EDP with strong anisotropy : transport, heat, waves equations

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Main goals

- Effective models when disparate scales occur ;
- Derivation of the limit models, well-posedness, balances, symmetries, etc ;
- Convergence, error estimates ;
- Applications : transport of charged particles (magnetic confinement), heat equations, Maxwell equations.

1. Multi-scale analysis for linear first order PDE

$$\begin{cases} \partial_t u^\varepsilon + a \cdot \nabla_y u^\varepsilon + \frac{1}{\varepsilon} b \cdot \nabla_y u^\varepsilon = 0, & (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m \\ u^\varepsilon(0,y) = u^{\mathrm{in}}(y), & y \in \mathbb{R}^m. \end{cases}$$

Hypotheses

$$a \in L^1_{\mathrm{loc}}(\mathbb{R}_+; W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m)), \ \ b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m) \implies \text{smooth flows}$$

$$\mathrm{div}_y a = 0, \ \ \mathrm{div}_y b = 0 \implies \text{measure preserving flows}$$

$$|a(t,y)| + |b(y)| \le C(1 + |y|) \implies \text{global flows}$$

Vlasov equation with strong magnetic field

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v_1\partial_{x_1} + v_2\partial_{x_2})f^\varepsilon + v_3\partial_{x_3} f^\varepsilon + \frac{q}{m}E\cdot\nabla_v f^\varepsilon + \frac{qB}{m\varepsilon}(v_2\partial_{v_1} - v_1\partial_{v_2})f^\varepsilon = 0$$

$$m = 6,\quad y = (x, v),\quad a\cdot\nabla_{x,v} = v_3\partial_{x_3} + \frac{q}{m}E(x)\cdot\nabla_v$$

$$\frac{1}{\varepsilon}b\cdot\nabla_{x,v} = \frac{1}{\varepsilon}(v_1\partial_{x_1} + v_2\partial_{x_2}) + \frac{\omega_c}{\varepsilon}(v_2\partial_{v_1} - v_1\partial_{v_2})$$

Question: behavior when $\varepsilon \searrow 0$?

Main idea: filtering out the fast oscillations

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = b(Y(s;y)), \quad Y(0;y) = y, \quad (s,y) \in \mathbb{R} \times \mathbb{R}^m$$

New coordinates

$$z = Y(-t/\varepsilon, y) \text{ or equivalently } y = Y(t/\varepsilon, z)$$

Search for a profile

$$u^\varepsilon(t,y) = v^\varepsilon(t, \underbrace{Y(-t/\varepsilon; y)}_{z})$$

Why this change of coordinates ?

$$\begin{cases} \partial_t v^\varepsilon(t,z) + \underbrace{\partial_y Y(-t/\varepsilon; Y(t/\varepsilon;z))a(t, Y(t/\varepsilon;z))}_{\varphi(t/\varepsilon)a(t)} \cdot \nabla_z v^\varepsilon(t,z) = 0, \\ v^\varepsilon(0,z) = u^{\text{in}}(z), \end{cases}$$

Stability for $(v^\varepsilon)_{\varepsilon>0}$

$$\varphi(s)a = \partial_y Y(-s; Y(s;\cdot))a(Y(s;\cdot))$$

Behavior when $\varepsilon \searrow 0$

$$\partial_y Y(-t/\varepsilon; Y(t/\varepsilon;z))a(t, Y(t/\varepsilon;z)) = \varphi(t/\varepsilon)a(t)$$

If involution between $a(t)$ and b

$$[b, a(t)] = 0 \implies \varphi(s)a(t) = a(t), s \in \mathbb{R}$$

$$\begin{cases} \partial_t v^\varepsilon(t,z) + a(t,z)\cdot\nabla_z v^\varepsilon(t,z) = 0, & (t,z)\in\mathbb{R}_+\times\mathbb{R}^m \\ v^\varepsilon(0,z) = u^{\mathrm{in}}(z), & z\in\mathbb{R}^m \end{cases}$$

$$v^\varepsilon(t,z) = u^{\mathrm{in}}(Z(-t;z)) = v(t,z), \;\; \frac{\mathrm{d}Z}{\mathrm{d}t} = a(t, Z(t;z))$$

$$u^\varepsilon(t,y) = v(t, Y(-t/\varepsilon;y)) = u^{\mathrm{in}}(Z(-t; Y(-t/\varepsilon;y)))$$

Splitting : advection along a and advection along $\frac{1}{\varepsilon}b$.

Two scale approach : t and $s = t/\varepsilon$

Use ergodicity

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \partial_y Y(-s; Y(s;\cdot))a(t, Y(s;\cdot))\,\mathrm{d}s = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \varphi(s)a(t)\,\mathrm{d}s$$

Key point

Emphasize a C^0-group of unitary transformations and use :

von Neumann's Ergodic Mean Theorem

Let $(G(s))_{s\in\mathbb{R}}$ be a C^0-group of unitary operators on a Hilbert space $(H, (\cdot,\cdot))$ and A be the infinitesimal generator of G. Then, for any $x \in H$, we have

$$\lim_{T\to+\infty} \frac{1}{T}\int_r^{r+T} G(s)x\,\mathrm{d}s = \mathrm{Proj}_{\ker A}x, \ \text{ strongly in } H$$

uniformly with respect to $r \in \mathbb{R}$.

Average vector field

$$X_Q = \{c(y) : \mathbb{R}^m \to \mathbb{R}^m \text{ measurable} : \int_{\mathbb{R}^m} Q(y) : c(y) \otimes c(y) \, \mathrm{d}y < +\infty\}$$

$$Q = P^{-1}, \;\; P = {}^tP > 0, [b, P] := (b \cdot \nabla_y)P - \partial_y b \, P - P \, {}^t\partial_y b = 0$$

$$(c, d)_Q = \int_{\mathbb{R}^m} Q(y) : c(y) \otimes d(y) \, \mathrm{d}y, \;\; c, d \in X_Q.$$

Proposition $(\varphi(s))_{s\in\mathbb{R}}$ is a C^0-group of unitary operators on X_Q.

Theorem

We denote by $\mathcal{L}$ the infinitesimal generator of the group $(\varphi(s))_{s\in\mathbb{R}}$. Then for any vector field $a \in X_Q$, we have the strong convergence in X_Q

$$\langle a \rangle := \lim_{T\to+\infty} \frac{1}{T} \int_r^{r+T} \partial_y Y(-s; Y(s; \cdot))a(Y(s; \cdot)) \, \mathrm{d}s = \mathrm{Proj}_{\ker \mathcal{L}} a$$

uniformly with respect to $r \in \mathbb{R}$.

Theorem (Long time behavior)

For any vector field $a \in X_Q$, we consider the problem

$$\begin{cases} \partial_t c - \mathcal{L}^2 c = 0, & t \in \mathbb{R}_+ \\ c(0,\cdot) = a(\cdot) \end{cases}$$

with $\mathcal{L}c = [b, c]$. Then the solution $c(t)$ converges weakly in X_Q, as $t \to +\infty$, toward the orthogonal projection on $\ker \mathcal{L}$

$$\lim_{t\to+\infty} c(t) = \mathrm{Proj}_{\ker \mathcal{L}} a, \text{ weakly in } X_Q.$$

Moreover, if the range of $\mathcal{L}$ is closed, then the previous convergence holds strongly in X_Q and has exponential rate.

Theorem (Convergence)

The family $(v^\varepsilon)_{\varepsilon>0}$ converges strongly in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$ to a weak solution $v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ of the transport problem

$$\begin{cases} \partial_t v + \langle a(t,\cdot)\rangle \cdot \nabla_z v = 0, & (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m \\ v(0,z) = u^{\mathrm{in}}(z), & z \in \mathbb{R}^m. \end{cases}$$

Moreover, if v is smooth enough, we have $v^\varepsilon = v + \mathcal{O}(\varepsilon)$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$, as $\varepsilon \searrow 0$.

Formal proof

$$\partial_t v^\varepsilon + \varphi(t/\varepsilon) a(t) \cdot \nabla_z v^\varepsilon = 0$$

$$v^\varepsilon(t,z) = v(t, s = t/\varepsilon, z) + \varepsilon v^1(t, s = t/\varepsilon, z) + \ldots$$

$$\partial_s v = 0, \;\; \partial_t v + \varphi(s) a(t) \cdot \nabla_z v + \partial_s v^1 = 0$$

$$\partial_t v + \left(\lim_{T \to +\infty} \frac{1}{T} \int_0^T \varphi(s) a(t) \, \mathrm{d}s \right) \cdot \nabla_z v = 0.$$

Rigorous proof

Lemma

Let $c \in L^\infty(\mathbb{R}_+; X_Q), d \in L^1(\mathbb{R}_+; X_P)$ such that

$$\exists\, \overline{c} := \lim_{T\to+\infty} \frac{1}{T}\int_r^{r+T} c(s)\,\mathrm{d}s \text{ strongly in } X_Q, \text{ uniformly w.r.t. } r \in \mathbb{R}.$$

Then we have

$$\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}_+} \langle d(t), c(t/\varepsilon)\rangle_{P,Q}\,\mathrm{d}t = \int_{\mathbb{R}_+} \langle d(t), \overline{c}\rangle_{P,Q}\,\mathrm{d}t$$

Application

$$\partial_t v^\varepsilon + \varphi(t/\varepsilon) a(t)\cdot\nabla_z v^\varepsilon = 0$$

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^m} v^\varepsilon(t,z)\varphi(t/\varepsilon)a(t)\cdot\nabla_z\xi(t,z)\,\mathrm{d}z\mathrm{d}t, \;\; \xi \in C_c^1(\mathbb{R}_+\times\mathbb{R}^m)$$

$$c(s) = \varphi(s)a, \;\; d(t) = v^\varepsilon(t,\cdot)\nabla_z\xi(t,\cdot), \;\; \lim_{T\to+\infty}\frac{1}{T}\int_r^{r+T} c(s)\,\mathrm{d}s = \langle a\rangle\,.$$

Non linear multi-scale problems

Vlasov-Poisson equations

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \frac{q}{m}\left(-\nabla_x \phi^\varepsilon + v \wedge \frac{B(t,x)}{\varepsilon}\right) \cdot \nabla_v f = 0$$

$$-\varepsilon_0 \Delta_x \phi^\varepsilon = \rho^\varepsilon := q \int f^\varepsilon(t,x,v)\,\mathrm{d}v$$

f^ε : particle presence density in the phase space

$f^\varepsilon(t,x,v)\mathrm{d}x\mathrm{d}v$: particle number inside the volume $\mathrm{d}x\mathrm{d}v$

$E^\varepsilon = -\nabla_x \phi^\varepsilon$: self consistent electric field

Main purposes

Homogenization, two scale convergence

Averaging with respect to the fast cyclotronic motion

Effective non linear Vlasov equation

Strong convergence results for any initial conditions (not necessarily well prepared)

Conservation laws, Hamiltonian structure

The finite Larmor radius regime

The reference time T is much larger than the cyclotronic period (strong magnetic field) *i.e.*,

$$T \frac{q|\mathbf{B}^\varepsilon|}{m} \approx \frac{1}{\varepsilon}, \text{ with } 0 < \varepsilon << 1.$$

The kinetic energy is much larger than the potential energy

$$\frac{m|V|^2}{q\phi} \approx \frac{1}{\varepsilon}$$

The Larmor radius is of the same order as the Debye length *i.e.*,

$$\lambda_D^2 = \frac{\varepsilon_0 \phi}{nq} \approx \rho_L^2.$$

Finite Larmor radius regime

strong uniform magnetic field $\mathbf{B}^\varepsilon = (0, 0, B/\varepsilon)$, perpendicular to x_1Ox_2.

$$x = (x_1, x_2), \quad v = (v_1, v_2), \quad {}^\perp v = (v_2, -v_1)$$

$\omega_c = \frac{qB}{m}$ is the rescaled cyclotronic frequency

$T_c = \frac{2\pi}{\omega_c}$ is the rescaled cyclotronic period

the real cyclotronic frequency is $\omega_c^\varepsilon = \omega_c/\varepsilon$

the real cyclotronic period is $T_c^\varepsilon = \frac{2\pi}{\omega_c^\varepsilon} = \varepsilon \frac{2\pi}{\omega_c} = \varepsilon T_c$

References

Frénod, Sonnendrücker, Golse, Han-Kwan, ...

Two dimensional setting

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v\cdot\nabla_x f^\varepsilon + \omega_c\, {}^\perp v\cdot\nabla_v f^\varepsilon) - \nabla_x\phi^\varepsilon\cdot\nabla_v f^\varepsilon = 0, (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2$$

$$-\Delta_x\phi^\varepsilon = \rho^\varepsilon(t,x) := \int_{\mathbb{R}^2} f^\varepsilon(t,x,v)\,\mathrm{d}v,\ \ (t,x)\in\mathbb{R}_+\times\mathbb{R}^2$$

$$f^\varepsilon(0,x,v) = f^{\mathrm{in}}(x,v),\ \ (x,v)\in\mathbb{R}^2\times\mathbb{R}^2$$

Main goal

What is the behavior of $(f^\varepsilon,\phi^\varepsilon)_{\varepsilon>0}$ when $\varepsilon \searrow 0$?

What is the limit Vlasov-Poisson system ?

The characteristic equations

The trajectories in (x, v) oscillate at the cyclotronic frequency

$$\frac{\mathrm{d}X^\varepsilon}{\mathrm{d}t} = \frac{V^\varepsilon(t)}{\varepsilon}, \quad \frac{\mathrm{d}V^\varepsilon}{\mathrm{d}t} = \frac{\omega_c}{\varepsilon} \ {}^\perp V^\varepsilon(t) - \nabla_x \phi^\varepsilon(t, X^\varepsilon(t))$$

But $\widetilde{X}^\varepsilon(t) = X^\varepsilon(t) + {}^\perp V^\varepsilon(t)/\omega_c$, $\widetilde{V}^\varepsilon(t) = \mathcal{R}\left(\omega_c t/\varepsilon\right) V^\varepsilon(t)$ are left invariant with respect to the cyclotronic dynamics

$$\frac{\mathrm{d}\widetilde{X}^\varepsilon}{\mathrm{d}t} = -\frac{{}^\perp\nabla_x\phi^\varepsilon(t, X^\varepsilon(t))}{\omega_c}, \quad \frac{\mathrm{d}\widetilde{V}^\varepsilon}{\mathrm{d}t} = -\mathcal{R}\left(\omega_c t/\varepsilon\right) \nabla_x \phi^\varepsilon(t, X^\varepsilon(t))$$

Main idea

stability for $(\widetilde{X}^\varepsilon, \widetilde{V}^\varepsilon)_{\varepsilon>0}$

$$\frac{\mathrm{d}\widetilde{X}}{\mathrm{d}t} = \mathcal{V}(t, \widetilde{X}(t), \widetilde{V}(t)), \quad \frac{\mathrm{d}\widetilde{V}}{\mathrm{d}t} = \mathcal{A}(t, \widetilde{X}(t), \widetilde{V}(t))$$

Change of phase space coordinates

Presence densities in the phase space $(\widetilde{x}, \widetilde{v})$

$$\tilde{f}^{\varepsilon}(t, \widetilde{x}, \widetilde{v}) = f^{\varepsilon}(t, x, v), \;\; \widetilde{x} = x + \frac{{}^{\perp}v}{\omega_c}, \;\; \widetilde{v} = \mathcal{R}\left(\omega_c t/\varepsilon\right) v$$

$$\partial_t \tilde{f}^{\varepsilon} - \frac{{}^{\perp}\nabla_x \phi^{\varepsilon}}{\omega_c} \cdot \nabla_{\widetilde{x}} \tilde{f}^{\varepsilon} - \mathcal{R}\left(\omega_c t/\varepsilon\right) \nabla_x \phi^{\varepsilon} \cdot \nabla_{\widetilde{v}} \tilde{f}^{\varepsilon} = 0, \; (t, \widetilde{x}, \widetilde{v}) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$$

$$\tilde{f}^{\varepsilon}(0, \widetilde{x}, \widetilde{v}) = f^{\mathrm{in}}\left(\widetilde{x} - \frac{{}^{\perp}\widetilde{v}}{\omega_c}, \widetilde{v}\right), \;\; (\widetilde{x}, \widetilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

We expect that $\lim_{\varepsilon \searrow 0} \tilde{f}^{\varepsilon} = \tilde{f}$ and

$$\partial_t \tilde{f} + \mathcal{V} \cdot \nabla_{\widetilde{x}} \tilde{f} + \mathcal{A} \cdot \nabla_{\widetilde{v}} \tilde{f} = 0, \;\; (t, \widetilde{x}, \widetilde{v}) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$$

The effective trajectories

$$e(z) = -\frac{1}{2\pi} \ln |z|, \quad z \in \mathbb{R}^2 \setminus \{0\}$$

The solution of the Poisson equation

$$\phi^\varepsilon(t,x) = \int_{\mathbb{R}^2} e(x-y)\rho^\varepsilon(t,y)\,\mathrm{d}y = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e(x-y) f^\varepsilon(t,y,w)\,\mathrm{d}w\mathrm{d}y$$

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{X}^\varepsilon}{\mathrm{d}t} &= -\frac{{}^\perp\nabla_x \phi^\varepsilon(t, X^\varepsilon(t))}{\omega_c} \\
&= -\frac{1}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}^\perp\nabla e\left(X^\varepsilon(t) - y\right) f^\varepsilon(t,y,w)\,\mathrm{d}w\mathrm{d}y \\
&= -\frac{1}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}^\perp\nabla e\left(\widetilde{X}^\varepsilon(t) - \widetilde{y} - \frac{1}{\omega_c}\mathcal{R}\left(-\omega_c t/\varepsilon\right){}^\perp(\widetilde{V}^\varepsilon(t) - \widetilde{w})\right) \tilde{f}^\varepsilon(t,\widetilde{y},\widetilde{w})
\end{aligned}$$

Average over a cyclotronic period

$$\frac{\widetilde{X}^\varepsilon(t+T_c^\varepsilon)-\widetilde{X}^\varepsilon(t)}{T_c^\varepsilon}$$

$$= -\frac{1}{\omega_c T_c^\varepsilon}\int_t^{t+T_c^\varepsilon}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}^\perp\nabla e\left(\widetilde{X}^\varepsilon(\tau)-\widetilde{y}-\mathcal{R}\left(-\frac{\omega_c\tau}{\varepsilon}\right)\frac{{}^\perp(\widetilde{V}^\varepsilon(\tau)-\widetilde{w})}{\omega_c}\right)$$

$$\times\ \tilde{f}^\varepsilon(\tau,\widetilde{y},\widetilde{w})\ \mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\mathrm{d}\tau$$

$$= -\frac{1}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{2\pi}\int_0^{2\pi} {}^\perp\nabla e\left(\widetilde{X}(t)-\widetilde{y}-\mathcal{R}(\theta)\frac{{}^\perp(\widetilde{V}(t)-\widetilde{w})}{\omega_c}\right)\mathrm{d}\theta\tilde{f}\ \mathrm{d}\tilde{w}\mathrm{d}\tilde{y}$$

$$= -\frac{{}^\perp\nabla_\xi}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{E}(\widetilde{X}(t)-\widetilde{y},\widetilde{V}(t)-\widetilde{w})\tilde{f}(t,\widetilde{y},\widetilde{w})\ \mathrm{d}\tilde{w}\mathrm{d}\tilde{y}+o(1),\ \ \varepsilon\searrow 0$$

$$\mathcal{E}(\xi,\eta)=\frac{1}{2\pi}\int_0^{2\pi} e\left(\xi-\omega_c^{-1}\mathcal{R}(\theta)^\perp\eta\right)\ \mathrm{d}\theta,\ \ (\xi,\eta)\in(\mathbb{R}^2\times\mathbb{R}^2)\setminus\{(0,0)\}$$

Effective velocity field

$$\frac{\mathrm{d}\widetilde{X}}{\mathrm{d}t} = \mathcal{V}[\tilde{f}(t)](\widetilde{X}(t), \widetilde{V}(t))$$

$$\mathcal{V}[\tilde{f}(t)](\widetilde{x}, \widetilde{v}) = -\frac{{}^\perp\nabla_\xi}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{E}(\widetilde{x}-\widetilde{y}, \widetilde{v}-\widetilde{w})\tilde{f}(t, \widetilde{y}, \widetilde{w})\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}, (\widetilde{x}, \widetilde{v}) \in \mathbb{R}^4$$

Effective acceleration field

$$\frac{\mathrm{d}\widetilde{V}}{\mathrm{d}t} = \mathcal{A}[\tilde{f}(t)](\widetilde{X}(t), \widetilde{V}(t))$$

$$\mathcal{A}[\tilde{f}(t)](\widetilde{x}, \widetilde{v}) = \omega_c\,{}^\perp\nabla_\eta\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{E}(\widetilde{x}-\widetilde{y}, \widetilde{v}-\widetilde{w})\tilde{f}(t, \widetilde{y}, \widetilde{w})\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}, (\widetilde{x}, \widetilde{v}) \in \mathbb{R}^4$$

Hamiltonian structure

$$\frac{\mathrm{d}\widetilde{X}}{\mathrm{d}t} = \mathcal{V}[\tilde{f}(t)](\widetilde{X}(t), \widetilde{V}(t)), \quad \frac{\mathrm{d}\widetilde{V}}{\mathrm{d}t} = \mathcal{A}[\tilde{f}(t)](\widetilde{X}(t), \widetilde{V}(t))$$

$$\mathcal{V}[\tilde{f}(t)](\widetilde{x}, \widetilde{v}) = -\omega_c^{-1}\, {}^{\perp}\nabla_{\widetilde{x}}\tilde{\phi}[\tilde{f}(t)], \quad \mathcal{A}[\tilde{f}(t)](\widetilde{x}, \widetilde{v}) = \omega_c\, {}^{\perp}\nabla_{\widetilde{v}}\tilde{\phi}[\tilde{f}(t)]$$

$$\tilde{\phi}[\tilde{f}(t)](\widetilde{x}, \widetilde{v}) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{E}(\widetilde{x} - \widetilde{y}, \widetilde{v} - \widetilde{w})\tilde{f}(t, \widetilde{y}, \widetilde{w})\, \mathrm{d}\tilde{w}\mathrm{d}\tilde{y}$$

This characteristic system is Hamiltonian, with respect to the conjugate variables $(\widetilde{x}_2, \omega_c^{-1}\widetilde{v}_1)$ and $(\omega_c\widetilde{x}_1, \widetilde{v}_2)$ and the Hamiltonian function $\tilde{\phi}[\tilde{f}]$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\, {}^{t}(\widetilde{X}_2, \omega_c^{-1}\widetilde{V}_1) = \nabla_{\omega_c\widetilde{x}_1, \widetilde{v}_2}\, \tilde{\phi}[\tilde{f}(t)](\widetilde{X}(t), \widetilde{V}(t))$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, {}^{t}(\omega_c\widetilde{X}_1, \widetilde{V}_2) = -\nabla_{\widetilde{x}_2, \omega_c^{-1}\widetilde{v}_1}\tilde{\phi}[\tilde{f}(t)](\widetilde{X}(t), \widetilde{V}(t)).$$

The function $\mathcal{E}$

$\mathcal{E}(\xi, \eta)$ is the average of the fundamental solution $e(\cdot)$ over the circle of center ξ and radius $|\eta|/|\omega_c|$

If $|\xi| > |\eta|/|\omega_c|$, the function $z \to e(z)$ is harmonic in the open set $\mathbb{R}^2 \setminus \{0\}$, which contains the disc $\{z \in \mathbb{R}^2 : |z - \xi| \leq |\eta|/|\omega_c|\}$ and thus, the mean property applied to the function $e(\cdot)$ and the circle of center ξ and radius $|\eta|/|\omega_c|$ yields

$$\mathcal{E}(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} e\left(\xi - \frac{\mathcal{R}(\theta)}{\omega_c} {}^\perp\eta\right) \, \mathrm{d}\theta = e(\xi) = -\frac{1}{2\pi} \ln |\xi|, \quad |\xi| > \frac{|\eta|}{|\omega_c|}.$$

The function $\mathcal{E}$

The function $\mathcal{E}$ has also the symmetry property $\mathcal{E}(\omega_c^{-1}\eta, \omega_c\xi) = \mathcal{E}(\xi,\eta)$ for any $(\xi,\eta) \in (\mathbb{R}^2 \times \mathbb{R}^2) \setminus \{(0,0)\}$.
If $|\xi| < |\eta|/|\omega_c|$, then

$$\mathcal{E}(\xi,\eta) = \mathcal{E}\left(\frac{\eta}{\omega_c}, \omega_c\xi\right) = e\left(\frac{\eta}{\omega_c}\right)$$

Finally

$$\mathcal{E}(\xi,\eta) = e\left(\frac{\eta}{\omega_c}\right)\mathbf{1}_{\{|\xi|\le|\eta|/|\omega_c|\}} + e(\xi)\mathbf{1}_{\{|\xi|>|\eta|/|\omega_c|\}}.$$

First order derivatives of $\mathcal{E}$

$$\nabla_\xi \mathcal{E}(\xi,\eta) = \nabla e(\xi) \mathbf{1}_{\{|\xi| > |\eta|/|\omega_c|\}}$$

$$\nabla_\eta \mathcal{E}(\xi,\eta) = \omega_c^{-1} \nabla e\left(\frac{\eta}{\omega_c}\right) \mathbf{1}_{\{|\xi| \leq |\eta|/|\omega_c|\}}$$

Second order derivatives of $\mathcal{E}$

$$\Delta_\xi \mathcal{E} = -\frac{\mathbf{1}_{\{|\xi|=|\eta|/|\omega_c|\}} \mathrm{d}\sigma(\xi,\eta)}{2\pi|\xi|\sqrt{1+\omega_c^{-2}}}, \quad \Delta_\eta \mathcal{E} = -\frac{\mathbf{1}_{\{|\xi|=|\eta|/|\omega_c|\}} \mathrm{d}\sigma(\xi,\eta)}{2\pi|\eta|\sqrt{1+\omega_c^{2}}}.$$

The velocity and acceleration fields associated to any density $\tilde{f} = \tilde{f}(\widetilde{x}, \widetilde{v})$ write

$$\mathcal{V}[\tilde{f}](\widetilde{x}, \widetilde{v}) = \frac{1}{2\pi\omega_c} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{^\perp(\widetilde{x} - \widetilde{y})}{|\widetilde{x} - \widetilde{y}|^2} \tilde{f}(\widetilde{y}, \widetilde{w}) \mathbf{1}_{\{|\widetilde{x} - \widetilde{y}| > \frac{|\widetilde{v} - \widetilde{w}|}{|\omega_c|}\}} \,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}$$

$$\mathcal{A}[\tilde{f}](\widetilde{x}, \widetilde{v}) = -\frac{\omega_c}{2\pi} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{^\perp(\widetilde{v} - \widetilde{w})}{|\widetilde{v} - \widetilde{w}|^2} \tilde{f}(\widetilde{y}, \widetilde{w}) \mathbf{1}_{\{|\widetilde{x} - \widetilde{y}| \leq \frac{|\widetilde{v} - \widetilde{w}|}{|\omega_c|}\}} \,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}.$$

Non linear limit model

$$\partial_t \tilde{f} + \mathcal{V}[\tilde{f}](\widetilde{x}, \widetilde{v}) \cdot \nabla_{\widetilde{x}} \tilde{f} + \mathcal{A}[\tilde{f}](\widetilde{x}, \widetilde{v}) \cdot \nabla_{\widetilde{v}} \tilde{f} = 0.$$

Convergence results

$f^{\text{in}} = f^{\text{in}}(x,v)$ non negative presence density satisfying

H1 $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^{\text{in}}(x,v)\,\mathrm{d}v\mathrm{d}x < +\infty$

H2 $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|v|^2}{2} f^{\text{in}}(x,v)\,\mathrm{d}v\mathrm{d}x < +\infty$

H3 there is a bounded, non increasing function $F^{\text{in}} = F^{\text{in}}(r) \in L^\infty \cap L^1(\mathbb{R}_+; r\mathrm{d}r)$, such that $f^{\text{in}}(x,v) \le F^{\text{in}}(|v|),\ \ (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$.

Let $(f^\varepsilon, \phi^\varepsilon)_{\varepsilon>0}$ be the weak solutions for the Vlasov-Poisson system

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v\cdot\nabla_x f^\varepsilon + \omega_c\, {}^\perp v\cdot\nabla_v f^\varepsilon) - \nabla_x\phi^\varepsilon\cdot\nabla_v f^\varepsilon = 0,\ \ (t,x,v) \in \mathbb{R}_+\times\mathbb{R}^4$$

$$-\Delta_x\phi^\varepsilon = \rho^\varepsilon(t,x) := \int_{\mathbb{R}^2} f^\varepsilon(t,x,v)\,\mathrm{d}v,\ \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$$

$$f^\varepsilon(0,x,v) = f^{\text{in}}(x,v),\ \ (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$$

Theorem (2D)

Let $(\tilde{f}^\varepsilon)_{\varepsilon>0}$ be the densities

$$\tilde{f}^\varepsilon(t,\widetilde{x},\widetilde{v}) = f^\varepsilon\left(t, \widetilde{x} - \frac{\mathcal{R}\left(-\omega_c t/\varepsilon\right)}{\omega_c}\,{}^\perp\widetilde{v}, \mathcal{R}\left(-\omega_c t/\varepsilon\right)\widetilde{v}\right)$$

Then $\exists\ (\varepsilon_k)_k \searrow 0$ s. t. $(\tilde{f}^{\varepsilon_k})_k$ converges strongly in $L^2([0,T];L^2(\mathbb{R}^2\times\mathbb{R}^2))$, for any $T\in\mathbb{R}_+$, to $\tilde{f}$

$$\partial_t\tilde{f}+\mathcal{V}[\tilde{f}(t)](\widetilde{x},\widetilde{v})\cdot\nabla_{\widetilde{x}}\tilde{f}+\mathcal{A}[\tilde{f}(t)](\widetilde{x},\widetilde{v})\cdot\nabla_{\widetilde{v}}\tilde{f} = 0,\ \ (t,\widetilde{x},\widetilde{v})\in\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2$$

with the initial condition

$$\tilde{f}(0,\widetilde{x},\widetilde{v}) = f^{\mathrm{in}}\left(\widetilde{x} - \frac{{}^\perp\widetilde{v}}{\omega_c},\widetilde{v}\right),\ \ (\widetilde{x},\widetilde{v})\in\mathbb{R}^2\times\mathbb{R}^2$$

where the velocity and acceleration vector fields $\mathcal{V},\mathcal{A}$ are given by

$$\mathcal{V}[\tilde{f}(t)](\widetilde{x},\widetilde{v}) = -\omega_c^{-1}\,{}^\perp\nabla_{\widetilde{x}}\tilde{\phi}[\tilde{f}(t)],\quad \mathcal{A}[\tilde{f}(t)](\widetilde{x},\widetilde{v}) = \omega_c\,{}^\perp\nabla_{\widetilde{v}}\tilde{\phi}[\tilde{f}(t)]$$

$$\tilde{\phi} = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left\{\ln\frac{|\widetilde{v}-\widetilde{w}|}{|\omega_c|}\,\mathbf{1}_{\{|\widetilde{x}-\widetilde{y}|\le\frac{|\widetilde{v}-\widetilde{w}|}{|\omega_c|}\}} + \ln|\widetilde{x}-\widetilde{y}|\,\mathbf{1}_{\{|\widetilde{x}-\widetilde{y}|>\frac{|\widetilde{v}-\widetilde{w}|}{|\omega_c|}\}}\right\}\tilde{f}$$

Lemma

Let $U = U(z,t,s) : \mathcal{O} \times \mathbb{R}_+ \times \mathbb{R}_s \to \mathbb{R}$ be a function in $L^1(\mathcal{O} \times \mathbb{R}_+; C_\#(\mathbb{R}_s))$, where $\mathcal{O}$ is an open set of $\mathbb{R}^N$ and $C_\#(\mathbb{R}_s)$ stands for the set of continuous periodic functions of (fixed) period $L > 0$. Then we have the convergence

$$\lim_{\varepsilon \searrow 0} \int_{\mathcal{O}} \int_{\mathbb{R}_+} |U(z,t,t/\varepsilon)| \, \mathrm{d}t\mathrm{d}z = \frac{1}{L} \int_{\mathcal{O}} \int_{\mathbb{R}_+} \int_0^L |U(z,t,s)| \, \mathrm{d}s\mathrm{d}t\mathrm{d}z.$$

Conservations

1. Let $\tilde{f} = \tilde{f}(t, \widetilde{x}, \widetilde{v})$ be a solution of the limit problem such that $1, \widetilde{x}, \widetilde{v}, |\widetilde{x}|^2, |\widetilde{v}|^2$ are integrable functions with respect to $\tilde{f}(0, \widetilde{x}, \widetilde{v})\mathrm{d}\widetilde{v}\mathrm{d}\widetilde{x} = f^{\mathrm{in}}(\widetilde{x} - \omega_c^{-1}{}^{\perp}\widetilde{v}, \widetilde{v})\mathrm{d}\widetilde{v}\mathrm{d}\widetilde{x}$. For any $t \in \mathbb{R}_+$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \{1, \widetilde{x}, \widetilde{v}, |\widetilde{x}|^2, |\widetilde{v}|^2\}\tilde{f}(t, \widetilde{x}, \widetilde{v}) \,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = 0$$

2. Let $\tilde{f} = \tilde{f}(t, \widetilde{x}, \widetilde{v})$ be a solution of the limit problem such that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{E}(\widetilde{x}-\widetilde{y}, \widetilde{v}-\widetilde{w})\tilde{f}(0, \widetilde{y}, \widetilde{w})\tilde{f}(0, \widetilde{x}, \widetilde{v}) \,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\, \mathrm{d}\tilde{v}\mathrm{d}\tilde{x} < +\infty.$$

The electric energy is preserved in time

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{\phi}[\tilde{f}(t)](\widetilde{x}, \widetilde{v})\tilde{f}(t, \widetilde{x}, \widetilde{v}) \,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{\phi}[\tilde{f}(t)]\partial_t\tilde{f} \,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = 0.$$

Lemma

Let $\tilde{f} = \tilde{f}(t, \widetilde{x}, \widetilde{v})$ be a solution of the limit model and $\psi = \psi(\widetilde{x}, \widetilde{v})$ be a smooth integrable function with respect to $\tilde{f}(0, \widetilde{x}, \widetilde{v})\mathrm{d}\widetilde{v}\mathrm{d}\widetilde{x} = f^{\mathrm{in}}(\widetilde{x} - \omega_c^{-1\perp}\widetilde{v}, \widetilde{v})\mathrm{d}\widetilde{v}\mathrm{d}\widetilde{x}$. For any $t \in \mathbb{R}_+$ we have

$$2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \psi(\widetilde{x}, \widetilde{v})\tilde{f}(t, \widetilde{x}, \widetilde{v})\,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\tilde{f}(t, \widetilde{y}, \widetilde{w})\tilde{f}(t, \widetilde{x}, \widetilde{v})$$
$$\times \left[\frac{1}{\omega_c}\left(\nabla_{\widetilde{y}}\psi(\widetilde{y}, \widetilde{w}) - \nabla_{\widetilde{x}}\psi(\widetilde{x}, \widetilde{v})\right) \cdot {}^\perp\nabla e(\widetilde{x} - \widetilde{y})\,\mathbf{1}_{\{|\widetilde{x}-\widetilde{y}| > |\widetilde{v}-\widetilde{w}|/|\omega_c|\}}\right.$$
$$\left. + \left(\nabla_{\widetilde{v}}\psi(\widetilde{x}, \widetilde{v}) - \nabla_{\widetilde{w}}\psi(\widetilde{y}, \widetilde{w})\right) \cdot {}^\perp\nabla e\left(\frac{\widetilde{v} - \widetilde{w}}{\omega_c}\right)\mathbf{1}_{\{|\widetilde{x}-\widetilde{y}| < |\widetilde{v}-\widetilde{w}|/|\omega_c|\}}\right]$$

Averaged Poisson equation

$$\widetilde{\rho}_s(t,\widetilde{x},\widetilde{v}) = \int_{\mathbb{R}^2} \tilde{f}(t,\widetilde{x} - \omega_c^{-1}\mathcal{R}(-\omega_c s)\,^{\perp}(\widetilde{v} - \widetilde{w}), \widetilde{w})\,\mathrm{d}\tilde{w}$$

We introduce the average charge density

$$\widetilde{\rho}(t,\widetilde{x},\widetilde{v}) = \frac{1}{T_c}\int_0^{T_c} \widetilde{\rho}_s(t,\widetilde{x},\widetilde{v})\,\mathrm{d}s, \quad (t,\widetilde{x},\widetilde{v}) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$$

The definition of the function $\mathcal{E}$ implies

$$\tilde{\phi}[\tilde{f}(t)](\widetilde{x},\widetilde{v}) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{E}(\widetilde{x} - \widetilde{y}, \widetilde{v} - \widetilde{w})\tilde{f}(t,\widetilde{y},\widetilde{w})\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}$$

$$= \frac{1}{T_c}\int_0^{T_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e(\widetilde{x} - \widetilde{y} - \omega_c^{-1}\mathcal{R}(-\omega_c s)\,^{\perp}(\widetilde{v} - \widetilde{w}))\tilde{f}(t,\widetilde{y},\widetilde{w})\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\mathrm{d}s$$

$$= \frac{1}{T_c}\int_0^{T_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e(\widetilde{x} - \tilde{z})\tilde{f}(t,\tilde{z} - \omega_c^{-1}\mathcal{R}(-\omega_c s)\,^{\perp}(\widetilde{v} - \widetilde{w}), \widetilde{w})\mathrm{d}\widetilde{w}\mathrm{d}\tilde{z}\mathrm{d}s$$

$$= \frac{1}{T_c}\int_0^{T_c}\int_{\mathbb{R}^2} e(\widetilde{x} - \tilde{z})\widetilde{\rho}_s(t,\tilde{z},\widetilde{v})\,\mathrm{d}\tilde{z}\mathrm{d}s = \int_{\mathbb{R}^2} e(\widetilde{x} - \tilde{z})\widetilde{\rho}(t,\tilde{z},\widetilde{v})\,\mathrm{d}\tilde{z} = (e \star \widetilde{\rho}(t,\cdot,\widetilde{v}))$$

2. Strongly anisotropic parabolic problems

$$\partial_t u^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u^\varepsilon) - \frac{1}{\varepsilon}\mathrm{div}_y(b(y)\otimes b(y)\nabla_y u^\varepsilon) = 0, \quad (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m \tag{1}$$

$$u^\varepsilon(0,y) = u^{\mathrm{in}}(y), \quad y \in \mathbb{R}^m \tag{2}$$

Limit model ($\varepsilon \searrow 0$)

$$\partial_t u - \mathrm{div}_y(\langle D \rangle \nabla_y u) = 0, \quad (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m$$

$$u(0,\cdot) = \left\langle u^{\mathrm{in}} \right\rangle$$

$$\langle D \rangle = \lim_{S\to+\infty} \frac{1}{S} \int_0^S \partial Y(-s; Y(s;\cdot)) D(Y(s;\cdot))\, {}^t\partial Y(-s; Y(s;\cdot))\,\mathrm{d}s.$$

Parabolic equations with stiff transport

$$\begin{cases} \partial_t u^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u^\varepsilon) + \dfrac{1}{\varepsilon}\, b(y)\cdot\nabla_y u^\varepsilon = 0, & (t,y)\in\mathbb{R}_+\times\mathbb{R}^m \\ u^\varepsilon(0,y) = u^{\mathrm{in}}(y), & y\in\mathbb{R}^m. \end{cases}$$

Change of variable

$$y = Y(t/\varepsilon; z),\ \ v^\varepsilon(t,z) = u^\varepsilon(t, Y(t/\varepsilon; z)),\ \ (t,z)\in\mathbb{R}_+\times\mathbb{R}^m,\ \ \varepsilon > 0$$

$$\mathrm{div}_y\{D\nabla_y(w\circ Y(-s,\cdot))\} = (\mathrm{div}_z(G(s)D\,\nabla_z w))\circ Y(-s,\cdot)$$

$$(G(s)D)(z) = \partial Y(-s; Y(s;z))\, D(Y(s;z))\, {}^t\partial Y(-s; Y(s;z))$$

$$\begin{cases} \partial_t v^\varepsilon - \mathrm{div}_z\left((G(t/\varepsilon)D)\nabla_z v^\varepsilon\right) = 0, & (t,z)\in\mathbb{R}_+\times\mathbb{R}^m \\ v^\varepsilon(0,z) = u^\varepsilon(0,z) = u^{\mathrm{in}}(z), & z\in\mathbb{R}^m,\ \varepsilon > 0 \end{cases}$$

Average matrix field

$$H_Q = \left\{ A(y) \ : \ \int_{\mathbb{R}^m} Q(y)A(y) : A(y)Q(y) \, \mathrm{d}y < +\infty \right\}$$

$$(\cdot,\cdot)_Q : H_Q \times H_Q \to \mathbb{R}, \ \ (A,B)_Q = \int_{\mathbb{R}^m} Q(y)A(y) : B(y)Q(y) \, \mathrm{d}y.$$

$$G(s) : H_Q \to H_Q, \ \ G(s)A = \partial_y Y^{-1}(s;\cdot)A(Y(s;\cdot)) \ {}^t\partial_y Y^{-1}(s;\cdot), s \in \mathbb{R}$$

Proposition $(G(s))_{s\in\mathbb{R}}$ is a C^0-group of unitary operators on H_Q.

Theorem

Let L be the infinitesimal generator of the group $(G(s))_{s\in\mathbb{R}}$. For any matrix field $A \in H_Q$ we have the strong convergence in H_Q

$$\langle A \rangle := \lim_{T\to+\infty} \frac{1}{T} \int_r^{r+T} \partial_y Y^{-1}(s;\cdot)A(Y(s;\cdot)) \ {}^t\partial_y Y^{-1}(s;\cdot) \, \mathrm{d}s = \mathrm{Proj}_{\ker L} A$$

uniformly with respect to $r \in \mathbb{R}$.

Theorem (Long time behavior)

For any matrix field $A \in H_Q$ we consider the problem

$$\begin{cases} \partial_t C - L^2 C = 0, & t \in \mathbb{R}_+ \\ C(0,\cdot) = A(\cdot) \end{cases}$$

where $L(C) = [b, C] = (b \cdot \nabla_y)C - \partial_y b\, C - C\, {}^t\partial_y b$. Then $C(t)$ converges weakly in H_Q, as $t \to +\infty$, toward the orthogonal projection on $\ker L$

$$\lim_{t \to +\infty} C(t) = \mathrm{Proj}_{\ker L} A, \quad \text{weakly in } H_Q.$$

Moreover, if the range of L is closed, then the previous convergence holds strongly in H_Q, and has exponential rate.

3. Strongly anisotropic wave equation

$$\partial_t^2 u^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u^\varepsilon) - \frac{1}{\varepsilon^2}\mathrm{div}_y(b(y)\otimes b(y)\nabla_y u^\varepsilon) = 0, (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m$$

$$u^\varepsilon(0,\cdot) = u^{\mathrm{in}}(\cdot), \;\; \partial_t u^\varepsilon(0,\cdot) = \dot{u}^{in}(\cdot)$$

- The limit model is a wave equation associated to the matrix field $\langle D \rangle$
- Strong convergences for well prepared initial conditions

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^m} \left(\frac{b \cdot \nabla_y u^\varepsilon(0,y)}{\varepsilon} \right)^2 \, \mathrm{d}y = 0$$

Finite speed propagation

Spectral analysis of $\langle D \rangle$

$$\left\langle \min_{\lambda \in \mathrm{Sp} Q^{1/2} D Q^{1/2}} \lambda \right\rangle \leq \min_{\lambda' \in \mathrm{Sp} Q^{1/2} \langle D \rangle Q^{1/2}} \lambda'$$

$$\max_{\lambda' \in \mathrm{Sp} Q^{1/2} \langle D \rangle Q^{1/2}} \lambda' \leq \left\langle \max_{\lambda \in \mathrm{Sp} Q^{1/2} D Q^{1/2}} \lambda \right\rangle$$

The limit solution propagates with speed c such that

$$c^2 \geq \max \max_{\lambda' \in \mathrm{Sp} Q^{1/2} \langle D \rangle Q^{1/2}} \lambda'$$

The limit solution propagates with speed

$$c \leq \left\langle \max_{\lambda \in \mathrm{Sp} Q^{1/2} D Q^{1/2}} \lambda \right\rangle^{1/2}$$

The Maxwell equations

$$\partial_t D - \mathrm{rot} H = 0, \ \ \partial_t B + \mathrm{rot} E = 0, \ \ \mathrm{div} D = 0, \ \ \mathrm{div} B = 0$$

$$D = \epsilon_0 \epsilon_r E, \ \ B = \mu_0 H$$

Energy balance

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{D \cdot E + B \cdot H\} + \mathrm{div}(E \wedge H) = 0.$$

Strongly anisotropic electric permittivity

$$\epsilon_r = \mathrm{diag}(n_1^2, n_2^2, n_3^2)$$

n_i = indice propre du milieux

$$\epsilon_r^{-1/2} = M + \frac{b \otimes b}{\varepsilon}$$

Maxwell equations

$$\partial_t D^\varepsilon - \mathrm{rot}\frac{B^\varepsilon}{\mu_0} = 0, \;\; \partial_t B^\varepsilon + \mathrm{rot}\left[\frac{1}{\epsilon_0}\left(M + \frac{b \otimes b}{\varepsilon}\right)^2 D^\varepsilon\right] = 0$$

$$\mathrm{div} B^\varepsilon = 0, \;\; \mathrm{div} D^\varepsilon = 0.$$

The case TM

$$\mathbf{B} = (0, 0, B), \quad \mathbf{D} = (D_1, D_2, 0)$$

$$M_\varepsilon = M + \frac{b \otimes b}{\varepsilon}, \quad M = \lambda\, {}^\perp b \otimes {}^\perp b$$

The limit wave equation for B

$$\partial_t^2 B - c_0^2 \mathrm{div} \left(\frac{\langle \lambda^2 |\xi|^4 \rangle}{|\xi|^4} (\xi \otimes \xi) \nabla B \right) = 0$$

$$\xi = |b| b, \quad \langle \cdot \rangle = \text{ the average operator along the flow of } {}^\perp \xi.$$

Perspectives

1. limit models with non uniform magnetic fields (magnetic confinement)
2. two scale convergence with non periodic fast variable
3. replace periodicity by ergodicity
4. gyrokinetic collisional models, averaged collision operators, invariants, equilibria (work in progress)
5. wave equation, Maxwell equations (general case)
6. spectral analysis of the average matrix field.