16 sec	229 MB/425 MB



$\#$ vertices $= 4{,}108$
$\#$ elements $= 8{,}054$

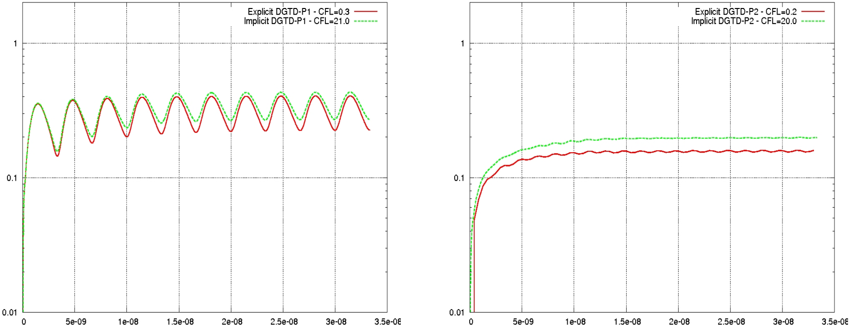
Cylinder: R=0.6 m
F=300 MHz, $\varepsilon_{r,1} = 1.0$ and $\varepsilon_{r,2} = 2.25$

$(\Delta t)_m = 2.09$ picosec
$(\Delta t)_M = 309.63$ picosec

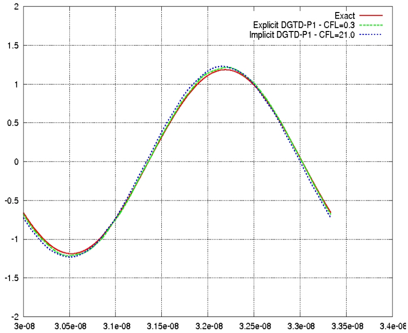


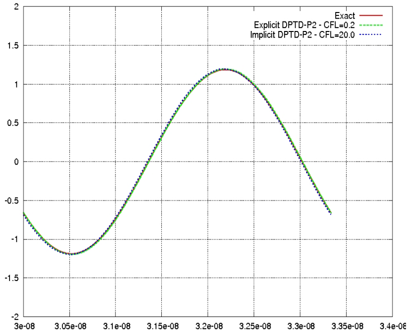
F=300 MHz: contour lines of E_z after 10 periods
Left: exact solution
Middle: implicit DGTD-$\mathbb{P}_1$ method - Right: implicit DGTD-$\mathbb{P}_2$ method

Numerical results
Scattering of a plane wave by a dielectric cylinder



F=300 MHz: time evolution of the L2 error
Left: DGTD-$\mathbb{P}_1$ method - Right: DGTD-$\mathbb{P}_2$ method

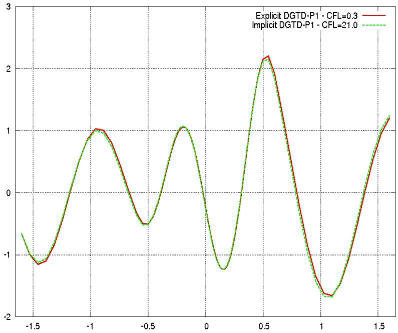


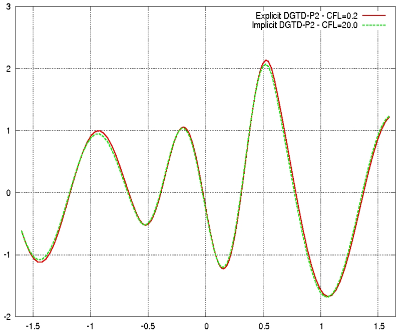


F=300 MHz: time evolution of E_z (zoom on the last of 10 periods)
Left: DGTD-$\mathbb{P}_1$ method - Right: DGTD-$\mathbb{P}_2$ method

Numerical results

Scattering of a plane wave by a dielectric cylinder





F=300 MHz: 1D distribution of DFT(E_z), $y = 0.0$ m
Left: DGTD-$\mathbb{P}_1$ method - Right: DGTD-$\mathbb{P}_2$ method

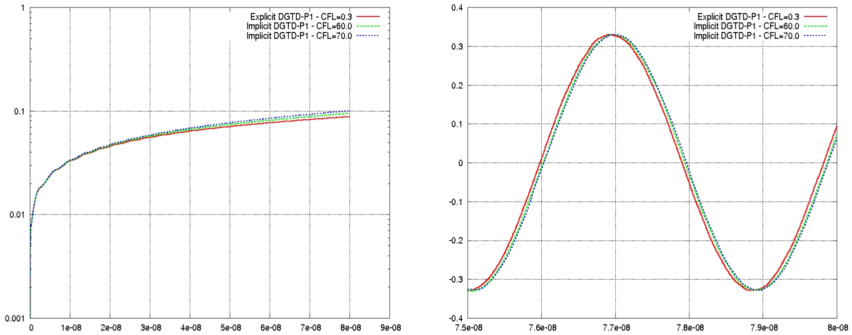
Computing times (AMD Opteron 2 GHz based workstation)

Time integration	Method	CFL-$\mathbb{P}_p$	CPU time
Explicit	DGTD-$\mathbb{P}_1$	0.3	542 sec
Implicit	-	21.0	102 sec
Explicit	DGTD-$\mathbb{P}_2$	0.2	1892 sec
Implicit	DGTD-$\mathbb{P}_2$	20.0	218 sec

Factorization phase

Method	CFL-$\mathbb{P}_p$	CPU time	RAM size (LU/total)
DGTD-$\mathbb{P}_1$	20.0	3 sec	40 MB/ 70 MB
DGTD-$\mathbb{P}_2$	20.0	6 sec	107 MB/181 MB

- # vertices = 3,815 and # tetrahedra = 19,540
- $(\Delta t)_m = 0.84$ picosec and $(\Delta t)_M = 107.10$ picosec



DGTD-$\mathbb{P}_1$ method: time evolutions of the L2 error (left) and of E_z (right)

Numerical results

Preliminary result in 3D: eigenmode in a metallic cavity

Computing times (AMD Opteron 2.2 GHz cluster with Myrinet)
Simulation for a duration of 80 nanosec, 8 processors

Time integration	Method	CFL-$\mathbb{P}_p$	CPU time
Explicit	DGTD-$\mathbb{P}_1$	0.3	2670 sec
Implicit	-	60.0	1312 sec
-	-	70.0	1049 sec

Factorization phase

CFL	CPU time	LU RAM size (min/max)	Total RAM size (mn/max)
60.0	1484 sec	343 MB/600 MB	1061 MB/1608 MB
70.0	1052 sec	-	-

Outline

Closure

- Implicit DGTD-$\mathbb{P}_p$ method
  - Second order accurate in time an $p + 1$-th order accurate in space
  - Non-dissipative and unconditionally stable
  - Allowable CFL (global time step) is dictated by physical considerations
- Performance issues
  - LU factorization is a viable option in 2D
  - For a fixed p, CPU explicit/CPU implicit $\searrow$ when frequency F $\nearrow$
  - For a fixed F, CPU explicit/CPU implicit $\nearrow$ when interpolation order p $\nearrow$
- Future works
  - Extension to the 3D case
    - Locally implicit DGTD-$\mathbb{P}_p$ method
    - Domain decomposition for a hybrid iterative/direct linear solver
  - Analytical dispersion analysis
  - High order time integration methods

Thank you for your attention!

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