Strongly anisotropic wave equations

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Main goals

- Effective models when disparate scales occur ;
- Derivation of the limit models, well-posedness, balances, symmetries ;
- Convergence results ;
- Applications : transport of charged particles (magnetic confinement), heat equations, Maxwell equations.

1. Multi-scale analysis for linear first order PDE

$$\begin{cases} \partial_t u^\varepsilon + a \cdot \nabla_y u^\varepsilon + \frac{1}{\varepsilon} b \cdot \nabla_y u^\varepsilon = 0, & (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m \\ u^\varepsilon(0,y) = u^{\mathrm{in}}(y), & y \in \mathbb{R}^m. \end{cases}$$

Hypotheses

$$a \in L^1_{\mathrm{loc}}(\mathbb{R}_+; W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m)), \;\; b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m) \implies \text{ smooth flows}$$

$$\mathrm{div}_y a = 0, \;\; \mathrm{div}_y b = 0 \implies \text{ measure preserving flows}$$

$$|a(t,y)| + |b(y)| \le C(1 + |y|) \implies \text{ global flows}$$

Dominant dynamics

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon} b \cdot \nabla_y u^\varepsilon = 0, \;\; (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m$$

Fast time variable

$$s = \frac{t}{\varepsilon}, \;\; \partial_s u + b \cdot \nabla_y u = 0, \;\; (s,y) \in \mathbb{R} \times \mathbb{R}^m$$

Question: behavior when $\varepsilon \searrow 0$?

Main idea: filtering out the fast oscillations

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = b(Y(s;y)), \quad Y(0;y) = y, \quad (s,y) \in \mathbb{R} \times \mathbb{R}^m$$

New coordinates

$$z = Y(-t/\varepsilon, y) \text{ or equivalently } y = Y(t/\varepsilon, z)$$

Search for a profile (solving the fast dynamics)

$$u^\varepsilon(t,y) = v^\varepsilon(t, \underbrace{Y(-t/\varepsilon; y)}_{z})$$

The equation in the new coordinates ?

$$\begin{cases} \partial_t v^\varepsilon(t,z) + \underbrace{\partial_y Y(-t/\varepsilon; Y(t/\varepsilon; z)) a(t, Y(t/\varepsilon; z))}_{\varphi(t/\varepsilon)a(t)} \cdot \nabla_z v^\varepsilon(t,z) = 0, \\ v^\varepsilon(0,z) = u^{\mathrm{in}}(z), \end{cases}$$

Stability for $(v^\varepsilon)_{\varepsilon>0}$

$$\varphi(s)a = \partial_y Y(-s; Y(s;\cdot)) a(Y(s;\cdot))$$

Behavior when $\varepsilon \searrow 0$

$$\partial_y Y(-t/\varepsilon; Y(t/\varepsilon; z)) a(t, Y(t/\varepsilon; z)) = \varphi(t/\varepsilon) a(t)$$

If involution between $a(t)$ and b

$$[b, a(t)] = 0 \implies \varphi(s)a(t) = a(t), s \in \mathbb{R}$$

$$\begin{cases} \partial_t v^\varepsilon(t,z) + a(t,z)\cdot\nabla_z v^\varepsilon(t,z) = 0, & (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m \\ v^\varepsilon(0,z) = u^{\mathrm{in}}(z), & z \in \mathbb{R}^m \end{cases}$$

$$v^\varepsilon(t,z) = u^{\mathrm{in}}(Z(-t;z)) = v(t,z), \;\; \frac{\mathrm{d}Z}{\mathrm{d}t} = a(t, Z(t;z))$$

$$u^\varepsilon(t,y) = v(t, Y(-t/\varepsilon;y)) = u^{\mathrm{in}}(Z(-t; Y(-t/\varepsilon;y)))$$

Splitting : advection along a and advection along $\frac{1}{\varepsilon}b$.

Two scale approach : t and $s = t/\varepsilon$

Use ergodicity

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \partial_y Y(-s; Y(s;\cdot)) a(t, Y(s;\cdot)) \,\mathrm{d}s = \lim_{T\to+\infty} \frac{1}{T} \int_0^T \varphi(s) a(t) \,\mathrm{d}s$$

Key point

Emphasize a C^0-group of unitary transformations and use :

von Neumann's Ergodic Mean Theorem

Let $(G(s))_{s\in\mathbb{R}}$ be a C^0-group of unitary operators on a Hilbert space $(H, (\cdot,\cdot))$ and A be the infinitesimal generator of G. Then, for any $x \in H$, we have

$$\lim_{T\to+\infty} \frac{1}{T} \int_r^{r+T} G(s)x \,\mathrm{d}s = \mathrm{Proj}_{\ker A} x, \text{ strongly in } H$$

uniformly with respect to $r \in \mathbb{R}$.

Average vector field

$$X_Q = \{c(y) : \mathbb{R}^m \to \mathbb{R}^m \text{ measurable} : \int_{\mathbb{R}^m} Q(y) : c(y) \otimes c(y)\,\mathrm{d}y < +\infty\}$$

$$Q = P^{-1}, \;\; P = {}^tP > 0, [b, P] := (b \cdot \nabla_y)P - \partial_y b\, P - P\, {}^t\partial_y b = 0$$

$$(c, d)_Q = \int_{\mathbb{R}^m} Q(y) : c(y) \otimes d(y)\,\mathrm{d}y, \;\; c, d \in X_Q.$$

Proposition $(\varphi(s))_{s\in\mathbb{R}}$ is a C^0-group of unitary operators on X_Q.

Theorem

We denote by $\mathcal{L}$ the infinitesimal generator of the group $(\varphi(s))_{s\in\mathbb{R}}$.

Then for any vector field $a \in X_Q$, we have the strong convergence in X_Q

$$\langle a \rangle := \lim_{T\to+\infty} \frac{1}{T} \int_r^{r+T} \partial_y Y(-s; Y(s; \cdot))a(Y(s; \cdot))\,\mathrm{d}s = \mathrm{Proj}_{\ker \mathcal{L}} a$$

uniformly with respect to $r \in \mathbb{R}$.

Theorem (Convergence)

The family $(v^\varepsilon)_{\varepsilon>0}$ converges strongly in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$ to a weak solution $v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ of the transport problem

$$\begin{cases} \partial_t v + \langle a(t,\cdot)\rangle \cdot \nabla_z v = 0, & (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m \\ v(0,z) = u^{\mathrm{in}}(z), & z \in \mathbb{R}^m. \end{cases}$$

Moreover, if v is smooth enough, we have $v^\varepsilon = v + \mathcal{O}(\varepsilon)$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$, as $\varepsilon \searrow 0$.

Formal proof

$$\partial_t v^\varepsilon + \varphi(t/\varepsilon) a(t) \cdot \nabla_z v^\varepsilon = 0$$

$$v^\varepsilon(t,z) = v(t, s = t/\varepsilon, z) + \varepsilon v^1(t, s = t/\varepsilon, z) + \ldots$$

$$\partial_s v = 0, \;\; \partial_t v + \varphi(s) a(t) \cdot \nabla_z v + \partial_s v^1 = 0$$

$$\partial_t v + \left(\lim_{T\to+\infty} \frac{1}{T} \int_0^T \varphi(s) a(t) \,\mathrm{d}s \right) \cdot \nabla_z v = 0.$$

2. Wave equation with disparate propagation speeds

$$\partial_t^2 u^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u^\varepsilon) - \frac{1}{\varepsilon^2}\mathrm{div}_y(b(y)\otimes b(y)\nabla_y u^\varepsilon) = 0, (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m$$

$$u^\varepsilon(0,y) = u^\varepsilon_{\mathrm{in}}(y), \quad \partial_t u^\varepsilon(0,y) = \dot{u}^\varepsilon_{\mathrm{in}}, \quad y \in \mathbb{R}^m$$

Variational solutions

$$a^\varepsilon(u,v) = \int_{\mathbb{R}^m} D(y)\nabla u \cdot \nabla v \,\mathrm{d}y + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^m} (b\cdot\nabla u)(b\cdot\nabla v)\,\mathrm{d}y, \quad u,v \in H^1_P$$

$$(u^\varepsilon(0), \partial_t u^\varepsilon(0)) = (u^\varepsilon_{\mathrm{in}}, \dot{u}^\varepsilon_{\mathrm{in}}) \in H^1_P \times L^2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m} \partial_t u^\varepsilon v(y)\,\mathrm{d}y + a^\varepsilon(u^\varepsilon(t), v) = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}_+), \ v \in H^1_P$$

Average matrix field

$$H_Q = \left\{ A(y) \ : \ Q^{1/2} A Q^{1/2} \in L^2 \right\}, \ \ H_Q^\infty = \left\{ A(y) \ : \ Q^{1/2} A Q^{1/2} \in L^\infty \right\}$$

$$(\cdot,\cdot)_Q : H_Q \times H_Q \to \mathbb{R}, \ \ (A,B)_Q = \int_{\mathbb{R}^m} Q(y)A(y) : B(y)Q(y)\,\mathrm{d}y.$$

$$G(s) : H_Q \to H_Q, \ \ G(s)A = \partial_y Y^{-1}(s;\cdot)A(Y(s;\cdot))\, {}^t\partial_y Y^{-1}(s;\cdot), s \in \mathbb{R}$$

Proposition $(G(s))_{s\in\mathbb{R}}$ is a C^0-group of unitary operators on H_Q.

Theorem Let L be the infinitesimal generator of the group $(G(s))_{s\in\mathbb{R}}$. For any matrix field $A \in H_Q$ we have the strong convergence in H_Q

$$\langle A \rangle := \lim_{T\to+\infty} \frac{1}{T} \int_r^{r+T} \partial_y Y^{-1}(s;\cdot)A(Y(s;\cdot))\, {}^t\partial_y Y^{-1}(s;\cdot)\,\mathrm{d}s = \mathrm{Proj}_{\ker L} A$$

uniformly with respect to $r \in \mathbb{R}$.

Localization

$$\psi \in C(\mathbb{R}^m), \ \ \psi \circ Y(s;\cdot) = \psi \ \text{ for any } s \in \mathbb{R}, \ \lim_{|y|\to+\infty} \psi(y) = +\infty.$$

Properties of $(G(s))_s$

1. If A is a field of symmetric matrices, then so is $G(s)A$, $s \in \mathbb{R}$.
2. If A is a field of non-negative matrices, then so is $G(s)A$, $s \in \mathbb{R}$.
3. Let $\mathcal{S} \subset \mathbb{R}^m$ be an invariant set of the flow of b, that is $Y(s;\mathcal{S}) = \mathcal{S}$, for any $s \in \mathbb{R}$. If there is $d > 0$ such that $Q^{1/2}(y)A(y)Q^{1/2}(y) \geq dI_m, y \in \mathcal{S}$, then for any $s \in \mathbb{R}$ we have $Q^{1/2}(y)(G(s)A)(y)Q^{1/2}(y) \geq dI_m, y \in \mathcal{S}$.
4. Moreover, the family of applications $(G(s))_{s\in\mathbb{R}}$ acts on $H_{Q,\mathrm{loc}}$, that is, if $A \in H_{Q,\mathrm{loc}}$, then $G(s)A \in H_{Q,\mathrm{loc}}$ for any $s \in \mathbb{R}$. We have

$$\mathbf{1}_{\{\psi \leq k\}} G(s)A = G(s)(\mathbf{1}_{\{\psi \leq k\}} A), \quad A \in H_{Q,\mathrm{loc}}, \quad s \in \mathbb{R}, \quad k \in \mathbb{N}.$$

Average of $H_{Q,\mathrm{loc}}$ matrix fields

1. If $A \in H_Q$ is a field of symmetric non-negative matrices, then so is $\langle A \rangle$.
2. Let $\mathcal{S} \subset \mathbb{R}^m$ be an invariant set of the flow of b. If $A \in H_Q$ and there is $d > 0$ such that $Q^{1/2}(y)A(y)Q^{1/2}(y) \geq dI_m, \;\; y \in \mathcal{S}$ therefore we have $Q^{1/2}(y) \langle A \rangle (y) Q^{1/2}(y) \geq dI_m, \;\; y \in \mathcal{S}$ and in particular, $\langle A \rangle (y)$ is definite positive, $y \in \mathcal{S}$.
3. If $A \in H_Q \cap H_Q^\infty$, then $\langle A \rangle \in H_Q \cap H_Q^\infty$ and

$$|\langle A \rangle|_{H_Q} \leq |A|_{H_Q}, \;\; |\langle A \rangle|_{H_Q^\infty} \leq |A|_{H_Q^\infty}.$$

Average of $H_{Q,\mathrm{loc}}$ matrix fields

4. For any matrix field $A \in H_{Q,\mathrm{loc}}$, the family

$$\left(\frac{1}{S}\int_r^{r+S} \partial Y(-s;Y(s;\cdot))A(Y(s;\cdot))\,{}^t\partial Y(-s;Y(s;\cdot))\,\mathrm{d}s\right)_{S>0}$$

converges in $H_{Q,\mathrm{loc}}$, when S goes to infinity, uniformly with respect to $r \in \mathbb{R}$, for any fixed $k \in \mathbb{N}$. Its limit, denoted by $\langle A \rangle$, satisfies

$$\mathbf{1}_{\{\psi \le k\}}\langle A \rangle = \left\langle \mathbf{1}_{\{\psi \le k\}} A \right\rangle, \text{ for any } k \in \mathbb{N}$$

where the symbol $\langle \cdot \rangle$ in the right hand side stands for the average operator on H_Q. In particular, any matrix field $A \in H_Q^\infty$ has an average in $H_{Q,\mathrm{loc}}$ and $|\langle A \rangle|_{H_Q^\infty} \le |A|_{H_Q^\infty}$. If $A \in H_{Q,\mathrm{loc}}$ is such that $Q^{1/2}(y)A(y)Q^{1/2}(y) \ge \alpha I_m, \ \ y \in \mathbb{R}^m$, for some $\alpha > 0$, then we have $Q^{1/2}(y)\langle A \rangle(y)Q^{1/2}(y) \ge \alpha I_m, \ \ y \in \mathbb{R}^m$.

Weighted H^1 space

$$H^1_P = \{u \in L^2 \;:\; P^{1/2}\nabla u \in L^2\}$$

$$(u,v)_{H^1_P} = \int_{\mathbb{R}^m} u(y)v(y)\,\mathrm{d}y + \int_{\mathbb{R}^m} P(y) : \nabla u \otimes \nabla v\,\mathrm{d}y, \quad u, v \in H^1_P.$$

Average of H^1_P functions

For any $s \in \mathbb{R}$ and $u \in H^1_P$ we have $u_s := u \circ Y(s;\cdot) \in H^1_P$ and $|u_s|_{H^1_P} = |u|_{H^1_P}$. The family of applications $u \in H^1_P \to u \circ Y(s;\cdot) \in H^1_P$ is a C^0-group of unitary operators on H^1_P. In particular, for any $u \in H^1_P$ we have $\langle u \rangle \in H^1_P$

$$\nabla_y \langle u \rangle = \lim_{S \to +\infty} \frac{1}{S} \int_r^{r+S} \nabla_y u_s \,\mathrm{d}s, \quad \text{in } X_P, \quad \text{uniformly w.r.t. } r \in \mathbb{R}$$

$$u - \langle u \rangle \perp \ker \mathcal{T} \cap H^1_P \;\text{ in } H^1_P, \quad |\nabla_y \langle u \rangle|_{X_P} \le |\nabla_y u|_{X_P}.$$

Lemma

For any matrix field $D \in H_Q^\infty$ and any vector field $c \in X_P$ we have the convergence

$$\lim_{S \to +\infty} \frac{1}{S} \int_r^{r+S} G(s) Dc \, \mathrm{d}s = \langle D \rangle \, c, \text{ strongly in } X_Q, \text{ unif. w.r.t. } r \in \mathbb{R}.$$

Proof

1. $D \in H_Q$
2. $D \in H_Q^\infty \subset H_{Q,\mathrm{loc}}$.

Asymptotic analysis

$$H^1_P \subset L^2, \quad a^\varepsilon : H^1_P \times H^1_P \to \mathbb{R}$$

$$a^\varepsilon(u,v) \int_{\mathbb{R}^m} D(y)\nabla u \cdot \nabla v \,\mathrm{d}y + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^m} (b\cdot\nabla u)(b\cdot\nabla v)\,\mathrm{d}y, \quad u,v \in H^1_P$$

$$Q^{1/2}(y)(D(y) + b(y)\otimes b(y))Q^{1/2}(y) \geq dI_m, \quad y \in \mathbb{R}^m$$

$$D \in H^\infty_Q, \quad b \in X^\infty_Q.$$

Proposition The bilinear forms a^ε are well defined, continuous, symmetric, non-negative. For any $\varepsilon \in]0,1]$, the forms a^ε are coercive on H^1_P, with respect to L^2.

$$a^\varepsilon(u,u) + d|u|^2_{L^2(\mathbb{R}^m)}$$

$$= \int_{\mathbb{R}^m} Q^{1/2}\left(D + \frac{b\otimes b}{\varepsilon^2}\right)Q^{1/2} : (P^{1/2}\nabla u)\otimes(P^{1/2}\nabla u)\,\mathrm{d}y + d|u|^2_{L^2(\mathbb{R}^m)}$$

$$\geq d|\nabla u|^2_{X_P} + d|u|^2_{L^2(\mathbb{R}^m)} = d|u|^2_{H^1_P}.$$

Well-posedness

Let $(u^{\varepsilon}_{\rm in}, \dot{u}^{\varepsilon}_{\rm in}) \in H^1_P \times L^2(\mathbb{R}^m)$. For any $\varepsilon \in]0,1]$ there is a unique variational solution *i.e.,* $u^{\varepsilon} \in L^{\infty}_{\rm loc}(\mathbb{R}_+; H^1_P), \partial_t u^{\varepsilon} \in L^{\infty}_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^m))$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^m} \partial_t u^{\varepsilon} v(y)\, \mathrm{d}y + a^{\varepsilon}(u^{\varepsilon}(t), v) = 0 \;\text{ in } \mathcal{D}'(\mathbb{R}_+),\; v \in H^1_P.$$

We have $u^{\varepsilon} \in C(\mathbb{R}_+; H^1_P), \partial_t u^{\varepsilon} \in C(\mathbb{R}_+; L^2(\mathbb{R}^m))$ and for any $t \in \mathbb{R}_+,\;\; 0 < \varepsilon \le 1$

$$|\partial_t u^{\varepsilon}(t)|^2_{L^2(\mathbb{R}^m)} + d|\nabla u^{\varepsilon}(t)|^2_{X_P} + \left(\frac{1}{\varepsilon^2} - 1\right)|b \cdot \nabla u^{\varepsilon}(t)|^2_{L^2(\mathbb{R}^m)} \le |\dot{u}^{\varepsilon}_{\rm in}|^2_{L^2(\mathbb{R}^m)}$$
$$+ |D|_{H^{\infty}_Q} |\nabla u^{\varepsilon}_{\rm in}|^2_{X_P} + \frac{1}{\varepsilon^2}|b \cdot \nabla u^{\varepsilon}_{\rm in}|^2_{L^2(\mathbb{R}^m)}$$

$$|u^{\varepsilon}(t)|^2_{L^2(\mathbb{R}^m)} \le 2|u^{\varepsilon}_{\rm in}|^2_{L^2(\mathbb{R}^m)}$$
$$+ 2t^2 \left[|\dot{u}^{\varepsilon}_{\rm in}|^2_{L^2(\mathbb{R}^m)} + |D|_{H^{\infty}_Q} |\nabla u^{\varepsilon}_{\rm in}|^2_{X_P} + \frac{1}{\varepsilon^2}|b \cdot \nabla u^{\varepsilon}_{\rm in}|^2_{L^2(\mathbb{R}^m)} \right].$$

Limit model

$$\langle a\rangle (u,v) = \int_{\mathbb{R}^m} \langle D\rangle (y) \nabla u \cdot \nabla v \,\mathrm{d}y, \quad u,v \in H^1_P$$

Proposition The bilinear forms $\langle a\rangle$ is well defined, continuous, symmetric, non-negative and coercive on H^1_P, with respect to $\mathrm{dom}\mathcal{T} \subset L^2(\mathbb{R}^m)$.

Proposition For any $(u^{\mathrm{in}}, \dot{u}_{\mathrm{in}}) \in H^1_P \times \mathrm{dom}\mathcal{T}$ there is a unique variational solution*i.e.*, $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^1_P), \partial_t u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{dom}\mathcal{T})$

$$(u(0) = u^{\mathrm{in}}, \partial_t u(0) = \dot{u}_{\mathrm{in}})$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^m} \partial_t u \; v(y) \,\mathrm{d}y + \langle a\rangle (u(t), v) = 0 \;\text{ in } \mathcal{D}'(\mathbb{R}_+), \; v \in H^1_P.$$

We have $u \in C(\mathbb{R}_+; H^1_P), \partial_t u \in C(\mathbb{R}_+; \mathrm{dom}\mathcal{T})$

Uniformly bounded energy

$$\sup_{0<\varepsilon\leq 1}\left\{|u^\varepsilon_{\rm in}|_{H^1_P}+|\dot{u}^\varepsilon_{\rm in}|_{L^2(\mathbb{R}^m)}+\frac{|b\cdot\nabla u^\varepsilon_{\rm in}|_{L^2(\mathbb{R}^m)}}{\varepsilon}\right\}<+\infty.$$

Theorem (Weak convergence)

$$\lim_{\varepsilon\searrow 0}u^\varepsilon_{\rm in}=u^{\rm in}\ \text{ weakly in } H^1_P,\quad \lim_{\varepsilon\searrow 0}\dot{u}^\varepsilon_{\rm in}=\dot{u}_{\rm in}\ \text{ weakly in } L^2(\mathbb{R}^m)$$

$$\sup_{0<\varepsilon\leq 1}\frac{|b\cdot\nabla u^\varepsilon_{\rm in}|_{L^2(\mathbb{R}^m)}}{\varepsilon}<+\infty.$$

Then $(u^\varepsilon)_\varepsilon,(\partial_t u^\varepsilon)_\varepsilon$ converge weakly $\star$ in $L^\infty([0,T],H^1_P),L^\infty([0,T];L^2(\mathbb{R}^m))$ respectively, $T\in\mathbb{R}_+$ toward the solution $(u,\partial_t u)\in L^\infty_{\rm loc}(\mathbb{R}_+;H^1_P\cap\ker\mathcal{T})\times L^\infty_{\rm loc}(\mathbb{R}_+;\ker\mathcal{T})$ of the problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^m}\partial_t u\, v\,\mathrm{d}y+\int_{\mathbb{R}^m}\langle D\rangle(y)\nabla u\cdot\nabla v\,\mathrm{d}y=0\ \text{ in } \mathcal{D}'(\mathbb{R}_+), v\in H^1_P$$

with $(u(0),\partial_t u(0))=(u^{\rm in},\langle\dot{u}_{\rm in}\rangle)$.

Proof For any $u(t), v \in H^1_P \cap \ker \mathcal{T}$ and any $s \in [0, S], S \in \mathbb{R}_+$

$$\int_{\mathbb{R}^m} D(y)\nabla u(t) \cdot \nabla v \,\mathrm{d}y = \int_{\mathbb{R}^m} D(y)\nabla(u(t))_{-s} \cdot \nabla v_{-s} \,\mathrm{d}y$$

$$= \int_{\mathbb{R}^m} D(y)\, {}^t\partial Y(-s;y)(\nabla u(t))_{-s} \cdot {}^t\partial Y(-s;y)(\nabla v)_{-s} \,\mathrm{d}y$$

$$= \int_{\mathbb{R}^m} \partial Y(-s;y) D(y)\, {}^t\partial Y(-s;y)(\nabla u(t))_{-s} \cdot (\nabla v)_{-s} \,\mathrm{d}y$$

$$= \int_{\mathbb{R}^m} \partial Y(-s; Y(s;z)) D(Y(s;z))\, {}^t\partial Y(-s; Y(s;z))\nabla u(t,z) \cdot \nabla v(z) \,\mathrm{d}z$$

$$= \int_{\mathbb{R}^m} G(s) D\nabla u(t) \cdot \nabla v \,\mathrm{d}y$$

$$= \int_{\mathbb{R}^m} \frac{1}{S}\int_0^S G(s) D \,\mathrm{d}s \nabla u(t) \cdot \nabla v \,\mathrm{d}y \to \int_{\mathbb{R}^m} \langle D \rangle (y) \nabla u(t) \cdot \nabla v \,\mathrm{d}y.$$

Proposition Assume that $A \in H^\infty_Q$ and that $(w^\varepsilon)_\varepsilon$ converges weakly in $L^2([0,T];X_P)$ toward w^0, when $\varepsilon \searrow 0$.

1. If $A = A(y)$ are non-negative, then

$$\int_0^T \int_{\mathbb{R}^m} A(y) w^0(t) \cdot w^0(t) \,\mathrm{d}y\mathrm{d}t \leq \liminf_{\varepsilon \searrow 0} \int_0^T \int_{\mathbb{R}^m} A(y) w^\varepsilon(t) \cdot w^\varepsilon(t) \,\mathrm{d}y\mathrm{d}t.$$

2. If there is $d > 0$ such that $Q^{1/2} A Q^{1/2} \geq d I_m$ and

$$\limsup_{\varepsilon \searrow 0} \int_0^T \int_{\mathbb{R}^m} A(y) w^\varepsilon(t) \cdot w^\varepsilon(t) \,\mathrm{d}y\mathrm{d}t \leq \int_0^T \int_{\mathbb{R}^m} A(y) w^0(t) \cdot w^0(t) \,\mathrm{d}y\mathrm{d}t$$

   then the family $(w^\varepsilon)_\varepsilon$ converges strongly in $L^2([0,T];X_P)$ toward w^0, when $\varepsilon \searrow 0$.

Theorem (Strong convergence) Assume that

$$\lim_{\varepsilon \searrow 0} u^\varepsilon_{\mathrm{in}} = u^{\mathrm{in}} \text{ strongly in } H^1_P, \quad \lim_{\varepsilon \searrow 0} \frac{b \cdot \nabla u^\varepsilon_{\mathrm{in}}}{\varepsilon} = 0 \text{ strongly in } L^2(\mathbb{R}^m)$$

$$\lim_{\varepsilon \searrow 0} \dot{u}^\varepsilon_{\mathrm{in}} = \dot{u}_{\mathrm{in}} \text{ strongly in } L^2(\mathbb{R}^m), \quad b \cdot \nabla \dot{u}_{\mathrm{in}} = 0.$$

Then we have the strong convergences

$$\lim_{\varepsilon \searrow 0} u^\varepsilon = u \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m)), \quad \lim_{\varepsilon \searrow 0} \nabla u^\varepsilon = \nabla u \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}_+; X_P)$$

$$\lim_{\varepsilon \searrow 0} \partial_t u^\varepsilon = \partial_t u \text{ in } 11 L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m)), \quad \lim_{\varepsilon \searrow 0} \frac{b \cdot \nabla u^\varepsilon}{\varepsilon} = 0 \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2$$

The Maxwell equations

$$\partial_t D - \mathrm{rot} H = 0, \;\; \partial_t B + \mathrm{rot} E = 0, \;\; \mathrm{div} D = 0, \;\; \mathrm{div} B = 0$$

$$D = \epsilon_0 \epsilon_r E, \;\; B = \mu_0 H$$

Energy balance

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \{ D \cdot E + B \cdot H \} + \mathrm{div}(E \wedge H) = 0.$$

Strongly anisotropic electric permittivity

$$\epsilon_r = \mathrm{diag}(n_1^2, n_2^2, n_3^2)$$

n_i = indice propre du milieux

$$\epsilon_r^{-1/2} = M + \frac{b \otimes b}{\varepsilon}$$

Maxwell equations

$$\partial_t D^\varepsilon - \mathrm{rot}\frac{B^\varepsilon}{\mu_0} = 0, \quad \partial_t B^\varepsilon + \mathrm{rot}\left[\frac{1}{\epsilon_0}\left(M + \frac{b \otimes b}{\varepsilon}\right)^2 D^\varepsilon\right] = 0$$

$$\mathrm{div} B^\varepsilon = 0, \quad \mathrm{div} D^\varepsilon = 0.$$

Perspectives

1. Maxwell equations with disparate permittivity eigenvalues
2. determine the effective permittivity
3. estimate the effective propagation speed
4. asymptotic behavior (weak/strong convergence results)

THANK YOU!