The Origin of the Multiscale Hybrid-Mixed Method: An Overview

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Laboratório Nacional de Computação Científica

INVENTEURS DU MONDE NUMÉRIQUE

Numerical Algorithms Scaling on Massive Computers

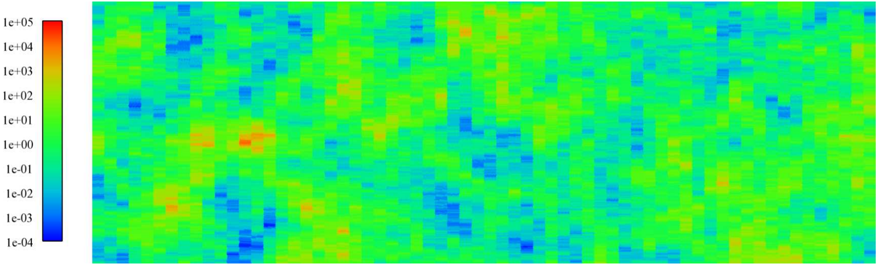
Sunway TaihuLight
100 Petaflop - Top 1

Santos Dumont - LNCC
1 Petaflop ⇒ 6 Petaflop

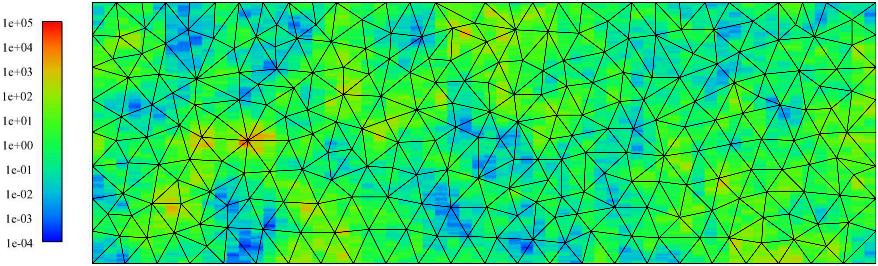
Motivation and Goals (from Fluids to Waves)

The MHM Method

Highly Heterogeneous Media

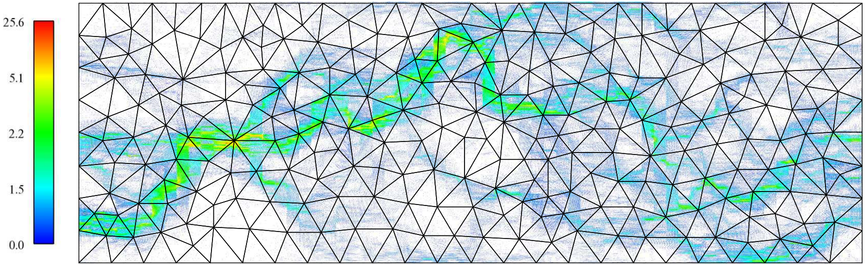


Upscaling Needed (Multiscale Basis)



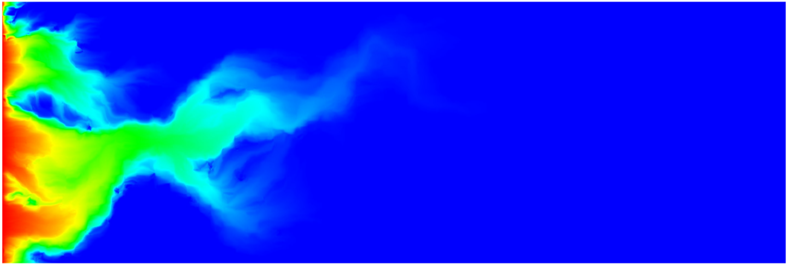
Flow Field

Darcy Velocity

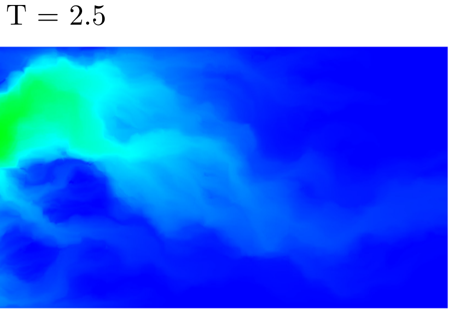


Transport in a Heterogenous Media

T = 1

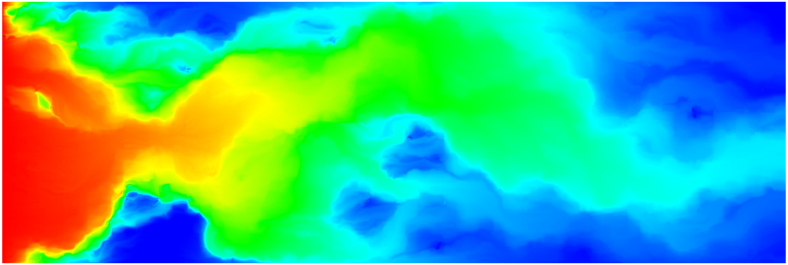


Transport in a Heterogenous Media

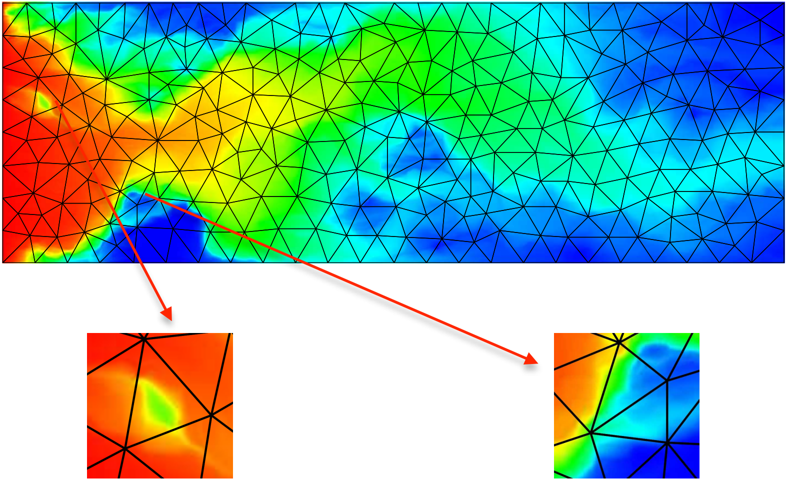


Transport in a Heterogenous Media

T = 5.0

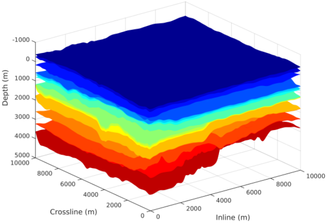


Transport in a Heterogenous Media

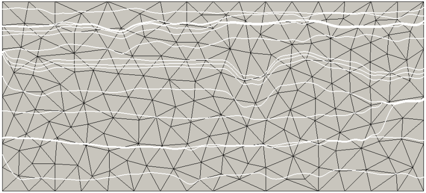


Highly Layered Media

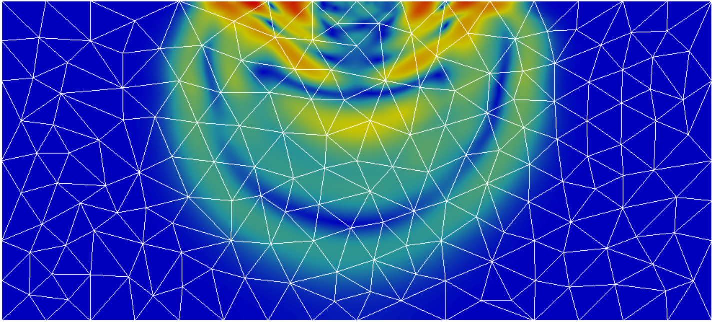
3D Domain



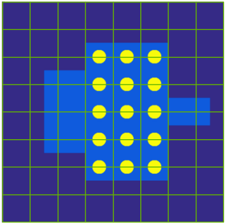
Non-Aligned Meshes



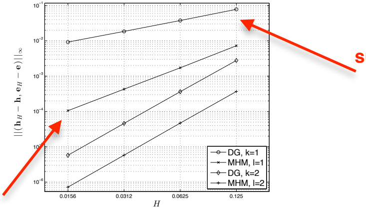
Elastic-Wave Propagation in Seismic



Electromagnetic Wave Propagation in Nano-Structures

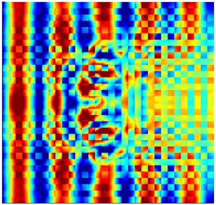


Upscaling may improve convergence

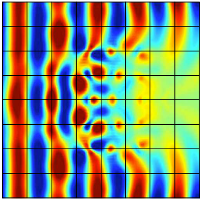


sub-optimal

optimal



No Upscalling



With Upscalling

Goals and Outline

- Upscalling through Multiscale Methods
  - Locality: Prompt to be used in parallel computers
  - Convergence: High-order accuracy
  - Robustness: "Coarse" meshes

- Outline (Part I)
  - Historical vision of numerical upscaling in FEM
  - Main ideas which originates the MHM method
  - Some open questions
- Outline (Part II)
  - The MHM method on general polygonal meshes
    - The construction and properties of the MHM method
    - New error analysis

80's - 90's

Stabilized Method, Bubble Function and GFEM

Boundary layers

Rapid changing coefficients

A Prototype: The Advection-Diffusion Model

$$\begin{cases} \mathcal{L}\,u := -\nabla\cdot(\mathcal{K}\,\nabla u) + \boldsymbol{\alpha}\cdot\nabla u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- $\boldsymbol{\alpha} \in W^{1,\infty}(\Omega)$ and $\nabla\cdot\boldsymbol{\alpha} = 0$ in Ω
- $|\xi|^2 C_{min} \le \xi^T\,\mathcal{K}(\mathbf{x})\,\xi \le C_{max}|\xi|^2 \quad \xi \in \mathbb{R}^d$ and $\mathbf{x} \in \Omega$

Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v)_\Omega \quad \forall v \in H_0^1(\Omega)$$

where

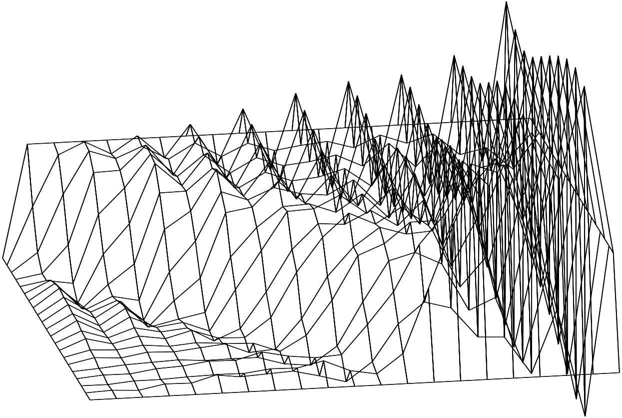
$$a(u,v) := (\mathcal{K}\,\nabla u,\, \nabla v)_\Omega + (\boldsymbol{\alpha}\cdot\nabla u,\, v)_\Omega$$

Coercivity:

$$a(v,v) := \|\mathcal{K}^{1/2}\,\nabla v\|_\Omega^2 + \cancelto{0}{(\boldsymbol{\alpha}\cdot\nabla v,\, v)_\Omega} \ge C_{min}\|\nabla v\|_\Omega^2$$

Advection Dominate Problem: $H|\boldsymbol{\alpha}| >> C_{min}$

GALERKIN METHOD



advection

Boundary Layer: SUPG (Hughes & Brooks '82)

Let $\mathcal{P}$ be a (regular) partition of Ω and K an element

$$V_1 := \left\{ v_1 \in H_0^1(\Omega) \,:\, v_1\,|_K \in \mathbb{P}^1(K),\ \forall K \in \mathcal{P} \right\}$$

Find $u_1 \in V_1$ *such that*

$$a(u_1, v_1) + \sum_{K \in \mathcal{P}} \tau_K \int_K (\mathcal{L}u_1 - f)\boldsymbol{\alpha} \cdot \nabla v_1 d\mathbf{x} = (f, v_1)_\Omega \quad \forall v_1 \in V_1$$

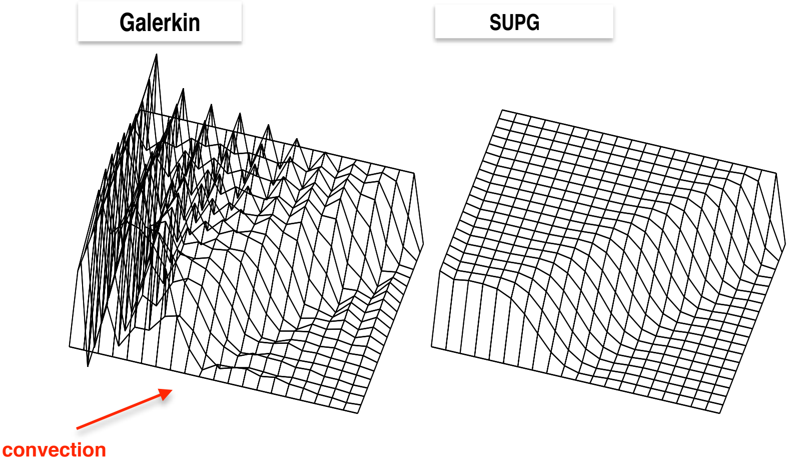
$$\tau_K = \begin{cases} \mathcal{O}(H_K^2/C_{min}) \\ \mathcal{O}(H_K/|\boldsymbol{\alpha}|) \end{cases}$$

diffusion dominates

convection dominates

$$a(v_1, v_1) \geq C_{min}\|\nabla v_1\|_\Omega^2 + \sum_{K \in \mathcal{P}} \tau_K \|\boldsymbol{\alpha} \cdot \nabla v_1\|_{0,K}^2$$

Boundary layer problem



Upscaling with Bubbles (Brezzi, Franca et al. '92)

$f\,|_K \in \mathbb{R}$, $\boldsymbol{\alpha}\,|_K \in \mathbb{R}^d$ and $\mathcal{K} \in \mathbb{R}^{d\times d}\,|_K$ and K is a triangle

Let B be the bubble space, i.e.,

$$B := \left\{ b \in H_0^1(\mathcal{P}) \,:\, b_K := b\,|_K \in \mathbb{P}^3(K),\ \forall K \in \mathcal{P} \right\}$$

Find $u := u_1 + u_b \in V_1 \oplus B$ *such that*

$$a(u,v) = (f,v)_\Omega \quad \forall v := v_1 + v_b \in V_1 \oplus B$$

Take $v\,|_K = b_K$ above and use $u_b\,|_K = C_K\, b_K$

$$C_K a_K(b_K, b_K) = (f, b_K)_K - a_K(u_1, b_K) = (f, b_K)_K - (\boldsymbol{\alpha}\cdot\nabla u_1, b_K)$$

$$\Rightarrow C_K = \frac{\int_K b_K\, d\mathbf{x}}{\|\mathcal{K}^{1/2}\nabla b_K\|_{0,K}} (f - \boldsymbol{\alpha}\cdot\nabla u_1)\,|_K$$

Recovering the SUPG (almost!)

- Take $v = v_1$ and use $u_b\,|_K = C_K\, b_K$

$$a(u_1, v_1) + a(u_b, v_1) = (f, v_1)_\Omega$$

$\Rightarrow$

$$a(u_1, v_1) + \sum_{K \in \mathcal{P}} (u_b, \mathcal{L}^*\, v_1)_K = (f, v_1)_\Omega$$

$\Rightarrow$

$$a(u_1, v_1) + \sum_{K \in \mathcal{P}} \tilde{\tau}_K (\boldsymbol{\alpha} \cdot \nabla u_1 - f, \boldsymbol{\alpha} \cdot \nabla v_1)_K = (f, v_1)_\Omega$$

We only recover the SUPG for the diffusive dominate cases

$$\tilde{\tau}_K := \frac{\left[\int_K b_K \, d\mathbf{x} \right]^2}{|K| \, \|\mathcal{K}^{1/2} \nabla b_K\|_{0,K}} = O(H_K^2 / C_{min})$$

Question: Is there an optimal enriching space B ?

A Rough Diffusive 1D Case: $\varepsilon \ll H$

Find u *such that*

$$\mathcal{L}\, u := -\frac{d}{d\,x}\left(\mathcal{K}^{\varepsilon}\,\frac{d\,u}{d\,x}\right) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

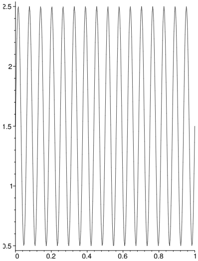
$\mathcal{K}^{\varepsilon} \in L^{\infty}(\Omega)$ has *multi-scale* features

$$C_{min} \leq \mathcal{K}^{\varepsilon}(x) \leq C_{max} \quad x \in \Omega$$

Find $u \in H_0^1(\Omega)$ such that

$$\int_{\Omega} \mathcal{K}^{\varepsilon}\,\frac{d\,u}{d\,x}\,\frac{d\,v}{d\,x} = \int_{\Omega} f\,v\,d\,x$$

$$\|\nabla(u - u_1)\|_{0,\Omega} \leq C\,\frac{H}{\varepsilon}\,\|f\|_{0,\Omega}$$



Generalized FEM: Babuska & Osborn ‘83

$$V_H := span\,\{\phi_i^{\varepsilon}\}_{i=1}^{N}\,, \quad -\frac{d}{d\,x}\mathcal{K}^{\varepsilon}(x)\,\frac{d\,\phi_i^{\varepsilon}}{d\,x} = 0 \text{ in } K, \quad \phi_i^{\varepsilon}\,|_{\partial K} \in \{0,\,1\}$$

Find $u_H \in V_H$ such that

$$\int_{\Omega} \mathcal{K}^{\varepsilon}\frac{d\,u_H}{d\,x}\,\frac{d\,v_H}{d\,x}\,dx \;=\; \int_{\Omega} f\,v_H\,dx \quad \forall v_H \in V_H$$

Nodal exactness

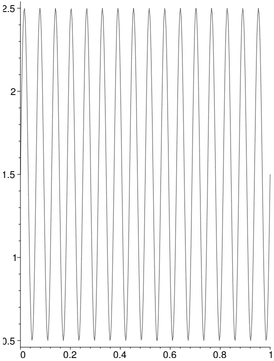
$$I_H u := \sum_{i=1}^{N} u(x_i)\phi_i^{\varepsilon}$$

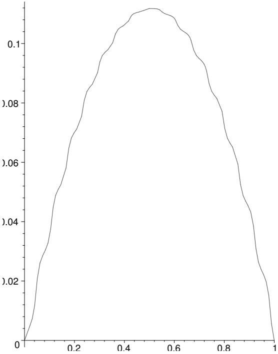
$$u_H(x_i) = I_H u(x_i) = u(x_i)$$

Robustness

$$\|\nabla(u - u_H)\|_{0,\Omega} \leq C\,H\|f\|_{0,\Omega}$$

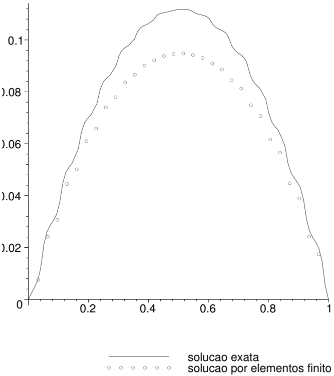
A Multi-Scale Benchmark

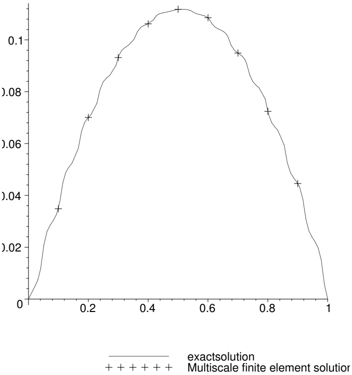




$$-\frac{d}{dx}(\mathcal{K}^{\boldsymbol{\varepsilon}}(x)\frac{du}{dx}) = f \text{ in } \Omega, \quad u\,|_{\partial\Omega} = 0$$

The GFEM





$$-\frac{d}{dx}(\mathcal{K}^{\varepsilon}(x)\frac{d\phi^{\boldsymbol{\varepsilon}}}{dx}) = 0 \text{ in } K, \quad \phi^{\boldsymbol{\varepsilon}}\,|_{\partial K} = \{0,1\}$$

90's – 00's

RFB, VMS and MsFEM

- Looking for "optimal" enrichments
- Extending GFEM to 2D/3D cases

RFB (Brezzi, Franca, Russo '94) and VMS (Hughes '95)

- Since the "optimal" B is unknown, take the whole space
$$B := \oplus_{K \in \mathcal{P}} H_0^1(K)$$
1D case: Linear + B = whole space
- *Find* $u_H = u_1 + u_b$ *such that*
$$a(u_H, v_H) = (f, v_H)_\Omega \quad \forall v_H = v_1 + v_b$$
- Take $v_H = v_b|_K$, and observe $u_b|_K$ solves
$$a_K(u_b, v_b) = (f, v_b)_K - a_K(u_1, v_b)$$
- $u_b|_K := \mathcal{M}_K(f - \mathcal{L}u_1)$ satisfies
$$\mathcal{L}\, u_b = f - \mathcal{L}\, u_1 \text{ in } K, \quad u_b|_{\partial K} = 0$$

$\Leftrightarrow$

$$u_b|_K = \mathcal{M}_K(1_K)\,(f - \mathcal{L}u_1)|_K, \quad (f - \mathcal{L}u_1)|_K \in \mathbb{R}$$

RFB (Brezzi, Franca, Russo '94) and VMS (Hughes '95)

- $\phi_K := \mathcal{M}_K(1_K)$ solves

$$\mathcal{L}\phi_K = 1_K \quad \text{in } K, \quad \phi_K = 0 \quad \text{on } \partial K$$

$$B := \{\phi_K\}_{K \in \mathcal{P}}$$

- *Find $u \in V_1$ such that*

$$a(u_1, v_1) + \sum_{K \in \mathcal{T}_H} \int_K a_K(u_b, v_1) =$$

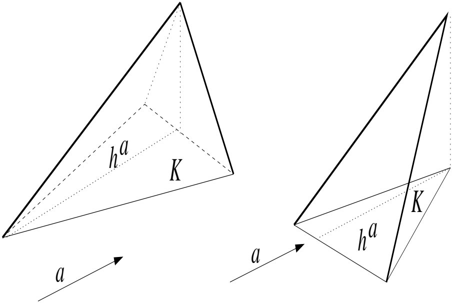
$$a(u_1, v_1) - \sum_{K \in \mathcal{T}_H} \int_K u_b \, \boldsymbol{\alpha} \cdot \nabla v_1 \, d\mathbf{x} =$$

$$a(u_1, v_1) + \sum_{K \in \mathcal{T}_H} \frac{\int_K \phi_K}{|K|} \int_K (\boldsymbol{\alpha} \cdot \nabla u_1 - f) \boldsymbol{\alpha} \cdot \nabla v_1 \, d\mathbf{x} = (f, v_1)_\Omega$$

RFB (Brezzi, Franca, Russo ‘94) and VMS (Hughes ‘95)

$$\|\boldsymbol{\alpha}\| H \gg \|\mathcal{K}\|$$

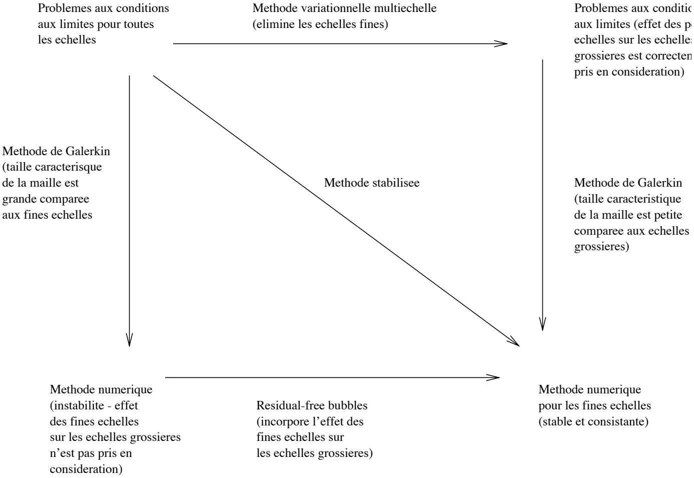
$$\frac{\int_K \phi_K}{|K|} = O(\frac{h^a}{\|\boldsymbol{\alpha}\|})$$



Remark: RFB $\rightleftharpoons$ VMS

$\phi_K = \int_K g$, Brezzi, F. and Franca, L. P. and Hughes, T. J. R. and Russo, A., CMAME, 1997

Stabilized Method $\rightleftharpoons$ RFB $\rightleftharpoons$ VMS



Problemes aux conditions
aux limites pour toutes
les echelles

Methode variationnelle multiechelle
(elimine les echelles fines)

Problemes aux conditic
aux limites (effet des p
echelles sur les echelle
grossieres est correcten
pris en consideration)

Methode de Galerkin
(taille caracterisque
de la maille est
grande comparee
aux fines echelles

Methode stabilisee

Methode de Galerkin
(taille caracteristique
de la maille est petite
comparee aux echelles
grossieres)

Methode numerique
(instabilite - effet
des fines echelles
sur les echelles grossieres
n'est pas pris en
consideration)

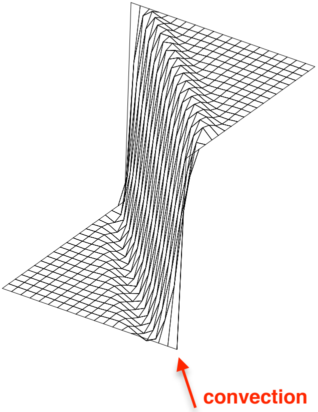
Residual-free bubbles
(incorpore l'effet des
fines echelles sur
les echelles grossieres)

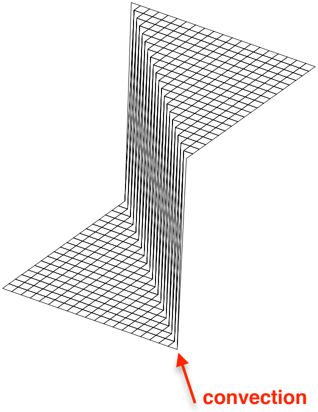
Methode numerique
pour les fines echelles
(stable et consistante)

Skew-Advection Case

SUPG

RFB





MsFEM (Hou, Wu and Cai ‘99): Oscillatory Coeff.

Find $u_H \in V_{MsFEM}$ *such that*

$$a(u_{ms}, v_{ms}) = (f, v_{ms})_\Omega \quad \forall v_{ms} \in V_{MsFEM}$$

$$V_{MsFEM} := span\,\{\phi_i^\varepsilon\}_{i=1}^N$$

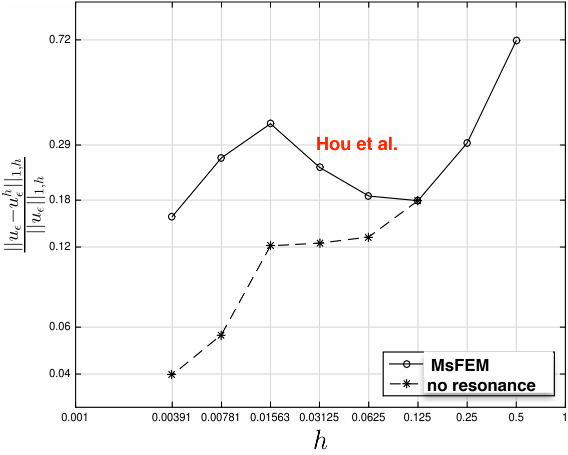
$$u_{ms} := \sum_i^N c_i \phi_i^\varepsilon$$

$$-\nabla \cdot (\mathcal{K}^\varepsilon \, \nabla \phi_i^\varepsilon) = 0 \quad \text{in } K, \quad \phi_i^\varepsilon = \psi_i \quad \text{on } \partial K$$

Resonance Error:

$$\|\nabla(u - u_{ms})\|_{0,\Omega} \leq C \left(H + \left(\frac{\varepsilon}{H}\right)^{1/2} \right) \|f\|_{0,\Omega}$$

Resonance Influence



Relationship: RFB and MsFEM (Sangalli '00)

$$\mathcal{L}^{\varepsilon} u := -\nabla \cdot (\mathcal{K}^{\varepsilon} \nabla u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega$$

$$u_{ms} = \sum_{i=1}^{N} c_i \phi_i^{\varepsilon} = \underbrace{\sum_{i}^{N} c_i \psi_i}_{u_1} + \sum_{i=1}^{N} c_i (\phi_i^{\varepsilon} - \psi_i) = u_1 - \sum_{K} \mathcal{M}_K (\mathcal{L}^{\varepsilon} u_1)$$

$$= u_1 + \sum_{K} \Big[\underbrace{\mathcal{M}_K (f - \mathcal{L}^{\varepsilon} u_1)}_{u_b\,|_K} - \mathcal{M}_K\, f \Big]$$

$$= \underbrace{u_{RFB}}_{u_1 + u_b} - \sum_{K} \mathcal{M}_K\, f$$

Resonance Error:

$$\|\nabla (u - u_{RFB})\|_{0,\Omega} \leq C \left(H + \left(\frac{\varepsilon}{H} \right)^{1/2} \right) \|f\|_{0,\Omega}$$

How to setup local upscalling without bubbles ?

Conforming

- PGEM and RELP
  Petrov-Galerkin enrichment strategies
- LOD
  (Quasi) local solutions decreasing exponentially
- HMM
  Local problems around integration points

Non-Conforming

- MsFEM
  Including oversampling to set up boundary conditions
- DEM
  Discontinuous enrichment through Lagrange multipliers
- MHM

PGEM (Franca, Madureira and Valentin '05) and RELP (Barrenechea and Valentin '10)

- $B := \oplus_{K \in \mathcal{T}_H} H^1_0(K)$ and $E \subset H^1_0(\Omega)$
- *Find* $u_H = u_1 + u_e \in V_1 + E$
$$\begin{cases} a_K(u_H, v_b) = (f, v_b)_K & \forall v_b \in B \\ \quad a(u_H, v_1) = (f, v_1)_\Omega & \forall v_1 \in V_1 \end{cases}$$
- $\mathcal{L}\, u_e = f - \mathcal{L} u_1$ in K and $u_e = g$ on ∂K
$$\begin{cases} \mathcal{L}_s g = f - \mathcal{L}_s u_1 & \text{PGEM} \\ \mathcal{L}_s g = F([\partial_{\boldsymbol{n}} u_1]) & \text{RELP} \end{cases}$$
- *Find* $u_1 \in V_1$ *such that*
$$\sum_K a_K((I - \tilde{\mathcal{M}}_K) u_1, v_1) = (f, v_1)_\Omega - \sum_K a_K(\tilde{\mathcal{M}} f, v_1)$$

PGEM: Reactive-Diffusive Equation

- Assume u solves

$$\mathcal{L}\, u := -\varepsilon\, \Delta\, u + u \,=\, f \quad \text{in } \Omega \quad \text{and} \quad u\,|_{\partial\Omega} = 0$$

- PGEM: *Find* $u_1 \in V_1$ *such that*

$$\sum_K a_K((I - \tilde{\mathcal{M}}_K)u_1, v_1) = (f, v_1)_\Omega - \sum_K a_K(\tilde{\mathcal{M}}_K f, v_1)$$

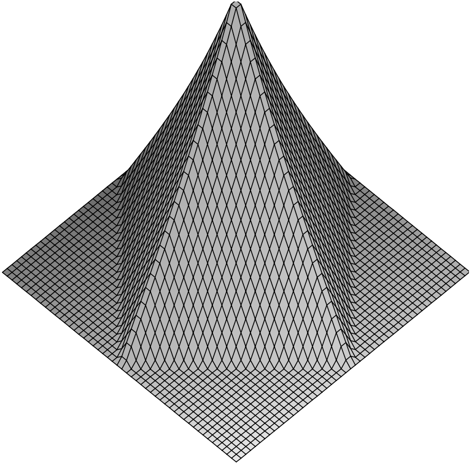
- The enrichment $u_e := \tilde{\mathcal{M}}_K(f - u_1)$ satisfies

$$\begin{cases} \mathcal{L}\, u_e = f - u_1 \text{ in } K \quad \text{and} \quad u_e = g \text{ on } \partial K \\ \mathcal{L}_s\, g := -\partial_{ss} g + g = f - u_1 \quad F \subset \partial K \end{cases}$$

- Convergence:

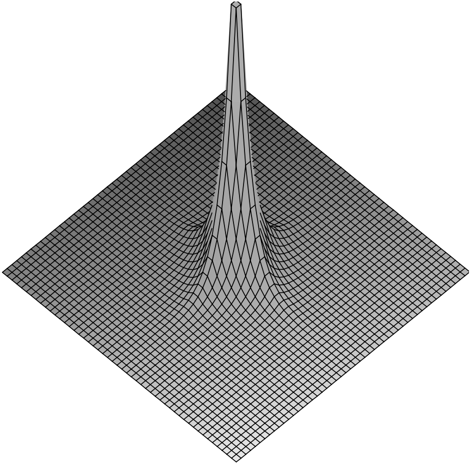
$$\|u - u_1\|_{1,\mathcal{P}} = \mathcal{O}(H + \sqrt{\frac{H_l^3}{\varepsilon}}), \quad H_l = \text{diam. first layer}$$

$\varepsilon = 1$



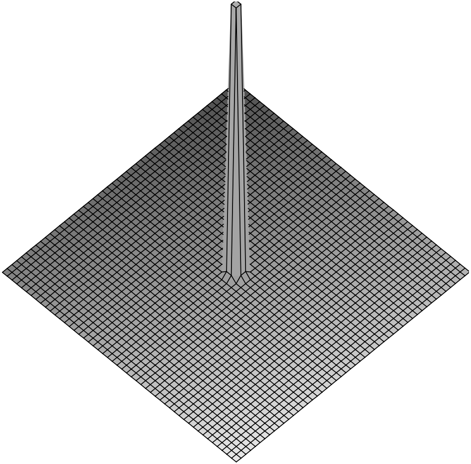
A multiscale base on a patch. $\mathcal{L}u := -\boldsymbol{\varepsilon}\Delta u + u = f$

$\varepsilon = 0.1$



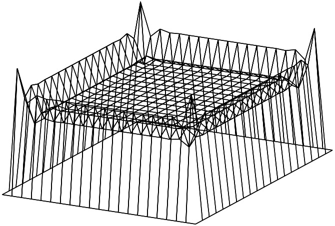
A multiscale base on a patch. $\mathcal{L}u := -\boldsymbol{\varepsilon}\Delta u + u = f$

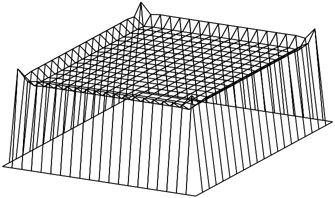
$\varepsilon = 0.01$ (mass lumping revisited)

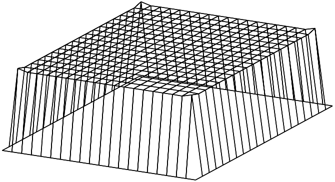


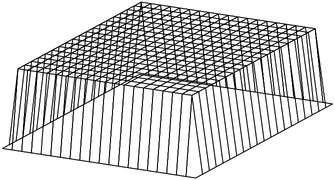
A multiscale base on a patch. $\mathcal{L}u := -\boldsymbol{\varepsilon}\Delta u + u = f$

$\varepsilon = 10^{-6}$ and $f = 1$


GALERKIN METHOD


RFB METHOD


UNUSUAL METHOD


NEW ENRICHED METHOD

Conclusion

- A survey of (some) multiscale methods from the 80's to today
- To compromise local upscaling (parallel computation) with accuracy and robustness is feasible
- Open questions. Can one stay local and...
  - get rid of resonance error completely ?
  - converge under low regularity ?
  - be robust with respect to high-contrast?
  - respect the maximum principle ?

MHM

Multiscale Hybrid-Mixed Method

A First Glance

Worldwide (Interdisciplinary) Collaboration

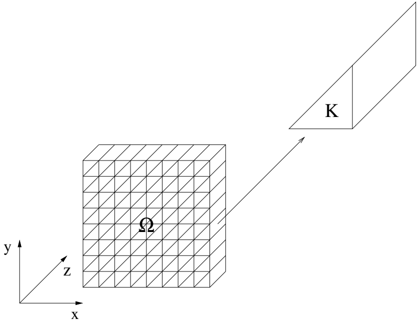
- America
  - A. T. Gomes, R. Souto, A. Madureira, W. Pereira and L. Martins - LNCC/Brazil
  - H. Fernando - UFF/Brazil
  - Henrique Versieux - UFMG/Brazil
  - S. Gomes and P. Devloo - UNICAMP/Brazil
  - Diego Paredes and Rodolfo Araya - UDEC/Chile
  - Abner Poza - USCS/Chile
  - Christopher Harder - MSDenver/USA
- Europe
  - Theophile Chaumont and Stephane Lanteri - Inria/France
  - Claire Scheid and Roland Masson - Université de Nice/France
  - Fabrice Jaillet - IUT Lyon1/France
  - Gabriel Barrenechea - University of Strathclyde/UK

MHM Method: A Brief Historical Perspective

Year / Article	Contribution
2013 JCP SINUM	Original idea. A priori and a posteriori analysis: Darcy model
2015 MMS	Extension to the reactive-advective dominated problem
2016 M2AN MathComp Springer	MHM for Elasticity Uniform Convergence Foundations of the MHM
2017 CMAME	MHM for Stokes equations
2018 MMS	MHM for Maxwell equations

Setting

Let $\mathcal{P}$ be a (coarse) partition of Ω and $\partial\mathcal{P}$ its boundary



$$V := \left\{ v \in L^2(\Omega) \,:\, v \in H^1(K) \right\}$$

$$\Lambda := \left\{ \sigma\, \boldsymbol{n}^K \,|_{\partial K} \,:\, \sigma \in H(div, \Omega) \right\}$$

Here $K \subset \mathcal{P}$ and $\partial K \subset \partial\mathcal{P}$

$$(\cdot,\cdot)_{\mathcal{P}} := \sum_{K \in \mathcal{P}} (\cdot,\cdot)_K \quad \text{and} \quad (\cdot,\cdot)_{\partial\mathcal{P}} := \sum_{K \in \mathcal{P}} (\cdot,\cdot)_{\partial K}$$

Hybrid Formulation (Raviart-Thomas '77)

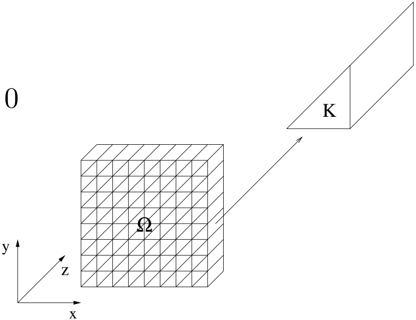
Find $(u, \lambda) \in V \times \Lambda$ such that

$$\begin{aligned} (\mathcal{K}\,\nabla u,\, \nabla\, v)_{\mathcal{P}} + (\lambda, v)_{\partial\mathcal{P}} &= (f,\, v)_{\mathcal{P}} \quad \forall v \in V \\ (\mu, u)_{\partial\mathcal{P}} &= 0 \quad \forall \mu \in \Lambda \end{aligned}$$

$\Rightarrow u \in H_0^1(\Omega)$ solves

$$-\nabla\cdot(\mathcal{K}\nabla u) = f \text{ in } \Omega \quad \text{and} \quad u\,|_{\partial\Omega} = 0$$

$$\Rightarrow \mathcal{K}\nabla\, u \cdot \boldsymbol{n}^K = -\lambda \text{ on } \partial K$$



The MHM Method

$$\textit{Find } (u_{0,H}, \lambda_H) \in V_0 \times \Lambda_H \textit{ such that}$$

Global

$$(\lambda_H, v_0)_{\partial\mathcal{T}_H} = (f, v_0)_{\mathcal{T}_H} \quad \forall v_0 \in V_0$$

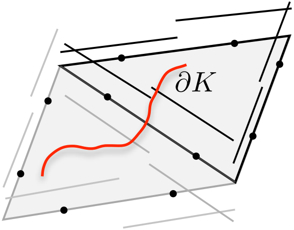
$$(\mu_H, T\,\lambda_H)_{\partial\mathcal{T}_H} + (\mu_H, u_{0,H})_{\partial\mathcal{T}_H} = -(\mu_H, \hat{T}\,f)_{\partial\mathcal{T}_H} \quad \forall \mu_H \in \Lambda_H$$

$$\textit{where } T\,\lambda_H = \textstyle\sum_i c_i \underbrace{T\,\psi_i}_{\eta_i} \text{ and } \hat{T}\,f$$

Local

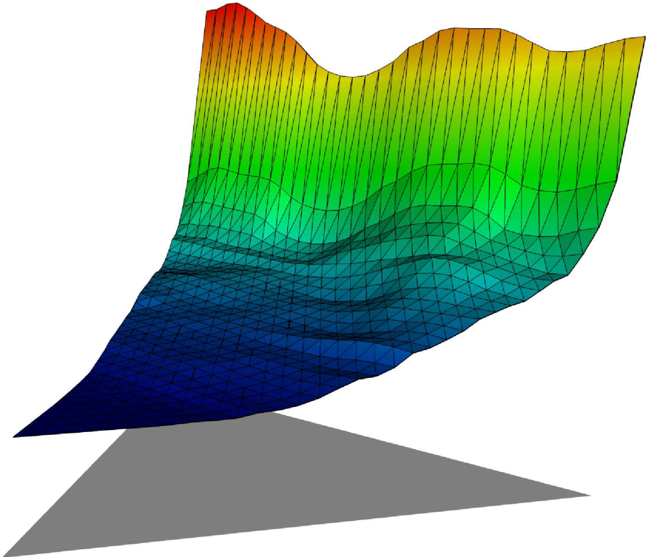
$$(\mathcal{K}^{-1}\nabla\,\eta_i, \nabla\,v^{\perp})_K = -(\psi_i, v^{\perp})_{\partial K}$$

$$(\mathcal{K}^{-1}\nabla\,\hat{T}\,f, \nabla\,v^{\perp})_K = (f, v^{\perp})_K$$



$$\lambda_H = \sum_i c_i\,\psi_i$$

A Typical Heterogenous Basis Function



Relationship with Other Methods

$l = 0,\ f \in \mathbb{R}$

$\mathcal{K} \equiv I$
$\mathrm{RT}_0/\mathbb{P}_0$

Multi-Scale $\mathcal{K}$
Hou at al. (2002)

$l = 0,\ 1$
Multi-Scale $\mathcal{K}$
Arbogast (2004)

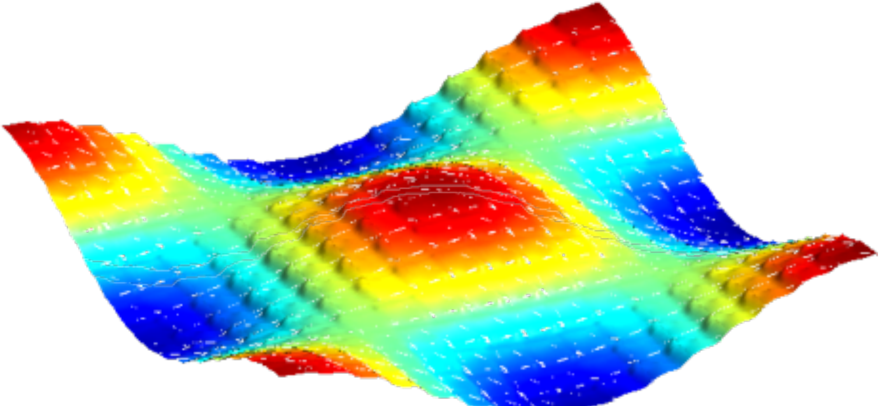
Numerical Validation
An Oscillatory Coefficient Case

The Highly-Oscillatory Problem ($\varepsilon = \frac{1}{16}$)

$$\mathcal{K}(\boldsymbol{x}) = \frac{2 + \frac{1.8 \sin 2\pi x}{\varepsilon}}{2 + \dfrac{1.8 \sin 2\pi y}{\varepsilon}} + \frac{2 + \dfrac{1.8 \sin 2\pi y}{\varepsilon}}{2 + \dfrac{1.8 \cos 2\pi x}{\varepsilon}} \,,$$

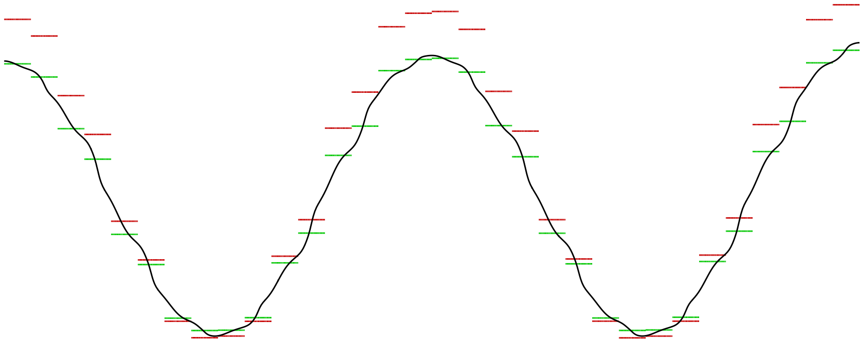
Space $\mathbb{P}_2(F)$ (16×16 Elements)

$$u_{0,H}^h + T_h\,\lambda_H + \hat{T}_h\,f$$



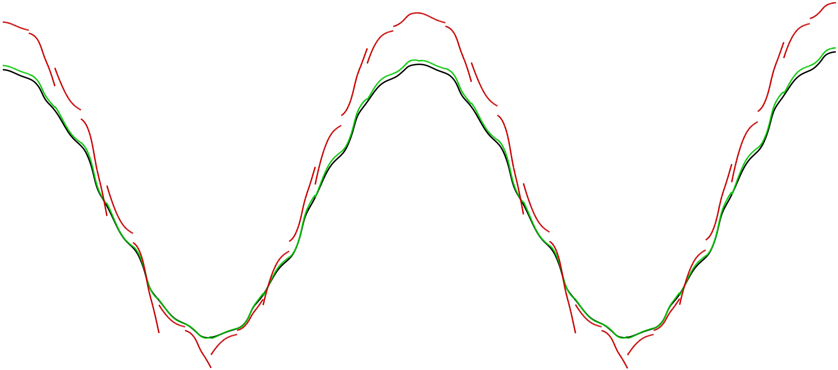
Spaces $\mathbb{P}_0(F)$ and $\mathbb{P}_2(F)$

$u^h_{0,H}$



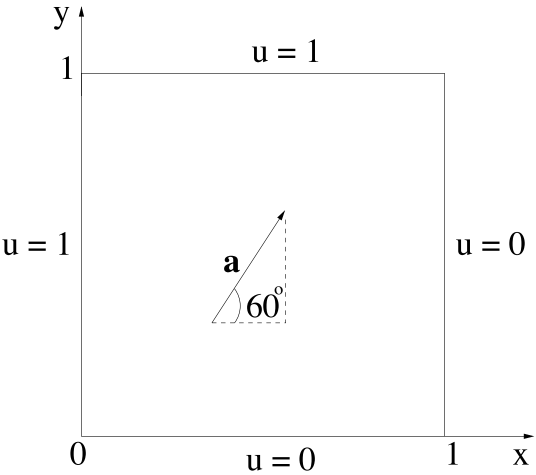
Spaces $\mathbb{P}_0(F)$ and $\mathbb{P}_2(F)$

$$u_{0,H}^h + T_h\,\lambda_H + \hat{T}_h\,f$$



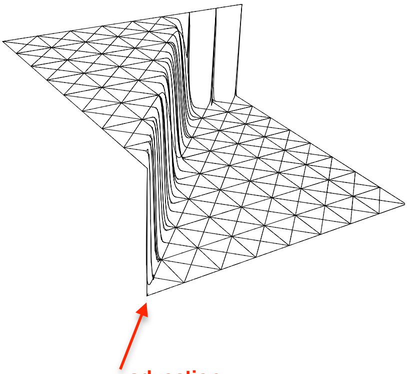
Numerical Validation
A Sharp Boundary Layer Case

Setting: $\varepsilon = 10^{-5}$

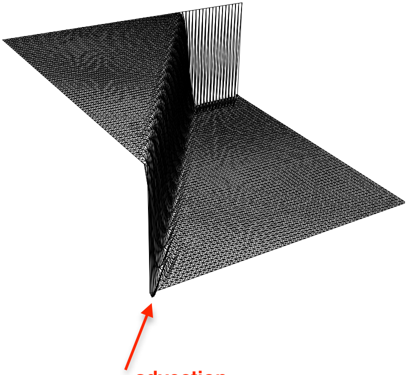


Same Number of D.O.Fs and Order of Approx.

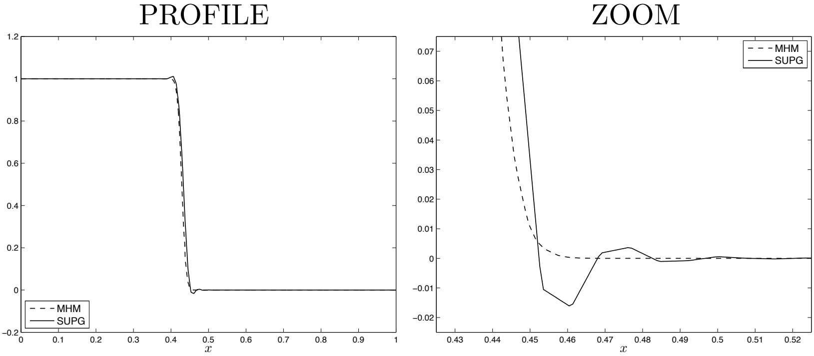
MHM



SUPG



Internal Layer: No Need of Shock-Capturing



To be continued in Part II

The MHM method

in

Polygon