Non-fitting meshes for Maxwell's equations

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Borehole Logging

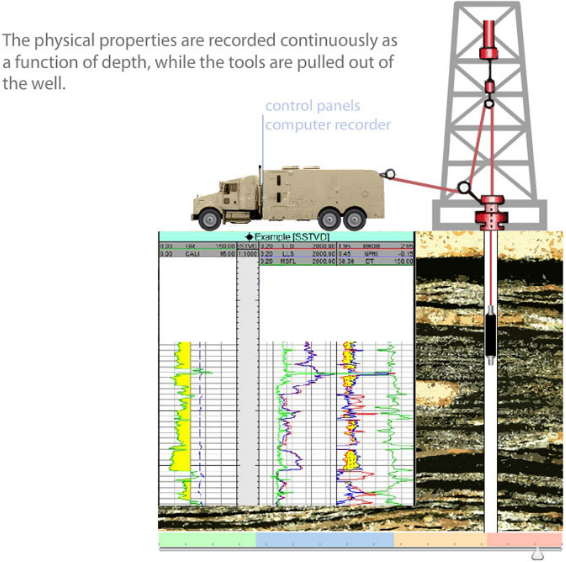
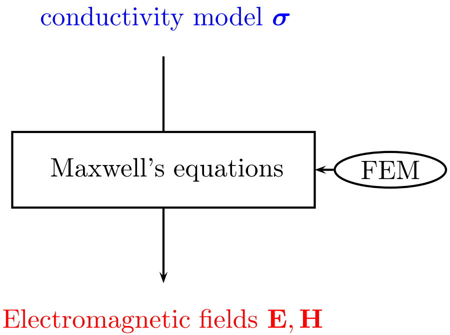


Figure: Borehole Logging

Objective

Given a conductivity model, compute a quantity of interest.



Applications: Magnetotellurics, Logging-While-Drilling...

Standard approach

FEMs are mesh-based methods.

Fitting meshes are usually used with FEMs.

The physical interfaces are fitted by mesh faces (edges).

The conductivity parameter is constant (smooth) inside each cell.

Fitting vs non-fitting meshes

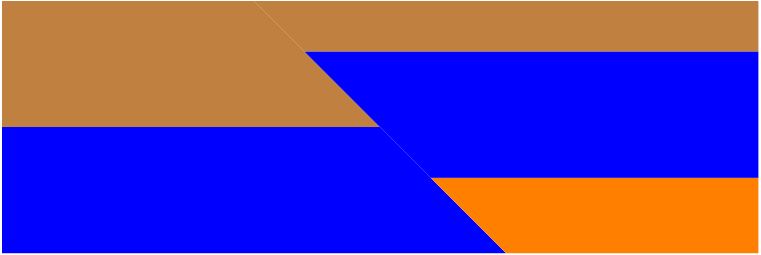


Figure: Example of conductivity model

There are two physical interfaces.

Fitting vs non-fitting meshes

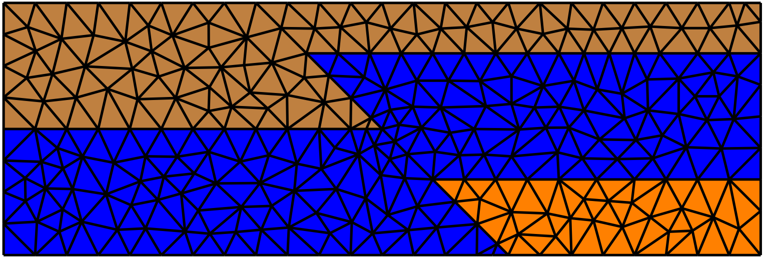


Figure: Example of fitting mesh

The physical interfaces aligned with mesh edges.

Fitting vs non-fitting meshes

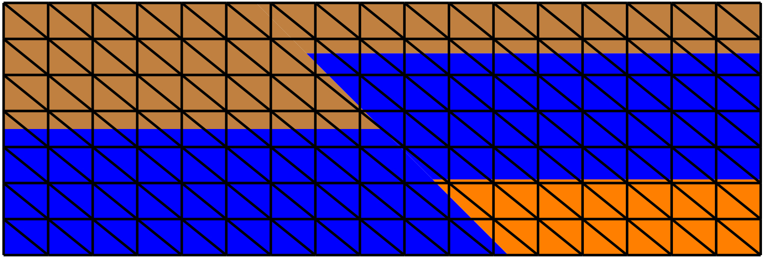


Figure: Example of non-fitting mesh

The physical interfaces are inside mesh cells.

Fitting meshes

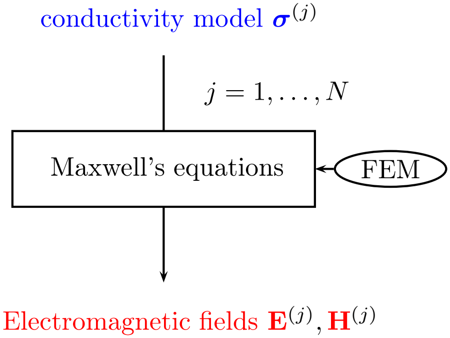
Fitting meshes are good because:

- matrix assembly is easy
- optimal convergence rates

However, they require to:

- use unstructured meshes
- re-generate a mesh for each conductivity model

Sequence of problems



Example: Logging while drilling

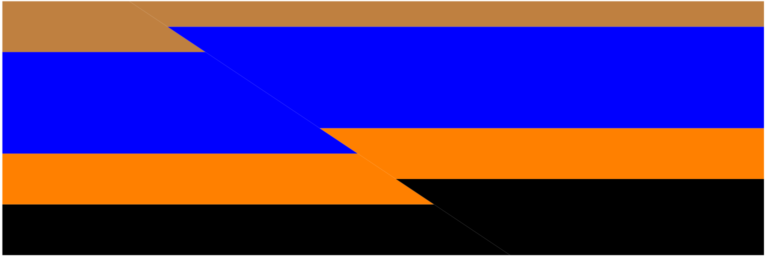


Figure: Logging while drilling

Example: Logging while drilling

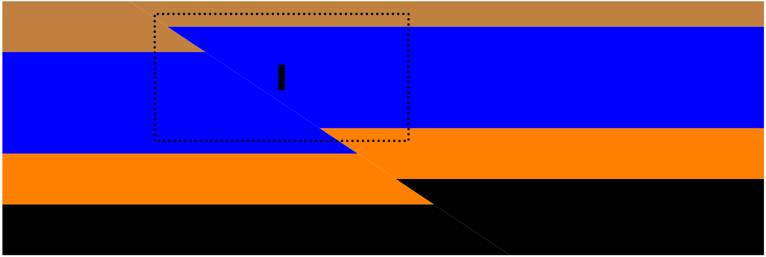


Figure: Logging while drilling

Example: Logging while drilling

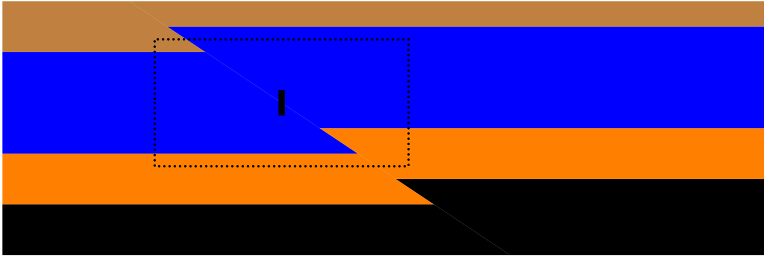


Figure: Logging while drilling

Example: Logging while drilling

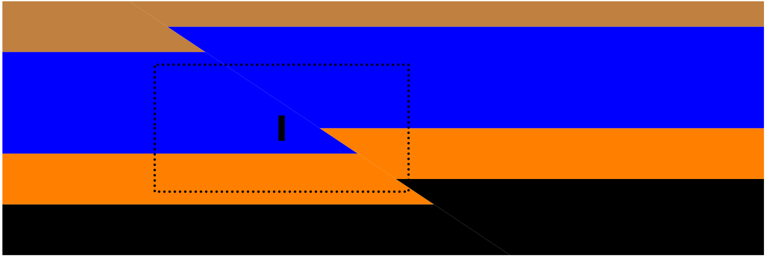


Figure: Logging while drilling

Fitting meshes

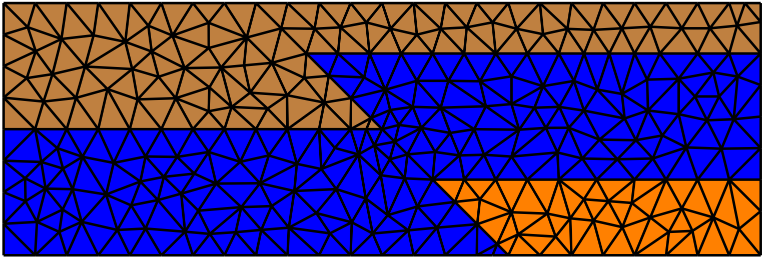


Figure: Position 1: fitting mesh

Fitting meshes

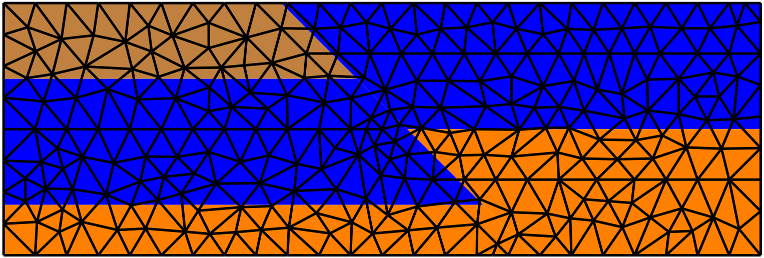


Figure: Position 2: the mesh becomes non-fitting

Fitting meshes

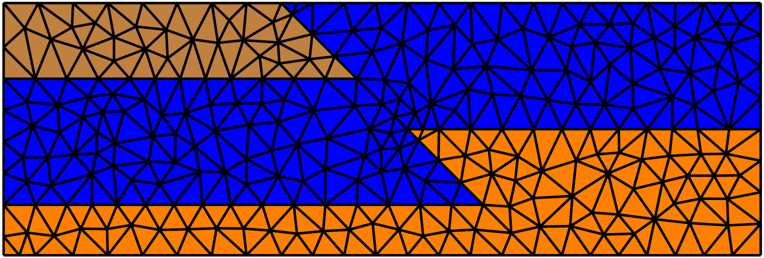


Figure: Position 2: remeshing required

Non-fitting meshes

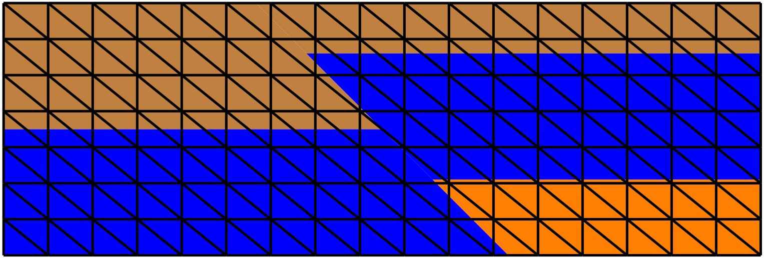


Figure: Non-fitting meshes avoid remeshing

Non-fitting meshes

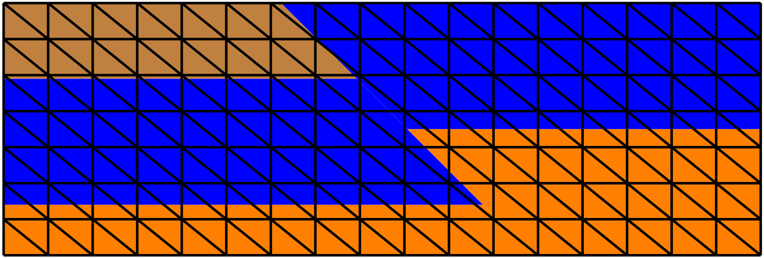


Figure: Non-fitting meshes avoid remeshing

Non-fitting meshes

The advantages of non-fitting meshes are:

- mesh generation is easier
- tensorial products are possible
- the same mesh can be used in a sequence of problems

The drawbacks of non-fitting meshes are:

- special quadrature schemes need to be used
- convergence rates might be decreased

Objectives

We focus on first-order Nedelec's edge elements.

Determine when non-fitting meshes can be used.

Analytical study: error estimates for non-fitting meshes.

Numerical study: 2D experiments.

Maxwell's equations

We solve the Maxwell's equation with a constant permeability $\mu = \mu_0$.

We assume that $i\omega\varepsilon + \boldsymbol{\sigma} \simeq \boldsymbol{\sigma}$, so that we set $\varepsilon = 0$.

To simplify, the conductivity $\boldsymbol{\sigma}$ is scalar and piecewise constant.

Maxwell's equations: the E-formulation

E is solution to

$$i\omega\mu_0\boldsymbol{\sigma}\mathbf{E} + \nabla \times \nabla \times \mathbf{E} = i\omega\mu_0\mathbf{J}.$$

H is obtained by

$$\mathbf{H} = \nabla \times \mathbf{E}.$$

Maxwell's equations: the H-formulation

H is solution to

$$i\omega\mu_0 \mathbf{H} + \nabla \times \left(\boldsymbol{\sigma}^{-1} \nabla \times \mathbf{H}\right) = \nabla \times \left(\boldsymbol{\sigma}^{-1} \mathbf{J}\right).$$

E is obtained by

$$\mathbf{E} = \boldsymbol{\sigma}^{-1} \left(\mathbf{J} - \nabla \times \mathbf{H}\right).$$

Standard error-estimates for fitting meshes

In a layered medium, the solution is piecewise smooth.

Hence, if a fitting mesh is used, we have

$$\|\mathbf{E} - \mathbf{E_h}\|_{L^2(\Omega)} = \mathcal{O}(\mathbf{h}),$$

and

$$\|\mathbf{H} - \mathbf{H_h}\|_{L^2(\Omega)} = \mathcal{O}(\mathbf{h}).$$

Standard error-estimates for non-fitting meshes

With a non-fitting mesh, the solution can jump inside mesh cells.

The solution being less regular, the standard error-estimates give

$$\|\mathbf{E} - \mathbf{E_h}\|_{L^2(\Omega)} = \mathcal{O}(\mathbf{h}^{1/2})$$

and

$$\|\mathbf{H} - \mathbf{H_h}\|_{L^2(\Omega)} = \mathcal{O}(\mathbf{h}^{1/2}).$$

At first sight, convergence rates are bad for **E** and **H**.

Actually, these error-estimates are pessismistic.

New error-estimates for non-fitting meshes

Our new result is that actually, we have

$$\|\mathbf{E} - \mathbf{E_h}\|_{L^2(\Omega)} = \mathcal{O}(\mathbf{h}^{1/2}),$$

and

$$\|\mathbf{H} - \mathbf{H_h}\|_{L^2(\Omega)} = \mathcal{O}(\mathbf{h}).$$

$\mathbf{H_h}$ converges lineary for fitting and non-fitting meshes.

Conclusion on error-estimates

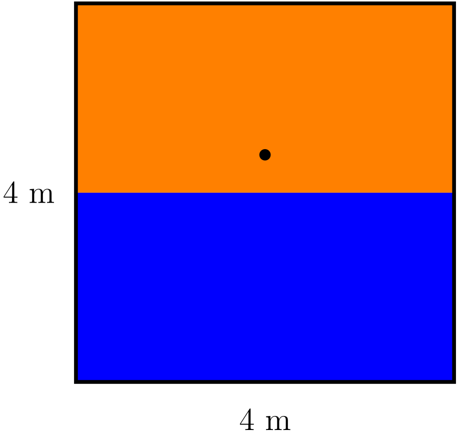
We have analyzed the general case, where the solution can have singularities.

Our new error-estimate have different convergence rates for $\mathbf{E_h}$ and $\mathbf{H_h}$.

In general $\mathbf{H_h}$ has a better convergence rate than “expected”.

The convergence rate for $\mathbf{H_h}$ is the same for fitting and non-fitting meshes.

Numerical experiments: Layered medium settings

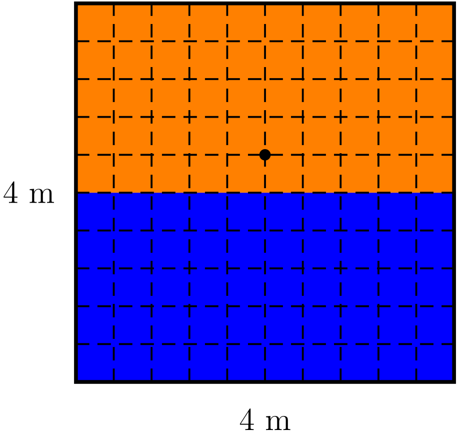


- (orange) $\boldsymbol{\sigma} = 10^{-2}\ \mathrm{Sm}^{-1}$ (Oil saturated rock)
- (blue) $\boldsymbol{\sigma} = 1\ \mathrm{Sm}^{-1}$ (Water saturated rock)
- $\bullet$ $\mathbf{y}$: source center

$f = 2 \times 10^6$ Hz

Figure: Layered medium

Numerical experiments: Layered medium settings



$\boldsymbol{\sigma} = 10^{-2}\ \mathrm{Sm}^{-1}$ (Oil saturated rock)

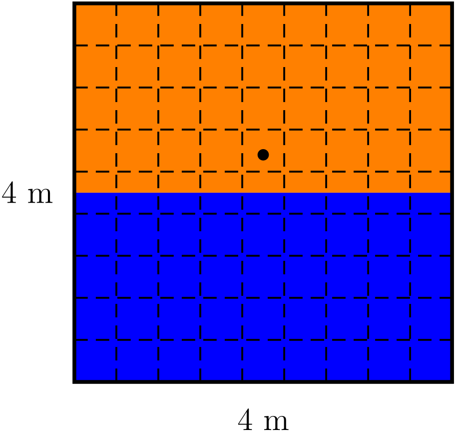
$\boldsymbol{\sigma} = 1\ \mathrm{Sm}^{-1}$ (Water saturated rock)

• $\mathbf{y}$: source center

$f = 2 \times 10^6$ Hz

Figure: Layered medium: fitting mesh

Numerical experiments: Layered medium settings



$\boldsymbol{\sigma} = 10^{-2}\ \mathrm{Sm}^{-1}$

(Oil saturated rock)

$\boldsymbol{\sigma} = 1\ \mathrm{Sm}^{-1}$

(Water saturated rock)

• $\mathbf{y}$: source center

$f = 2 \times 10^{6}$ Hz

Figure: Layered medium: non-fitting mesh

TE-polarization

The current density is polarized as $\mathbf{J} = (0, \mathbf{J}_y, 0)$.

The electric field is a scalar $\mathbf{E}_y$.

The magnetic field is a vector $\mathbf{H} = (\mathbf{H}_x, \mathbf{H}_z)$.

$\mathbf{H}$ is approximated by edge finite elements.

We compute $\mathbf{E} = \boldsymbol{\sigma}^{-1} (\mathbf{J} - \nabla \times \mathbf{H})$ by post-processing.

TE-polarization

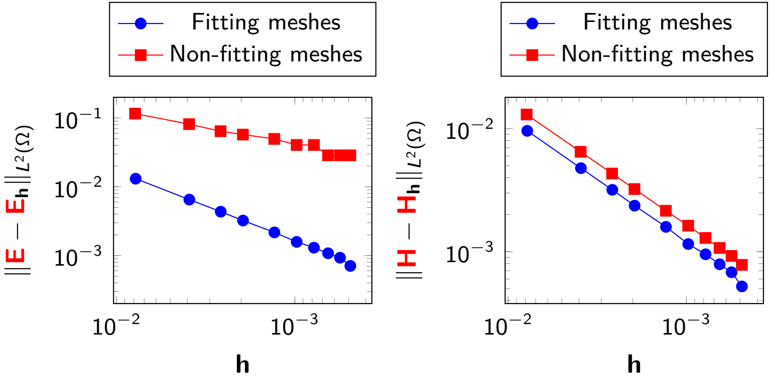


Figure: Numerical errors in TE-polarization

TM-polarization

The current density is polarized as $\mathbf{J} = (0, 0, \mathbf{J}_z)$.

The electric field is a vector $\mathbf{E} = (\mathbf{E}_x, \mathbf{E}_z)$.

The magnetic field is a scalar $\mathbf{H}_y$.

We approximate $\mathbf{E}$ with edge finite elements.

We obtain $\mathbf{H}_y = \nabla \times \mathbf{E}$ by post-processing.

TM-polarization

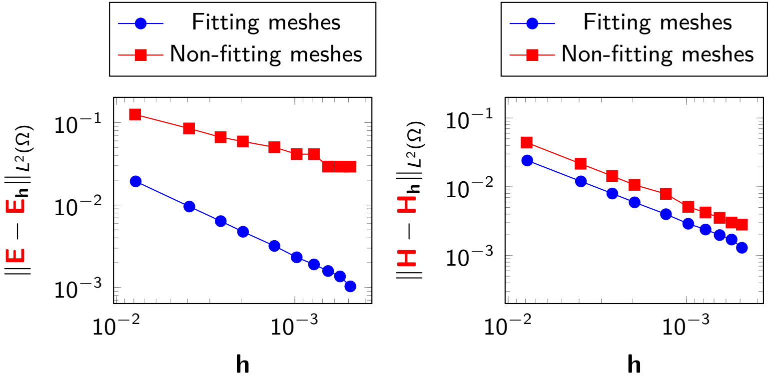


Figure: Numerical errors in TM-polarization

Conclusion

The predicted convergence rates are observed numerically.

The convergence rate of $\mathbf{E_h}$ is decreased for non-fitting meshes.

The convergence rate of $\mathbf{H_h}$ is same for both types of meshes.

Conclusion

For geophysical applications ($\mu = \mu_0$), non-fitting meshes can be used to approximate $\mathbf{H}$.

The accuracy loss due to non-fitting meshes is "reasonable".

In our numerical experiments, the error is at most multiplied by 2.

On-going work

Sharper error estimates for layered media.

More realistic simulations (borehole logging, MT...).

2.5D and 3D Maxwell's equations.

Comparison between the **E** and **H** formulations.