Numerical assessment of a high-order non-conforming discontinuous Galerkin method for electromagnetic wave propagation

H. Fahs[1] S. Lanteri[1] F. Rapetti[2]

[1]INRIA, NACHOS project-team
2004 Route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex, France

[2]University of Nice/Sophia Antipolis
J.A. Dieudonné Mathematics Laboratory, 06108 Nice Cedex, France

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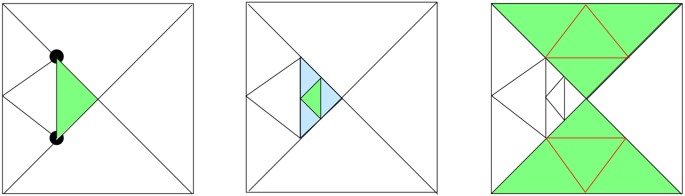
Context and motivations

- Time-domain electromagnetic wave propagation
- Irregulary shaped geometries, heterogeneous media
  - Non-conforming, locally refined, triangular (2D)/tetrahedral (3D) meshes
- Numerical ingredients (starting point to this study)

  L. Fezoui, S. Lanteri, S. Lohrengel and S. Piperno: ESAIM, M2AN, 2005
  - Discontinuous Galerkin time-domain (DGTD) methods
  - Nodal (Lagrange type) polynomial interpolation
  - Explicit time integration
- Overall objectives of this study
  - Investigate strengthes and weaknesses of explicit DGTD methods using non-conforming simplicial meshes with arbitrary level hanging nodes
    - Theoretical and numerical aspects (stability, dispersion error, convergence)
    - Computational aspects

Context and motivations
Non-conforming simplicial meshes



- Red (non-conforming) refinement
  - Each triangle is split into 4 similar triangles
  - Each tetrahedron is split into 8 non-similar tetrahedra
- Can be used for more flexibility in the discretisation of :
  - complex domains,
  - heterogeneous media.
- Expected to reduce memory consumption and computing time

Content

DGTD-$\mathbb{P}_{p_i}$ method

- Time-domain Maxwell's equations

$$\epsilon\partial_t \mathbf{E} - curl(\mathbf{H}) = 0 \quad \text{and} \quad \mu\partial_t \mathbf{H} + curl(\mathbf{E}) = 0$$

- Boundary conditions : $\partial\Omega = \Gamma_a \cup \Gamma_m$

$$\begin{cases} \mathbf{n} \times \mathbf{E} = 0 \;\; \text{on} \;\; \Gamma_m \\ \mathbf{n} \times \mathbf{E} - \sqrt{\dfrac{\mu}{\epsilon}}\mathbf{n} \times (\mathbf{H} \times \mathbf{n}) = \mathbf{n} \times \mathbf{E}_{\text{inc}} - \sqrt{\dfrac{\mu}{\epsilon}}\mathbf{n} \times (\mathbf{H}_{\text{inc}} \times \mathbf{n}) \;\; \text{on} \;\; \Gamma_a \end{cases}$$

- Triangulation of Ω: $\overline{\Omega_h} \equiv \mathcal{T}_h = \bigcup_{T_i \in \mathcal{T}_h} \overline{T_i}$
  - Hanging nodes are allowed
  - $a_{ik} = T_i \cap T_k$ (interface)
  - $p = \{p_i : T_i \in \mathcal{T}_h\}$, p_i is the local polynomial degree
  - Approximation space: $V_p(\mathcal{T}_h) := \{\mathbf{v} \in [L^2(T_i)]^3 : \mathbf{v}_{|T_i} \in \mathbb{P}_{p_i}(T_i), \;\; \forall T_i \in \mathcal{T}_h\}$

DGTD-$\mathbb{P}_{p_i}$ method
Discretizations

- Variational formulation: $\forall\, \varphi \in \text{Span}\{\varphi_{ij}, 1 \leq j \leq d_i\}$

$$\left\{ \begin{array}{lcl} \displaystyle\int_{T_i} \epsilon_i \varphi . \partial_t \mathbf{E} & = & \displaystyle\int_{T_i} curl(\varphi).\mathbf{H} - \int_{\partial T_i} \varphi.(\mathbf{H} \times \mathbf{n}) \\ \displaystyle\int_{T_i} \mu_i \varphi . \partial_t \mathbf{H} & = & \displaystyle -\int_{T_i} curl(\varphi).\mathbf{E} + \int_{\partial T_i} \varphi.(\mathbf{E} \times \mathbf{n}) \end{array} \right.$$

- Centered fluxes [M. Remaki : 2000, COMPEL]

$$\{\mathbf{E}_h\}_{ik} = \frac{\mathbf{E}_{i|a_{ik}} + \mathbf{E}_{k|a_{ik}}}{2}, \quad \{\mathbf{H}_h\}_{ik} = \frac{\mathbf{H}_{i|a_{ik}} + \mathbf{H}_{k|a_{ik}}}{2}$$

- Leap-frog time scheme
  - Unknowns related to **E** are approximated at $t^n = n\Delta t$
  - Unknowns related to **H** are approximated at $t^{n+\frac{1}{2}} = (n + \frac{1}{2})\Delta t$

DGTD-$\mathbb{P}_{p_i}$ method

- Matrix form of the DGTD-$\mathbb{P}_{p_i}$ scheme :

$$\begin{cases} \epsilon_i \mathbb{M}_i \dfrac{\mathbf{E}_i^{n+1} - \mathbf{E}_i^n}{\Delta t} = \mathbb{K}_i \mathbf{H}_i - \displaystyle\sum_{k \in \mathcal{V}_i} \mathbb{S}_{ik} \mathbf{H}_k \\ \mu_i \mathbb{M}_i \dfrac{\mathbf{H}_i^{n+3/2} - \mathbf{H}_i^{n+1/2}}{\Delta t} = -\mathbb{K}_i \mathbf{E}_i - \displaystyle\sum_{k \in \mathcal{V}_i} \mathbb{S}_{ik} \mathbf{E}_k \end{cases}$$

  - $\mathbb{M}_i$ is the positive definite symmetric mass matrix of size d_i
  - $\mathbb{K}_i$ is the symmetric stiffness matrix of size d_i
  - $\mathbb{S}_{ik}$ is the interface matrix of size $d_i \times d_k$:

$$(\mathbb{S}_{ik})_{jl} = \frac{1}{2} \int_{a_{ik}} \varphi_j . (\psi_l \times \mathbf{n}_{ik})$$

    - If a_{ik} is a conforming interface $\Rightarrow$ no problem
    - If a_{ik} is a non-conforming interface $\Rightarrow$ we calculate $\mathbb{S}_{ik}$ using the Gauss-Legendre numerical quadrature [Fahs *et al* : 2007, RR-6162, INRIA]

Properties of the DGTD-$\mathbb{P}_{p_i}$ scheme

- Stable under a CFL-like condition on Δt
  - $\mathcal{E}^n = \frac{1}{2}\sum_{i=1}^{N} \epsilon_i {}^t\mathbf{E}_i^n \mathbb{M}_i \mathbf{E}_i^n + \mu_i {}^t\mathbf{H}_i^{n-\frac{1}{2}} \mathbb{M}_i \mathbf{H}_i^{n+\frac{1}{2}}$ is exactly conserved (when $\Gamma_a = \emptyset$)
  - H. Fahs, S. Lanteri and F. Rapetti: INRIA, RR-6023, 2006

$$\begin{cases} \forall \text{ internal face } \ a_{ik} \ (k \in \mathcal{V}_i) \\ c_i \Delta t (2\alpha_i + \beta_{ik}) \leq 4 \min\left(\frac{|T_i|}{P_i^x}, \frac{|T_i|}{P_i^y} \right) \end{cases}$$

  - Numerical CFL values for the DGTD-$\mathbb{P}_p$ method

p	0	1	2	3	4	5	6
CFL	1.0	0.3	0.2	0.1	0.08	0.06	0.045

- Convergence analysis [Fezoui *et al* : 2005, M2AN]

$$\mathcal{O}(Th^{\min(s,p)}) + \mathcal{O}(\Delta t^2)$$

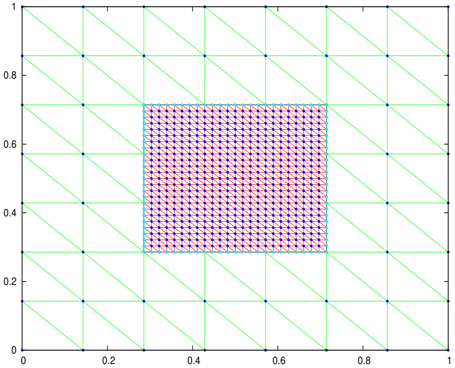
  - The asymptotic convergence order is bounded by 2 independently of p
  - This convergence result seems sub-optimal

- Two-dimensional Maxwell's equation (TMz)

$$\begin{cases} \epsilon \dfrac{\partial E_z}{\partial t} - \dfrac{\partial H_y}{\partial x} + \dfrac{\partial H_x}{\partial y} = 0 \\ \mu \dfrac{\partial H_x}{\partial t} + \dfrac{\partial E_z}{\partial y} = 0 \\ \mu \dfrac{\partial H_y}{\partial t} - \dfrac{\partial E_z}{\partial x} = 0 \end{cases}$$

  - Eigenmode in a unitary PEC square cavity
  - $f = 0.212$ GHz, $p_i = p =$ constant
  - 7-irregular non-conforming meshes (a centered zone is refined 3 times)
    - For $p = 0, 1 \Rightarrow$ 10 points per wavelength in the coarse mesh and 80 in the refined zone
    - For $p = 2, 3, 4 \Rightarrow$ 6 points per wavelength in the coarse mesh and 48 in the refined zone

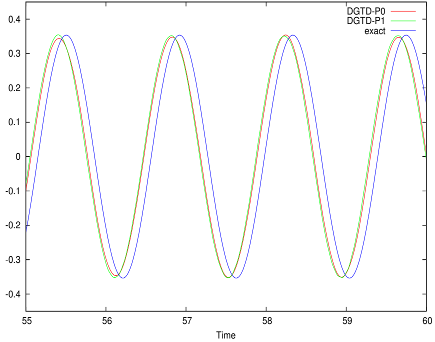
Resonance in a PEC cavity
Non-conforming mesh



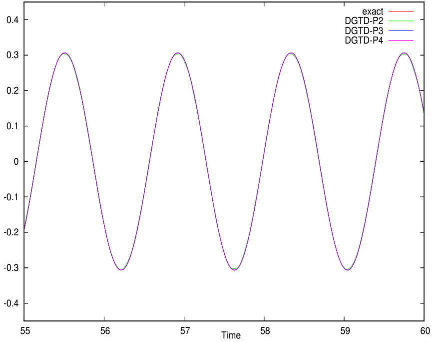
The centered zone is locally refined 3 times
7-irregular mesh

Numerical dispersion

DGTD-$\mathbb{P}_p$, $p \leq 1$ method

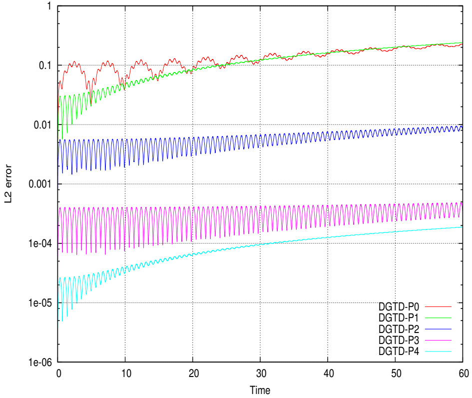


DGTD-$\mathbb{P}_p$, $p \geq 2$ method



DGTD-$\mathbb{P}_p$ method : time evolution of the H_x component
Zoom on the last 5 periods

Numerical dispersion



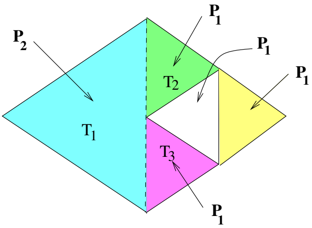
DGTD-$\mathbb{P}_p$ method : time evolution of the L^2 error

DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ method

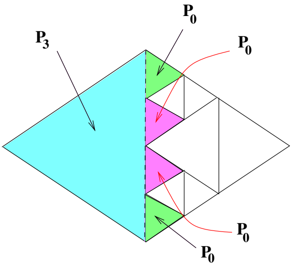
H. Fahs, S. Lanteri and F. Rapetti: INRIA, RR-6162, 2007

H. Fahs, L. Fezoui, S. Lanteri and F. Rapetti: IEEE Trans. Magn., 2008

- The DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ method consists in using:
  - high polynomial degrees "p_c" in the coarse elements,
  - low polynomial degrees "p_f" in the refined elements.
- If $p_c = p_f = p$, we find again the classical DGTD-$\mathbb{P}_p$ scheme



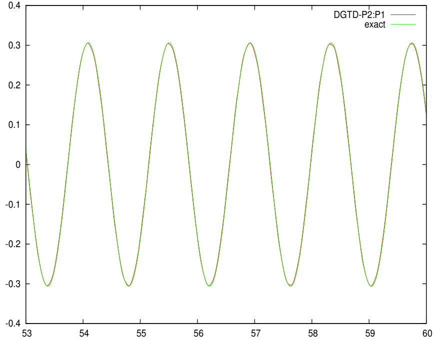
Quadratic polynomials in the coarse elements and linear polynomials in the fine elements



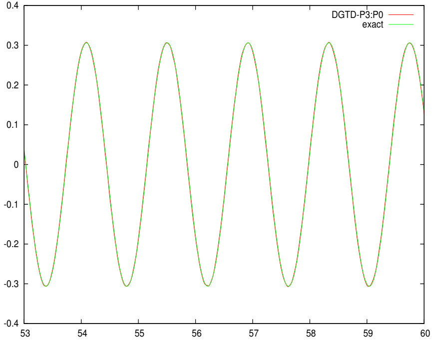
Cubic polynomials in the coarse elements and constant polynomials in the fine elements

DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ method
Eigenmode in a PEC cavity

DGTD-$\mathbb{P}_2$:$\mathbb{P}_1$ method



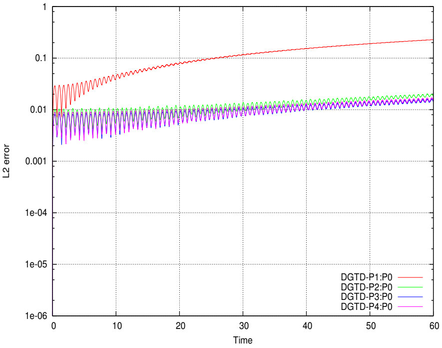
DGTD-$\mathbb{P}_3$:$\mathbb{P}_0$ method

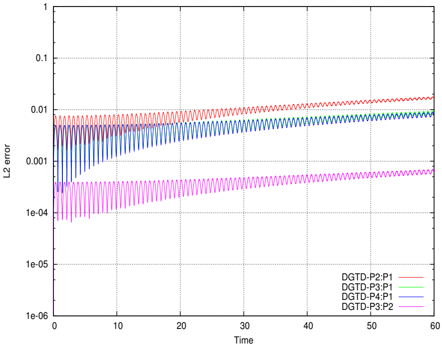


DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ method : time evolution of the H_x component
Zoom on the last 5 periods

DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ method

Eigenmode in a PEC cavity





DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ method : time evolution of the L^2 error

Comparison in terms of # DOF between DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ and DGTD-$\mathbb{P}_p$ methods using the 7-irregular mesh

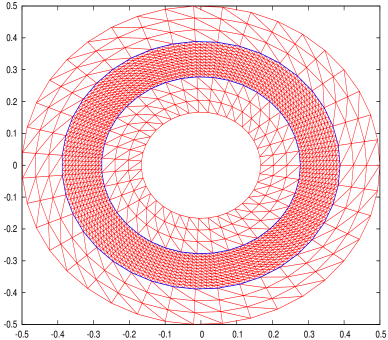
DGTD-$\mathbb{P}_p$	$\mathbb{P}_0$	$\mathbb{P}_1$	$\mathbb{P}_2$	$\mathbb{P}_3$	$\mathbb{P}_4$
# DOF	2304	6912	3456	5760	8640

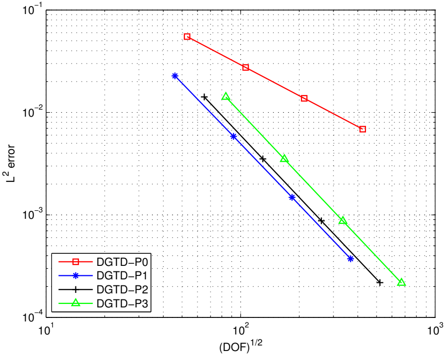
DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$	$\mathbb{P}_2$:$\mathbb{P}_0$	$\mathbb{P}_3$:$\mathbb{P}_0$	$\mathbb{P}_2$:$\mathbb{P}_1$	$\mathbb{P}_3$:$\mathbb{P}_1$	$\mathbb{P}_3$:$\mathbb{P}_2$
# DOF	896	1152	1920	2176	3712

Percentage of # DOF saved by the DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ method

	DGTD-$\mathbb{P}_2$	DGTD-$\mathbb{P}_3$	DGTD-$\mathbb{P}_4$
DGTD-$\mathbb{P}_2$:$\mathbb{P}_0$	−74%	−84%	−90%
DGTD-$\mathbb{P}_3$:$\mathbb{P}_0$	−64%	−80%	−87%
DGTD-$\mathbb{P}_2$:$\mathbb{P}_1$	−46%	−67%	−78%
DGTD-$\mathbb{P}_3$:$\mathbb{P}_1$	−37%	−62%	−75%
DGTD-$\mathbb{P}_3$:$\mathbb{P}_2$	+7%	−35%	−57%

H. Fahs: IJNAM, submitted. Also INRIA, RR-6311, 2007





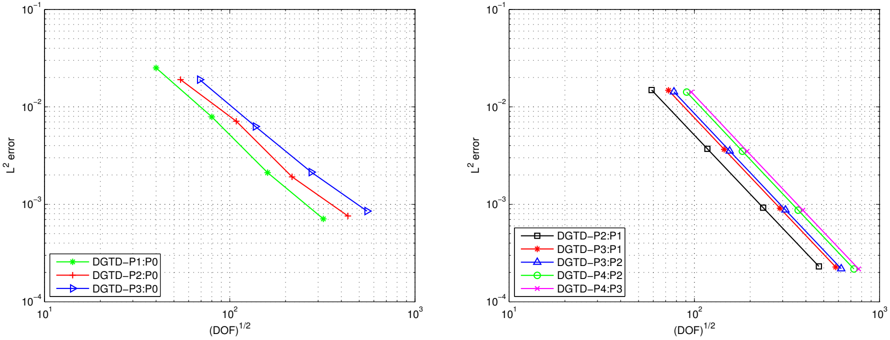
Numerical convergence of the DGTD-$\mathbb{P}_p$ method

Global (space and time) L^2 error versus the square root of # DOF

Non-conforming meshes

Slopes: 1.0 (DGTD-$\mathbb{P}_0$ method) and 2.0 (DGTD-$\mathbb{P}_p$, $\forall\, p \geq 1$, method)

Numerical Results
Two-Concentric PEC cylinders

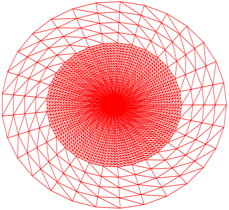


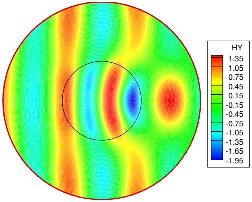
Numerical convergence of the DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ method

Global (space and time) L^2 error versus the square root of # DOF

Non-conforming meshes

Slopes: 1.7 ($\forall\ p_c > p_f,\ p_f = 0$) and 2.2 ($\forall\ p_c > p_f,\ p_f \geq 1$)





DGTD-$\mathbb{P}_p$: Conforming triangular mesh

method	DGTD-$\mathbb{P}_0$	DGTD-$\mathbb{P}_1$	DGTD-$\mathbb{P}_2$	DGTD-$\mathbb{P}_3$
L^2 error, CPU (min) # DOF	13.6%, 20 11920	7.15%, 178 35760	5.20%, 542 71520	5.22%, 1817 119200

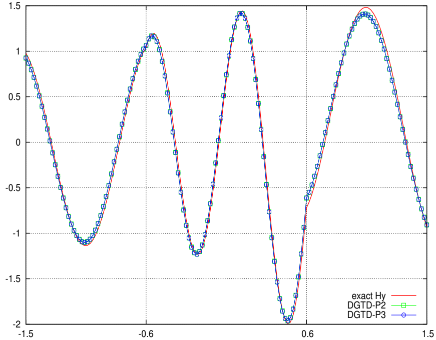
DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$: Non-conforming triangular mesh

method	DGTD-$\mathbb{P}_1$:$\mathbb{P}_0$	DGTD-$\mathbb{P}_2$:$\mathbb{P}_0$	DGTD-$\mathbb{P}_2$:$\mathbb{P}_1$	DGTD-$\mathbb{P}_3$:$\mathbb{P}_2$
L^2 error, CPU (min) # DOF	11.6%, 9 11450	5.36%, 25 19700	5.39%, 33 26100	5.37%, 179 46700

Scattering of a plane wave by a dielectric cylinder

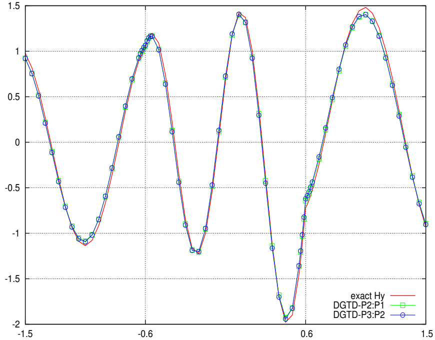
DGTD-$\mathbb{P}_2$ & DGTD-$\mathbb{P}_3$ methods

Conforming mesh



DGTD-$\mathbb{P}_2$:$\mathbb{P}_1$ & DGTD-$\mathbb{P}_3$:$\mathbb{P}_2$ methods

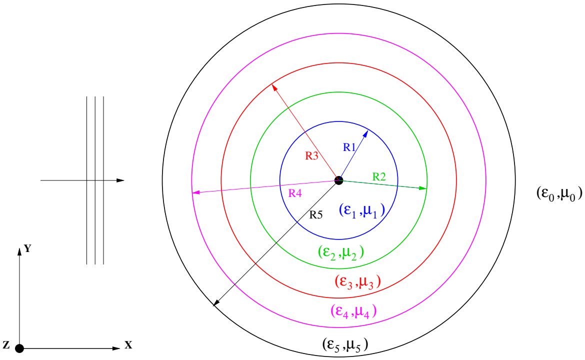
Non-conforming mesh



1D distribution of H_y along $y = 0.0$ at $t = 5$

Numerical Results

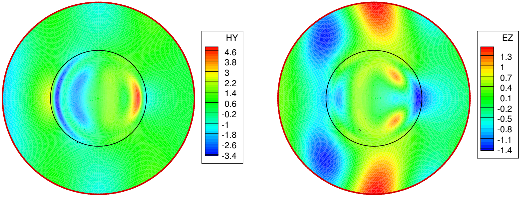
Scattering of a plane wave by multilayered dielectric cylinders



Region	Region 1	Region 2	Region 3	Region 4	Region 5	Region 6
ϵ_r	$\epsilon_1 = 1$	$\epsilon_2 = 4$	$\epsilon_3 = 9$	$\epsilon_4 = 16$	$\epsilon_5 = 64$	$\epsilon_0 = 1$
λ (m)	1	0.5	0.33	0.25	0.125	1
Radii (m)	$R_1 = 0.1$	$R_2 = 0.2$	$R_3 = 0.3$	$R_4 = 0.4$	$R_5 = 0.5$	$R_0 = 1.0$

Scattering by multilayered dielectric cylinders
Reference solution

Contour lines at t=5s

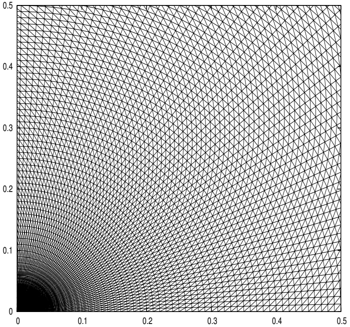


- Ω is a cylinder of radius one
- First order Silver-Müller ABC at the artificial boundary
- No exact solution is available
- Reference solution is constructed in a very fine conforming mesh using DGTD-$\mathbb{P}_4$ method

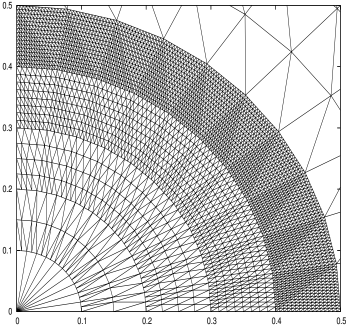
# nodes	# triangles	# DOF
25001	49750	746250

Scattering by multilayered dielectric cylinders

Conforming mesh
14401 nodes & 28560 triangles



Non-conforming mesh
14441 nodes (920 hanging nodes) & 27640 triangles



Region	Region 1	Region 2	Region 3	Region 4	Region 5	Region 6
Interpolation order	p_1	p_2	p_3	p_4	p_5	p_6
Level of refinement	0	1	2	3	4	0
# triangles non-conforming mesh	40	320	1280	5120	20480	400
# triangles conforming mesh	2640	2880	2880	2880	2880	14400

Scattering by multilayered dielectric cylinders

DGTD-$\mathbb{P}_p$ method - Conforming mesh

$p_i = p$	Error on H_y	Error on E_z	CPU (min)	# DOF
0	8.6 %	12.7 %	25	28560
1	7.6 %	7.80 %	137	85680
2	2.2 %	1.30 %	286	171360
3	2.2 %	1.20 %	842	285600

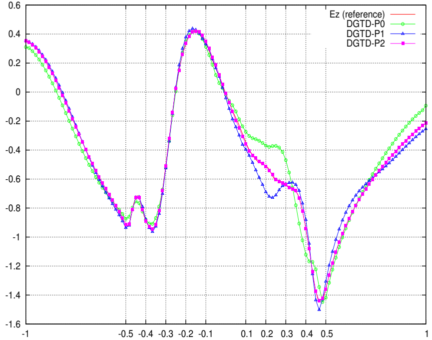
DGTD-$\mathbb{P}_{p_i}$ method - Non-conforming mesh

$p_1, p_2, p_3, p_4, p_5, p_6$	Error on H_y	Error on E_z	CPU (min)	# DOF
4,3,2,1,0,2	5.0 %	1.7 %	12	49720
4,3,2,2,0,2	4.8 %	1.8 %	13	65080
4,3,2,2,1,4	3.5 %	2.6 %	17	109640
4,2,2,4,1,4	3.2 %	2.6 %	21	154440
2,2,2,2,0,2	4.8 %	1.8 %	10	63440
2,2,2,2,1,2	3.5 %	2.5 %	14	104400
2,2,2,2,2,4	2.5 %	1.6 %	20	169440

Scattering by multilayered dielectric cylinders

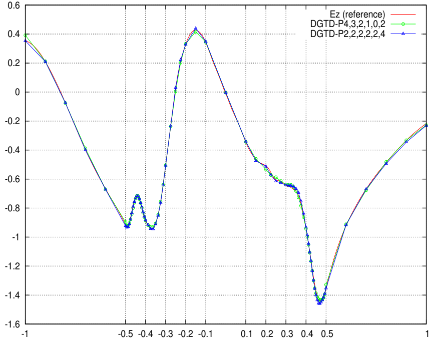
DGTD-$\mathbb{P}_p$ method

Conforming mesh



DGTD-$\mathbb{P}_{p_i}$ method

Non-conforming mesh

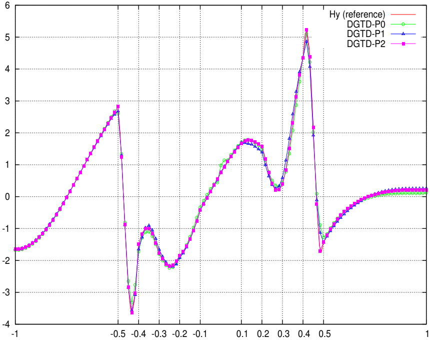


1D distribution of E_z along $y = 0.0$ at $t = 5$

Scattering by multilayered dielectric cylinders

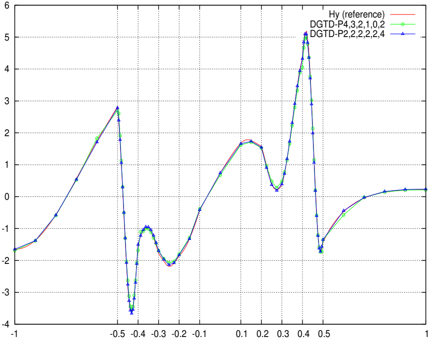
DGTD-$\mathbb{P}_p$ method

Conforming mesh



DGTD-$\mathbb{P}_{p_i}$ method

Non-conforming mesh



1D distribution of H_y along $y = 0.0$ at $t = 5$

Fourth-order leap-frog scheme

GOAL : Increase the order of the existing Leap-Frog (LF) scheme

- Accuracy increased with respect to the LF2 scheme
  - 12 to 20 times more accurate than the LF2 scheme
- Very easy to implement
- CFL(LF4) = 2.85$\times$CFL(LF2)
  - Requires 1.4 times less CPU time than the LF2 scheme
- Requires 2 times more memory than the LF2 scheme

Test case: Eigenmode in a unitary square cavity

DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$ method using non-conforming mesh

		LF2		LF4	
DGTD-$\mathbb{P}_{p_c}$:$\mathbb{P}_{p_f}$	# DOF	Error	CPU (min)	Error	CPU (min)
DGTD-$\mathbb{P}_3$:$\mathbb{P}_2$	6668	1.3E-03	17	2.3E-05	12
DGTD-$\mathbb{P}_4$:$\mathbb{P}_2$	9138	1.3E-03	27	1.5E-05	19
DGTD-$\mathbb{P}_4$:$\mathbb{P}_3$	10290	3.2E-04	61	1.5E-05	44

Closure

- Works in progress
  - Preliminary extension to 3D
    - Non-conforming local refinement of tetrahedral meshes
    - Evaluation of the non-conforming interface matrix $\mathbb{S}_{ik}$
  - Stability analysis of the 4th-order DGTD method
- Future works
  - Design of an *a posteriori* error estimator for an *hp*-adaptive DGTD method

Thank you for your attention!

Closure

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Thank you for your attention!