The Relation Between Optimized Schwarz Methods for Scalar and Systems of Partial Differential Equations

Victorița Dolean and Martin Gander

University of Nice/Sophia Antipolis
J.A. Dieudonné Mathematics Laboratory

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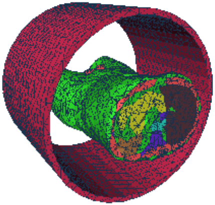
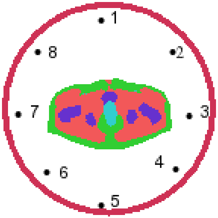
Outline of the talk

Motivation

Local hyperthemia using electromagnetic waves

Treatment of a cancerous tumour by rising locally the temperature of the tumour.
Tool : use an electromagnetic field radio-frequencies or micro-waves.

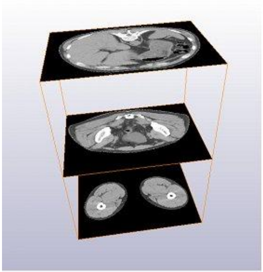


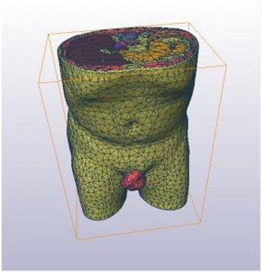


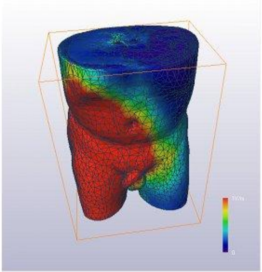
Motivation

Therapeutical planning

1. Segmentation of scanners images
2. Mesh of the body
3. Electromagnetic and thermic computations + optimization of the parameters.







Electromagnetic computation

- Mathematical model:

$$\begin{cases} (\mathrm{i}\omega\epsilon + \sigma)\mathbf{E} - \operatorname{curl}\mathbf{H} = -\mathbf{J}_{\mathrm{imp}}, \\ \mathrm{i}\omega\mu\mathbf{H} + \operatorname{curl}\mathbf{E} = 0. \end{cases}$$

- Other features:
  - Unbounded domain.
  - Antennas : courant source terms inside the domain.
  - Linear isotropic material for a given frequency.
  - Unstructured mesh and heterogeneous media.

Optimized Schwarz: from scalar problems to systems

- Schwarz algorithms that are convergent without overlap: Lions '90.
- Approximate radiation conditions for Helmholtz: Despres, '91.
- The use of non-local operators first invocated in Hagstrom, '88.
- Approximations of non-local interface conditions for advection-diffusion equation Charton '91, Nataf '95, optimized transmission conditions Japhet '00.
- Helmholtz equations: Chevalier '98, Gander, Magoules, Nataf '01.
- Optimized conditions for symmetric, positive definite problems Gander '06, time-dependent problems Gander, Halpern, Nataf '03.
- Maxwell's equations (curl-curl) Alonso,Gerardo-Giorda '06.
- Derivation of optimized conditions for Cauchy-Riemann equations using the equivalence with a scalar problem V.D., Gander '06.

Systematic approach: from scalar problems to systems using Smith factorization for Euler V.D., Nataf '05 and Stokes V.D., Nataf, Rapin '06

Mathematical model

Scattering problem - total field

$$\begin{cases} (\mathrm{i}\omega G_0 + G_0')\mathbf{W} + G_x\partial_x\mathbf{W} + G_y\partial_y\mathbf{W} + G_z\partial_z\mathbf{W} = 0, \text{ in } \Omega, \\ (M_{\Gamma_m} - G_{\mathbf{n}})\mathbf{W} = 0 \text{ on } \Gamma_m, \\ (M_{\Gamma_a} - G_{\mathbf{n}})(\mathbf{W} - \mathbf{W}_{\mathrm{inc}}) = 0 \text{ on } \Gamma_a. \end{cases}$$

- Unknown electromagnetic vector field $\mathbf{W}$: $\mathbf{W} = (\mathbf{E}, \mathbf{H})^T$.
- Properties of different media :

$$G_0 = \begin{pmatrix} \epsilon I_3 & 0_{3\times 3} \\ 0_{3\times 3} & \mu I_3 \end{pmatrix} \text{ and } G_0' = \begin{pmatrix} \sigma I_3 & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{pmatrix}.$$

- Vector product with $\mathbf{e}_l$, l in $\{x, y, z\}$:

$$G_l = \begin{pmatrix} O_{3\times 3} & N_{\mathbf{e}_l} \\ N_{\mathbf{e}_l}^t & O_{3\times 3} \end{pmatrix} \text{ with } N_{\mathbf{n}} = \begin{pmatrix} 0 & \mathbf{n}_z & -\mathbf{n}_y \\ -\mathbf{n}_z & 0 & \mathbf{n}_x \\ \mathbf{n}_y & -\mathbf{n}_x & 0 \end{pmatrix}$$

Mathematical model

- Taking into account the boundary conditions :

$$M_{\Gamma_m} = \begin{pmatrix} 0_{3\times3} & N_{\mathbf{n}} \\ -N_{\mathbf{n}}^t & 0_{3\times3} \end{pmatrix} \text{ and } M_{\Gamma_a} = |G_{\mathbf{n}}|.$$

- Characteristic variables in direction $\mathbf{n}$: $\mathbf{w} = T_{\mathbf{n}}^{-1}\mathbf{W}$ (where $G_{\mathbf{n}} = G_x n_x + G_y n_y + G_z n_z = T_{\mathbf{n}} \Lambda_{\mathbf{n}} T_{\mathbf{n}}^{-1}$) used to impose a simple approximation of absorbing boundary conditions at $\partial\Omega$ where $\mathbf{n}$ is the outward normal.

Characteristic variables associated with the direction $\widetilde{\mathbf{n}} = (1, 0, 0)$

$$w_1 = -\frac{1}{2}(E_2 - H_3),\; w_2 = \frac{1}{2}(E_3 + H_2),\; w_3 = H_1,$$
$$w_4 = E_1,\; w_5 = \frac{1}{2}(E_2 + H_3),\; w_6 = -\frac{1}{2}(E_3 - H_2)$$
$$\mathbf{w}_- = (w_1, w_2)^T,\; \mathbf{w}_0 = (w_3, w_4)^T,\; \mathbf{w}_+ = (w_5, w_6)^T.$$

Relation to a scalar equation

Consider a simplified Maxwell system on the domain $\Omega = [0,1] \times \mathbb{R}^2$:

- No conductivity: $\sigma = 0$.
- Homogeneous media (ϵ, μ constants) and normalization of the variables: equivalent system with $\epsilon = \mu = 1$.
- No source of courant: $\mathbf{J} = 0$.

Maxwell system written in characteristic variables:

$$\begin{array}{rcl}
(i\omega - \partial_x)w_1 + \frac{1}{2}\partial_z w_3 - \frac{1}{2}\partial_y w_4 & = & 0 \\
(i\omega - \partial_x)w_2 + \frac{1}{2}\partial_y w_3 + \frac{1}{2}\partial_z w_4 & = & 0 \\
i\omega w_3 + \partial_z w_1 + \partial_y w_2 - \partial_z w_5 - \partial_y w_6 & = & 0 \\
i\omega w_1 - \partial_y w_1 + \partial_z w_2 - \partial_y w_5 + \partial_z w_6 & = & 0 \\
(i\omega + \partial_x)w_5 - \frac{1}{2}\partial_z w_3 - \frac{1}{2}\partial_y w_4 & = & 0 \\
(i\omega + \partial_x)w_6 - \frac{1}{2}\partial_y w_3 + \frac{1}{2}\partial_z w_4 & = & 0
\end{array}$$

with the characteristic boundary conditions + radiation condition $\Rightarrow$ well-posed problem.

$$\mathbf{w}_+(0,y,z) = \mathbf{r}(y,z), \quad \mathbf{w}_-(1,y,z) = \mathbf{s}(y,z), \quad (y,z) \in \mathbb{R}^2,$$

Equivalence result

Let **w** be the characteristic variables. Any component $\widetilde{w}_j$, $j = 1, \ldots, 6$, of the characteristic variables of the Maxwell system satisfies, in the interior of $\Omega = [0,1] \times \mathbb{R}^2$, the Helmholz equation,

$$-(\omega^2 + \Delta)w_j = 0, \quad j = 1, 2, \ldots, 6,$$

together with the boundary conditions

$$\begin{array}{rcl} (\partial_x - i\omega)\widetilde{w}_j(0,y,z) & = & \widetilde{r}_j(y,z), \\ \widetilde{w}_j(1,y,z) & = & \widetilde{s}_j(1,y,z), (y,z) \in \mathbb{R}^2. \end{array}$$

Equivalence between Schwarz algorithms

Classical Schwarz algorithm for Maxwell

Decomposition into domains: $\Omega_1 = [0, \alpha] \times \mathbb{R}^2$, $\Omega_2 = [\beta, 1] \times \mathbb{R}^2$

$$i\omega\,\mathbf{w}^{1,n} + \sum_{l=x,y,z} G_l \partial_l \mathbf{w}^{1,n} = 0, \Omega_1 \qquad i\omega\,\mathbf{w}^{2,n} + \sum_{l=x,y,z} \mathbf{G}_l \partial_l \mathbf{w}^{2,n} = 0, \Omega_2,$$

$$\mathbf{w}_+^{1,n} = \mathbf{r},\ \partial\Omega_1 \cap \Omega, \qquad \mathbf{w}_-^{2,n} = \mathbf{s},\ \partial\Omega_2 \cap \Omega,$$

$$\mathbf{w}_-^{1,n} = \mathbf{w}_-^{2,n-1},\ \partial\Omega_1 \cap \Omega_2, \qquad \mathbf{w}_+^{2,n} = \mathbf{w}_+^{1,n-1},\ \partial\Omega_2 \cap \Omega_1,$$

Schwarz algorithm for Helmholtz

$$-(\omega^2 + \Delta)\widetilde{w}_j^{1,n} = 0, \Omega_1, \qquad -(\omega^2 + \Delta)\widetilde{w}_j^{2,n} = \widetilde{0}, \Omega_2,$$

$$(\partial_x - i\omega)\widetilde{w}_j^{1,n} = \widetilde{r}_j,\ \partial\Omega_1 \cap \Omega, \qquad \widetilde{w}_j^{2,n} = \widetilde{s}_1, \partial\Omega_2 \cap \Omega,$$

$$\widetilde{w}_j^{1,n} = \widetilde{w}_j^{2,n-1},\ \partial\Omega_1 \cap \Omega_2, \qquad (\partial_x - i\omega)\widetilde{w}_j^{2,n} = (\partial_x - i\omega)\widetilde{w}_j^{1,n-1},\ \partial\Omega_2 \cap \Omega_1.$$

Convergence rate of the algorithm

Proposition Let $\Omega = \mathbb{R}^3$, and consider the Maxwell system in Ω with the radiation condition

$$\lim_{r\to\infty} r(\mathbf{n}\times\mathbf{E} + \mathbf{n}\times(\mathbf{n}\times\mathbf{H})) = 0,$$

where $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$. Let Ω be decomposed into $\Omega_1 := (-\infty, L)\times\mathbb{R}^2$ and $\Omega_2 := (0, +\infty)\times\mathbb{R}^2$, $(L \geq 0)$. For any given initial guess $\mathbf{w}^{1,0} \in (L^2(\Omega_1))^6$, $\mathbf{w}^{2,0} \in (L^2(\Omega_2))^6$, the Schwarz algorithm applied to system converges for all Fourier modes such that $k_y^2 + k_z^2 \neq \omega^2$. The convergence factor is

$$R_{th} = \begin{cases} \left|\frac{\sqrt{\omega^2-(k_y^2+k_z^2)}-\omega}{\sqrt{\omega^2-(k_y^2+k_z^2)}+\omega}\right|, & \text{for } k_y^2+k_z^2 < \omega^2, \\ e^{-\sqrt{k_y^2+k_z^2-\omega^2}L}, & \text{for } k_y^2+k_z^2 > \omega^2. \end{cases}$$

Absorbing boundary conditions for Maxwell's equations

The exact absorbing boundary conditions for the time harmonic Maxwell equations on the domain $\Omega = (0,1) \times \mathbb{R}^2$:

$$(\mathbf{w}_+ + \mathcal{S}_1 \mathbf{w}_-)(0,y,z) = 0, \quad (\mathbf{w}_- + \mathcal{S}_2 \mathbf{w}_+)(1,y,z) = 0, \qquad (y,z) \in \mathbb{R}^2,$$

where the operators $\mathcal{S}_l$, $l = 1,2$, are general, pseudodifferential operators acting in the y and z directions.

Absorbing boundary conditions for Maxwell

Lemma If the operators $\mathcal{S}_l$, $l = 1,2$ have the Fourier symbol

$$\mathcal{F}(\mathcal{S}_l) = \frac{1}{(\sqrt{|k|^2 - \omega^2} + i\omega)^2} \begin{bmatrix} k_z^2 - k_y^2 & -2k_y k_z \\ -2k_y k_z & k_y^2 - k_z^2 \end{bmatrix}, \qquad j = 1,2,$$

then the solution of the Maxwell equations in Ω coincides with the restriction on Ω of the solution of the Maxwell system on $\mathbb{R}^3$.

Absorbing boundary conditions for Helmholtz's equation

Absorbing boundary conditions for the Helmholtz equation in $\Omega = (0,1) \times \mathbb{R}^2$

$$(\partial_x - \widetilde{\mathcal{S}}_1)\mathbf{u}(0,y,z) = 0, \quad (\partial_x + \widetilde{\mathcal{S}}_2)\mathbf{u}(1,y,z) = 0, \qquad (y,z) \in \mathbb{R}^2,$$

where $\widetilde{\mathcal{S}}_j$ $(j = 1,2)$ are general, pseudodifferential operators acting in the y and z directions.

Absorbing boundary conditions for Helmholtz

Lemma If the operators $\widetilde{\mathcal{S}}_l$ $(l = 1,2)$ have the Fourier symbol

$$\widetilde{\sigma}_l = \mathcal{F}(\widetilde{\mathcal{S}}_l) = \sqrt{|k|^2 - \omega^2}$$

then the solution of Helmholtz equation in Ω coincides with the restriction on Ω of the solution of the Helmholtz equation on $\mathbb{R}^3$.

More general interface conditions

$$\Omega_1 : [(\mathbf{w}_- + \mathcal{S}_1 \mathbf{w}_+)(L,y,z)]^{1,n} = [(\mathbf{w}_- + \mathcal{S}_1 \mathbf{w}_+)(L,y,z)]^{2,n-1}, \quad (y,z) \in \mathbb{R}^2,$$

$$\Omega_2 : [(\mathbf{w}_+ + \mathcal{S}_2 \mathbf{w}_-)(0,y,z)]^{2,n} = [(\mathbf{w}_+ + \mathcal{S}_2 \mathbf{w}_-)(0,y,z)]^{1,n-1}, \quad (y,z) \in \mathbb{R}^2.$$

BUT the operators $\mathcal{S}_l, l = 1,2$, which lead to this optimal performance, are non-local operators $\longrightarrow$ the necessity of approximating operators in the transmission conditions.

General form of the interface conditions

The operators $\mathcal{S}_1$ and $\mathcal{S}_2$ have the Fourier symbol

$$\sigma_l := \mathcal{F}(\mathcal{S}_l) = \gamma_l \begin{bmatrix} k_z^2 - k_y^2 & -2k_y k_z \\ -2k_y k_z & k_y^2 - k_z^2 \end{bmatrix}, \qquad \gamma_l \in \mathbb{C}(k_y, k_z) \quad (l = 1,2),$$

Equivalence between optimized methods

Different choices of transmission conditions in the optimized Schwarz algorithm

Maxwell equations	Helmholtz
Case 1: Dirichlet/Dirichlet	Després conditions
Case 2: Optimized 1 param	Optimized/Després
Case 3: Optimized 2 param	Optimized/Optimized
Exact absorbing	Exact absorbing

Asymptotical behavior of the OSM

Five variants of the optimized Schwarz method applied to Maxwell's equations, when the mesh parameter h is small, and the maximum numerical frequency is estimated by $k_{\max} = \frac{C}{h}$, and where $C_\omega = \min\left(k_+^2 - \omega^2, \omega^2 - k_-^2\right), \bar{C}_\omega = k_+^2 - \omega^2$.

Convergence rate after optimization

	without overlap, $L = 0$	
Case	ρ	parameters
1	1	none
2	$1 - \frac{\sqrt{2}C_\omega^{\frac{1}{4}}}{\sqrt{C}}\sqrt{h}$	$p = \frac{\sqrt{C}C_\omega^{\frac{1}{4}}}{\sqrt{2}\sqrt{h}}$
3	$1-\frac{\sqrt{2}(\bar{C}_\omega)^{\frac{1}{4}}}{\sqrt{C}}\sqrt{h}$	$p = \frac{\sqrt{C}(\bar{C}_\omega)^{\frac{1}{4}}}{\sqrt{2}\sqrt{h}}$
4	$1 - \frac{C_\omega^{\frac{1}{8}}}{C^{\frac{1}{4}}}h^{\frac{1}{4}}$	$p_1 = \frac{C_\omega^{\frac{3}{8}} \cdot C^{\frac{1}{4}}}{2 \cdot h^{\frac{1}{4}}}$, $p_2 = \frac{C_\omega^{\frac{1}{8}} \cdot C^{\frac{3}{4}}}{h^{\frac{3}{4}}}$
5	$1 - \frac{(\bar{C}_\omega)^{\frac{1}{8}}}{C^{\frac{1}{4}}}h^{\frac{1}{4}}$	$p_1 = \frac{(\bar{C}_\omega)^{\frac{3}{8}} \cdot C^{\frac{1}{4}}}{2 \cdot h^{\frac{1}{4}}}$, $p_2 = \frac{(\bar{C}_\omega)^{\frac{1}{8}} \cdot C^{\frac{3}{4}}}{h^{\frac{3}{4}}}$

Comparison of the different methods

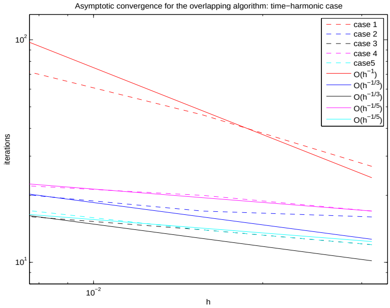
- Two-dimensional case: transverse electric waves
- unit square decomposed into two subdomains $\Omega_1 = (0, \beta) \times (0, 1)$ and $\Omega_2 = (\alpha, 1) \times (0, 1)$, where $0 < \alpha \leq \beta < 1$
- tolerance fixed at $\varepsilon = 10^{-6}$

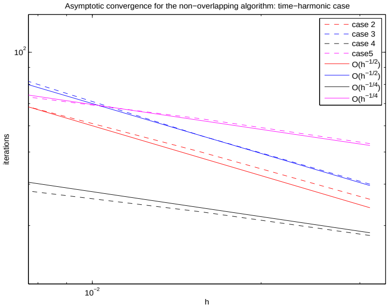
Non-overlapping and overlapping optimized methods

	with overlap, $L = h$				without overlap, $L = 0$			
h	1/16	1/32	1/64	1/128	1/16	1/32	1/64	1/128
Case 1	18	27	46	71	-	-	-	-
Case 2	16	16	17	20	28	36	50	68
Case 3	10	12	14	16	31	40	56	81
Case 4	17	17	20	22	26	28	33	38
Case 5	10	12	14	17	41	53	63	73

Comparison of the different methods

Theoretical and numerical asymptotics for overlapping and non-overlapping optimized methods



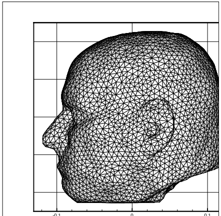


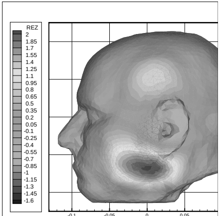
Preliminary three-dimensional results: a bioelectromagnetism example

Propagation of a plane wave in realistic geometrical models of head tissues: collaboration with S. Lanteri (INRIA Sophia-Antipolis)

Table: Characteristics of the tetrahedral meshes.

Mesh	# tetraheda	L_{min} (m)	L_{max} (m)	L_{avg} (m)
M1	361,848	0.00185	0.04537	0.01165
M2	1,853,832	0.00158	0.02476	0.00693





Conclusions

- derivation of all possible optimized conditions for first order Maxwell system and optmization whenever no equivalence with scalar algorithm was found.
- validation on a simple 2d geometry and implementation in three-dimensions.

Ongoing work

- The use of optimized interface conditions with high order discretization methods (DG methods) in collaboration with R. Perrussel (Lab. Ampère, Lyon) and S. Lanteri (INRIA).
- Promising results concerning the robustness of optimized parameters with respect to the polynomial order (2d numerical simulations).

Future works

- Optimization of the parameters in the general case by taking into account the conductivity and application to realistic three-dimensional bioelectromagnetism simulations.