The MHM Framework

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Setting

Linear PDE

Find u satisfying

$$\mathcal{L}u = f, \quad \text{in } \Omega \subset \mathbb{R}^d$$

Boundary conditions

- Herein assume homogeneous essential boundary conditions.
- Natural boundary conditions can also be considered.

Some applications

- 2nd-order Elliptic
- Elasticity
- Reaction-Advection-Diffusion

Solutions and Their Approximation

Weak Form

Find $u \in V$ such that

$$a(u, v) = f(v), \quad \forall v \in V$$

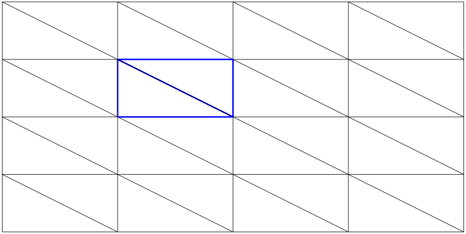

FIGURE: Triangulation of the domain.

Solutions and Their Approximation

Weak Form

Find $u \in W$ *such that*

$$a(u, v) = f(v), \quad \forall\, v \in W$$

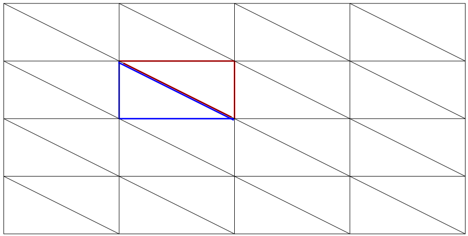

Figure: Triangulation of the domain.

Problem Statement

Laplace Problem

Find u satisfying

$$\begin{aligned} -\nabla\cdot\mathcal{K}\nabla u &= f, \quad \text{in } \Omega \subset \mathbb{R}^d \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

Weak Form

Find $u \in H_0^1(\Omega)$ *such that*

$$(\mathcal{K}\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \forall\, v \in H_0^1(\Omega)$$

Hybrid Formulation

Weak Form

Find $u \in \Pi_K H^1(K)$ *(and* $\lambda \in M$*) such that*

$$\sum_K (\mathcal{K}\nabla u, \nabla v)_K + \sum_K (\lambda, v)_{\partial K} = \sum_K (f, v)_K \quad \forall v \in \Pi_K H^1(K)$$

$$\sum_K (\mu, u)_{\partial K} = 0, \quad \forall \mu \in M$$

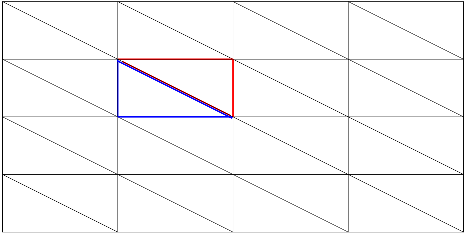

Figure: Triangulation of the domain.

Partial Approximation

Weak Form

Find $u_s \in W_s$ (and $\lambda \in M$) such that

$$\sum_K (\mathcal{K}\nabla u_s, \nabla v)_K + \sum_K (\lambda, v)_{\partial K} = \sum_K (f, v)_K \quad \forall\, v \in W_s$$

$$\sum_K (\mu, u)_{\partial K} = 0, \quad \forall\, \mu \in M$$

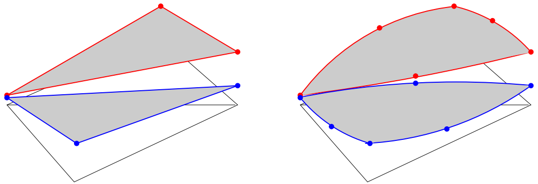

Figure: Linear space (left) and quadratic space (right).

PARTIAL APPROXIMATION

WEAK FORM

Find $u_s \in W_s$ (and $\lambda \in M$) such that

$$\sum_K (\mathcal{K}\nabla u_s, \nabla v)_K + \sum_K (\lambda, v)_{\partial K} = \sum_K (f, v)_K \quad \forall\, v \in W_s$$

$$\sum_K (\mu, u)_{\partial K} = 0, \quad \forall\, \mu \in M$$

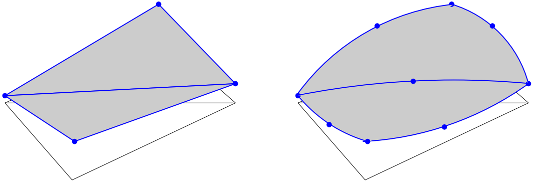

FIGURE: Solution belongs to continuous spaces.

Full Approximation

Weak Form

Find $u_s \in W_s$ (and $\lambda_s \in M_s$) such that

$$\sum_K (\mathcal{K}\nabla u_s, \nabla v)_K + \sum_K (\lambda_s, v)_{\partial K} = \sum_K (f, v)_K \quad \forall\, v \in W_s$$

$$\sum_K (\mu, u_s)_{\partial K} = 0, \quad \forall\, \mu \in M_s$$

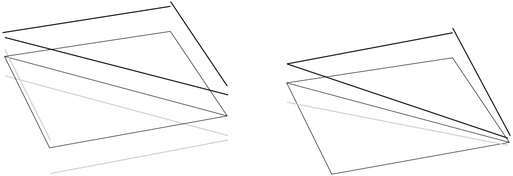

Figure: Constant and linear spaces on the boundaries.

Assumptions

- K are all triangles.
- $\mathcal{K}$ is the identity.
- f is piecewise constant.
- W_s is piecewise quadratic.
- M_s is piecewise constant.

Weak Form

Find $u \in W_s$ (and $\lambda \in M_s$) such that

$$\sum_K (\nabla u, \nabla v)_K + \sum_K (\lambda, v)_{\partial K} = \sum_K (f, v)_K \quad \forall\, v \in W_s$$

$$\sum_K (\mu, u)_{\partial K} = 0, \quad \forall\, \mu \in M_s$$

Localization

Local Statement

Find $u_s \in W_s$ *(depending on* f *and* $\lambda_s \in M_s$*) such that*

$$\underbrace{(\nabla u_s, \nabla v)_K = (f, v)_K - (\lambda_s, v)_{\partial K} \quad \forall\, v \in W_s}_{\textit{ill posed in each } K}$$

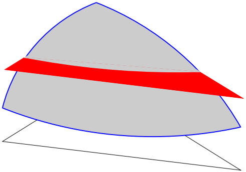

Figure: W_s

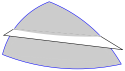

Figure: $W_s^{\perp}$

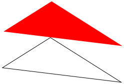

Figure: W_0

Localization

Local Statement

Find $u_\perp \in W_s^\perp$ *(depending on* $\lambda_s \in M_s$*) such that*

$$\underbrace{(\nabla u_\perp, \nabla v)_K = -(\lambda_s, v)_{\partial K} \quad \forall\, v \in W_s^\perp}_{\text{well posed in each } K}$$

Nice properties

- On each K, $u_\perp$ is the solution to the infinite-dimensional problem

$$-\triangle u_\perp = C_K$$
$$\nabla u_\perp \cdot \mathbf{n} = -\lambda_s$$

- $\sigma_u = \nabla u_\perp$ expands using the lowest-order Raviart-Thomas basis!!!

Mixed Form

Weak Form

Find $u \in W_s$ *(and* $\lambda \in M_s$*) such that*

$$\sum_K (\nabla u_s, \nabla v)_K + \sum_K (\lambda_s, v)_{\partial K} = \sum_K (f, v)_K \quad \forall\, v \in W_s$$

$$\sum_K (\mu, u_s)_{\partial K} = 0, \quad \forall\, \mu \in M_s$$

Reduced

Find $u_0 \in W_0$ *(and* $\lambda_s \in M_s$*) such that*

$$\sum_K (\lambda_s, v_0)_{\partial K} = \sum_K (f, v)_K \quad \forall\, v \in W_s$$

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Mixed Form

Weak Form

Find $u_s \in W_s$ *(and* $\lambda_s \in M_s$*) such that*

$$\sum_K (\nabla u_s, \nabla v)_K + \sum_K (\lambda_s, v)_{\partial K} = \sum_K (f, v)_K \quad \forall\, v \in W_s$$

$$\sum_K (\mu, u_s)_{\partial K} = 0, \quad \forall\, \mu \in M_s$$

Reduced, in Mixed Form

Find $u_0 \in W_0$ *(and* $\lambda_s \in M_s$*) such that*

$$\sum_K (\nabla \cdot \sigma_u, v_0)_K = \sum_K (f, v_0)_K \quad \forall\, v \in W_s$$

$$\sum_K (\sigma_u, \sigma_v)_K + \sum_K (\nabla \cdot \sigma_v, u_0)_K = 0, \quad \forall\, \mu \in M_s$$

A Posteriori Estimator

Reduced

Find $u_0 \in W_0$ *(and* $\lambda \in M_s$*) such that*

$$\sum_K (\lambda, v_0)_{\partial K} = \sum_K (f, v)_K \quad \forall\, v \in W_s$$

$$\sum_K (\mu, u_\perp + u_0)_{\partial K} = 0, \quad \forall\, \mu \in M_s$$

Reduced

Find $u \in W_s$ *(and* $\lambda \in M_s$*) such that*

$$\sum_F (\lambda \mathbf{n}, [\![v_0]\!])_F = \sum_K (f, v)_K \quad \forall\, v \in W_s$$

$$\sum_F (\mu \mathbf{n}, [\![u_\perp + u_0]\!])_F = 0, \quad \forall\, \mu \in M_s$$

A Posteriori Estimator

$$R_F := \begin{cases} -\frac{1}{2}[\![u_\perp + u_0]\!], & F \text{ is an interior edge;} \\ (-u_\perp + u_0)\boldsymbol{n}_\Omega, & F \text{ is on the boundary.} \end{cases}$$

The a posteriori estimator is given by

$$\eta := [\sum_K \eta_K^2]^{1/2},$$

$$\eta_K^2 := \sum_F \eta_F^2,$$

$$\eta_F := \frac{C}{h_F^{1/2}} \| R_F \|_{0,F}^2.$$

Summary

Under the assumptions:

- K are all triangles.
- $\mathcal{K}$ is the identity.
- f is piecewise constant.
- W_s is piecewise quadratic.
- M_s is piecewise constant.

Properties

- Localization with exact solution
- Approximation of variables for mixed form
- A posteriori estimator in terms of jumps

Summary

Under the assumptions:

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Summary

Under the assumptions:

- ~~K are all triangles.~~
- ~~$\mathcal{K}$ is the identity.~~
- ~~f is piecewise constant.~~
- ~~W_s is piecewise quadratic.~~
- ~~M_s is piecewise constant.~~

Properties of the Multiscale Hybrid-Mixed Method

- Localization ~~with exact solution~~
- Approximation of variables for mixed form
- A posteriori estimator in terms of jumps

LOCALIZATION

LOCAL STATEMENT

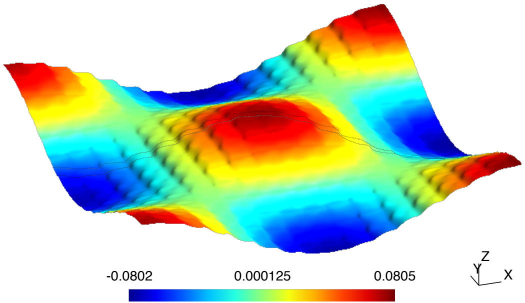
Find $u_\perp$ (depending on f and λ_s) such that

$$(\mathcal{K}\nabla u_\perp, \nabla v_\perp)_K = (f, v_\perp)_K - (\lambda_s, v_\perp)_{\partial K} \quad \forall\, v_\perp$$

- Write $u_\perp = u_\perp^\lambda + u_\perp^f$

OSCILLATORY COEFFICIENT

- Unit square domain
- $\mathcal{K} = \frac{2+1.8\sin\frac{2\pi x}{\varepsilon}}{2+1.8\sin\frac{2\pi y}{\varepsilon}} + \frac{2+1.8\sin\frac{2\pi y}{\varepsilon}}{2+1.8\cos\frac{2\pi x}{\varepsilon}}$, $\varepsilon = \frac{1}{16}$
- Homogeneous Neumann boundary conditions
- $f(x, y) = 2\pi^2 \cos(2\pi x)\cos(2\pi y)$
- Let M_s be M_0 or M_2.



Comparison with Lowest-order RT

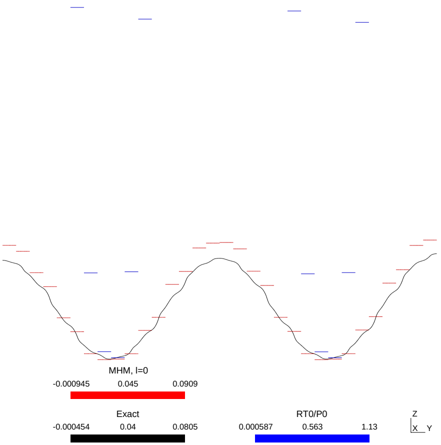


FIGURE: Comparing lowest-order Raviart-Thomas to lowest-order MHM.

Comparison Constant Solution

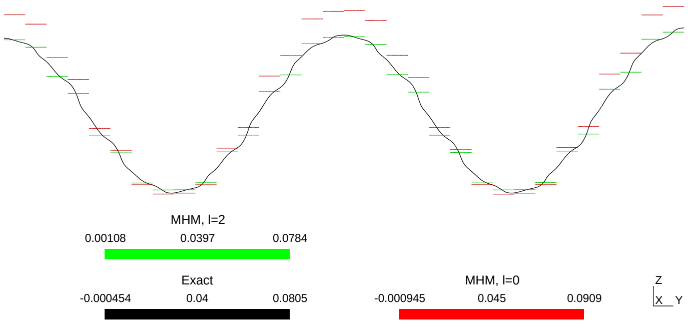


FIGURE: Comparing u_0 for $l = 0, 2$

COMPARISON FULL SOLUTION

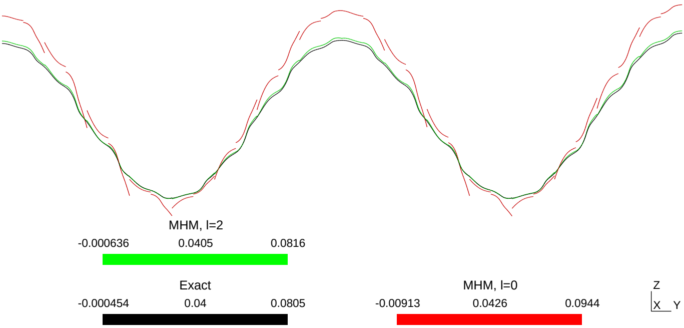


FIGURE: Comparing $u_0 + u_\perp^{\lambda_l} + u_\perp^f$ for $l = 0, 2$

Primal Hybrid Formulation

Weak Form

Find $(u, \lambda) \in W \times M$ *such that*

$$a(u,v) + \sum_K (\lambda, v)_{\partial K} = f(v), \quad \forall\, v \in W$$

$$\sum_K (\mu, u)_{\partial K} = 0, \quad \forall\, \mu \in M.$$

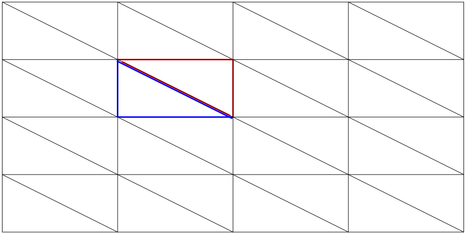


Figure: Triangulation of the domain.

MHM Formulation

From the Primal Hybrid formulation:

- $W = W_0 \oplus W_\perp$.
- Rewrite the Primal Hybrid formulation as
    - locally-defined problems (using $W_\perp$);
    - a globally-defined problem (using W_0).

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Decomposition

- Define $W_0 \subset W$ by the property

$$u_0 \in W_0 \iff a(u_0, v) = 0, \quad \forall\, v \in W.$$

  - Laplace: W_0 consists of piecewise constants.
  - Elasticity: W_0 consists of piecewise rigid-body modes.
  - Advection-Reaction-Diffusion: $W_0 = \{0\}$
- Define $W_\perp$ the L^2-orthogonal complement in W of W_0.
- $W = W_0 \oplus W_\perp$.

MHM Formulation

From the Primal Hybrid formulation:

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  - a globally-defined problem (using W_0).

Primal Hybrid Rewritten

Find $(u_0 + u_\perp, \lambda) \in (W_0 \oplus W_\perp) \times M$ *such that*

$$a(u_\perp, v_\perp) + \sum_K (\lambda, v_0 + v_\perp)_{\partial K} = f(v_0 + v_\perp), \quad \forall\, v_0 + v_\perp \in W_0 \oplus W_\perp$$

$$\sum_K (\mu, u_0 + u_\perp)_{\partial K} = 0, \qquad \forall\, \mu \in M.$$

- *Find* $u_\perp \in W_\perp$ *such that*

$$a(u_\perp, v_\perp) + \sum_K (\lambda, v_\perp)_{\partial K} = f(v_\perp), \quad \forall\, v_\perp \in W_\perp.$$

- *Find* $(u_0, \lambda) \in W_0 \times M$ *such that*

$$\sum_K (\lambda, v_0)_{\partial K} = f(v_0), \quad \forall\, v_0 \in W_0$$

$$\sum_K (\mu, u_0)_{\partial K} + \sum_K (\mu, u_\perp)_{\partial K} = 0, \quad \forall\, \mu \in M.$$

Primal Hybrid Rewritten

Find $(u_0 + u_\perp, \lambda) \in (W_0 \oplus W_\perp) \times M$ *such that*

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$$\sum_K (\mu, u_0 + u_\perp)_{\partial K} = 0, \qquad \forall\, \mu \in M.$$

- *Find* $u_\perp \in W_\perp$ *such that*

$$a(u_\perp, v_\perp) + \sum_K (\lambda, v_\perp)_{\partial K} = f(v_\perp), \quad \forall\, v_\perp \in W_\perp.$$

- *Find* $(u_0, \lambda) \in W_0 \times M$ *such that*

$$\sum_K (\lambda, v_0)_{\partial K} = f(v_0), \quad \forall\, v_0 \in W_0$$

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Primal Hybrid Rewritten

Find $(u_0 + u_\perp, \lambda) \in (W_0 \oplus W_\perp) \times M$ *such that*

$$a(u_\perp, v_\perp) + \sum_K (\lambda, v_0 + v_\perp)_{\partial K} = f(v_0 + v_\perp), \quad \forall\, v_0 + v_\perp \in W_0 \oplus W_\perp$$

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- *Find* $u_\perp \in W_\perp$ *such that*

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- *Find* $(u_0, \lambda) \in W_0 \times M$ *such that*

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$$\sum_K (\mu, u_0)_{\partial K} + \sum_K (\mu, u_\perp)_{\partial K} = 0, \quad \forall\, \mu \in M.$$

Local Problems

Find $u_\perp \in W_\perp$ *such that*

$$a(u_\perp, v_\perp) + \sum_K (\lambda, v_\perp)_{\partial K} = f(v_\perp), \quad \forall\, v_\perp \in W_\perp.$$

- Eliminate $u_\perp$ in terms of f and the solution λ.
  - $u_\perp = u_\perp^\lambda + u_\perp^f$
- Well posed

Local Problems

Find $u_\perp \in W_\perp$ *such that*

$$a(u_\perp, v_\perp) + \sum_K (\lambda, v_\perp)_{\partial K} = f(v_\perp), \quad \forall\, v_\perp \in W_\perp.$$

- Eliminate $u_\perp$ in terms of f and the solution λ.
  - $u_\perp = u_\perp^\lambda + u_\perp^f$
- Well posed

Global Problem

Find $(u_0, \lambda) \in V_0 \times M$ *such that*

$$\sum_K (\lambda, v_0)_{\partial K} = f(v_0), \quad \forall\, v_0 \in V_0$$

$$\sum_K (\mu, u_0)_{\partial K} + \sum_K (\mu, u_\perp)_{\partial K} = 0, \quad \forall\, \mu \in M$$

- Substitute $u_\perp = u_\perp^\lambda + u_\perp^f$:

Find $(u_0, \lambda) \in V_0 \times M$ *such that*

$$\sum_K (\lambda, v_0)_{\partial K} = f(v_0), \qquad \forall\, v_0 \in V_0$$

$$\sum_K (\mu, u_0)_{\partial K} + \sum_K (\mu, u_\perp^\lambda)_{\partial K} = \sum_K (\mu, u_\perp^f)_{\partial K}, \quad \forall\, \mu \in M.$$

Global Problem

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- Substitute $u_\perp = u_\perp^\lambda + u_\perp^f$:

Find $(u_0, \lambda) \in V_0 \times M$ *such that*

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MHM Methods

MHM Formulation

Find $(u_0, \lambda) \in V_0 \times M$ *such that* $\forall\, (v_0, \mu) \in V_0 \times M$,

$$\sum_K (\lambda, v_0)_{\partial K} = f(v_0)$$

$$\sum_K (\mu, u_0)_{\partial K} + \sum_K (\mu, u_\perp^\lambda)_{\partial K} = \sum_K (\mu, u_\perp^f)_{\partial K}$$

where $\forall\, v_\perp \in W_\perp$,

$$a(u_\perp^\lambda, v_\perp) = \sum_K (\lambda, v_\perp)_{\partial K}$$

$$a(u_\perp^f, v_\perp) + \sum_K (\lambda, v_\perp)_{\partial K} = f(v_\perp).$$

MHM Method

Find $(u_0^s, \lambda_s) \in V_0 \times M_s$ *such that* $\forall\, (v_0, \mu) \in V_0 \times M_s$,

$$\sum_K (\lambda_s, v_0)_{\partial K} = f(v_0)$$

$$\sum_K (\mu, u_0^s)_{\partial K} + \sum_K (\mu, u_\perp^{\lambda_h})_{\partial K} = \sum_K (\mu, u_\perp^f)_{\partial K}.$$

where $\forall\, v_\perp \in W_\perp$,

$$a(u_\perp^\lambda, v_\perp) = \sum_K (\lambda, v_\perp)_{\partial K},$$

$$a(u_\perp^f, v_\perp) + \sum_K (\lambda, v_\perp)_{\partial K} = f(v_\perp).$$

MHM Methods

MHM Formulation

Find $(u_0, \lambda) \in V_0 \times M$ *such that* $\forall\, (v_0, \mu) \in V_0 \times M$,

$$\sum_K (\lambda, v_0)_{\partial K} = f(v_0)$$

$$\sum_K (\mu, u_0)_{\partial K} + \sum_K (\mu, u_\perp^{\lambda})_{\partial K} = \sum_K (\mu, u_\perp^{f})_{\partial K}$$

where $\forall\, v_\perp \in W_\perp$,

$$a(u_\perp^{\lambda}, v_\perp) = \sum_K (\lambda, v_\perp)_{\partial K}$$

$$a(u_\perp^{f}, v_\perp) + \sum_K (\lambda, v_\perp)_{\partial K} = f(v_\perp).$$

MHM Method

Find $(u_0^s, \lambda_s) \in V_0 \times M_s$ *such that* $\forall\, (v_0, \mu) \in V_0 \times M_s$,

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where $\forall\, v_\perp \in W_\perp$,

$$a(u_\perp^{\lambda}, v_\perp) = \sum_K (\lambda, v_\perp)_{\partial K},$$

$$a(u_\perp^{f}, v_\perp) + \sum_K (\lambda, v_\perp)_{\partial K} = f(v_\perp).$$

Properties

Must choose M_s (and possibly spaces for two-level approximation);

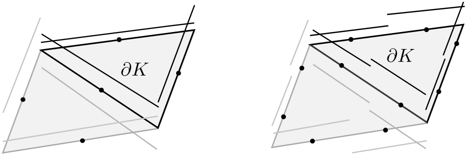
- $M_s \approx M$;
- $M_s \supset M_0$;
  - M_0 guarantees invertibility.

Reaction-Advection-Diffusion

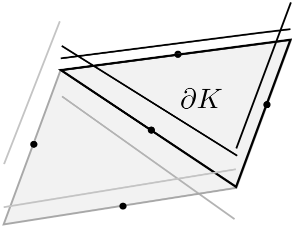
- Let

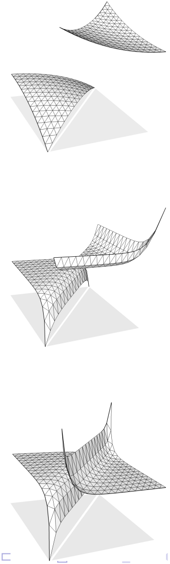
$$\mathcal{L} = \nabla \cdot (-\mathcal{K}\nabla u + \alpha u) + \sigma u$$

- Possible M_s

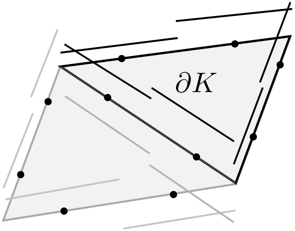


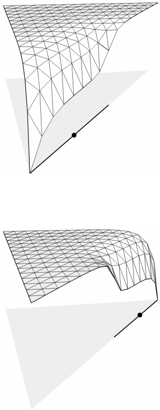
Solutions to local problems





Solutions to local problems





Sample problem statement

Find u such that

$$-\epsilon \triangle u + \alpha \cdot \nabla u + \sigma u = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$

$|\alpha| = 1$, and $f = 0$

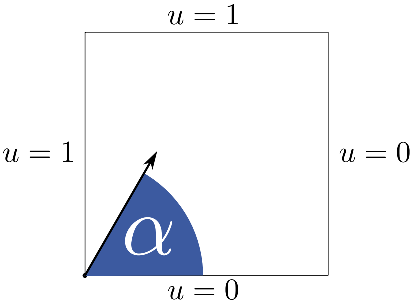
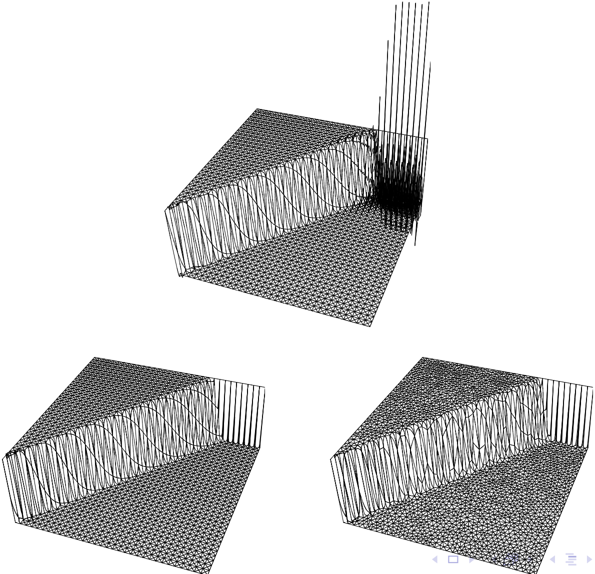
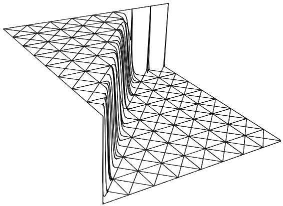


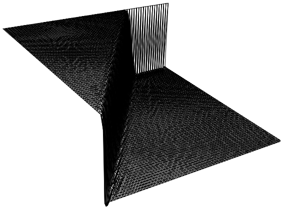
FIGURE: Setup of the problem.

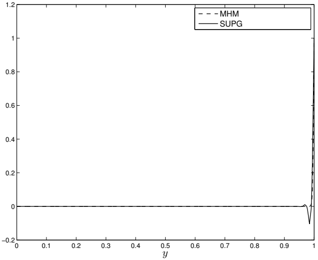
Classical Galerkin vs. MHM
$\epsilon = 1e - 4$ and $\sigma = 0$

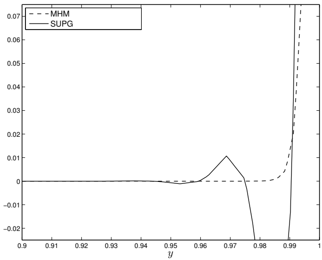


SUPG vs. MHM
$\epsilon = 1e-4$ and $\sigma = 0$



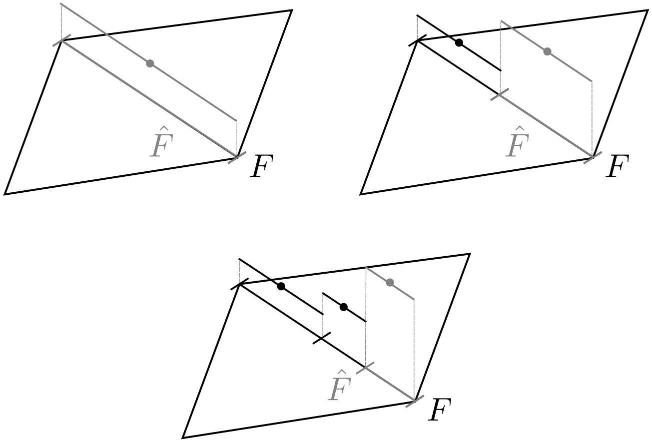






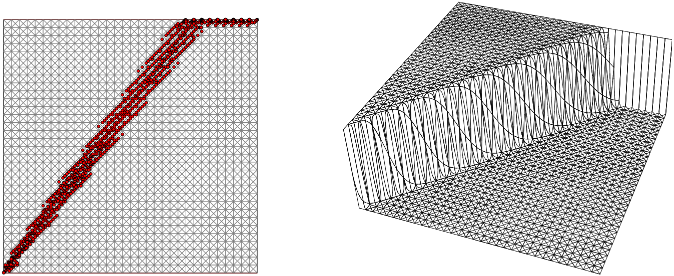
A Posteriori Estimator

Recall the a posteriori estimator depends on $-\frac{1}{2}[\![u_\perp + u_0]\!]$

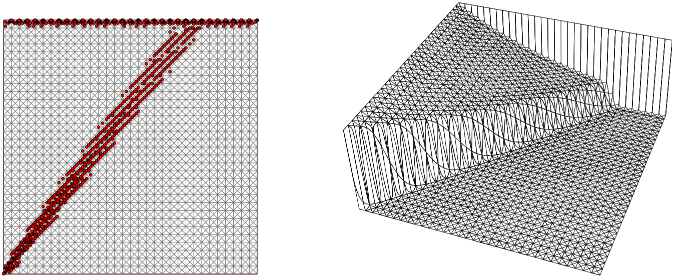


A Posteriori Estimator

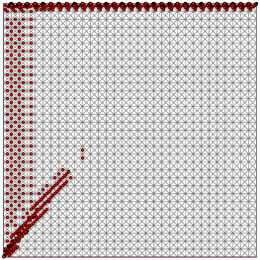
$\epsilon = 1e-4$ AND $\sigma = 0$

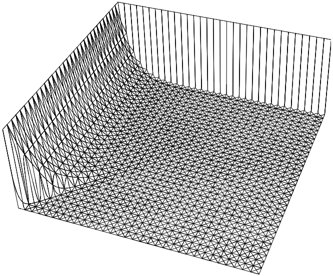


A Posteriori Estimator
$\epsilon = 1e-4$ and $\sigma = 1$



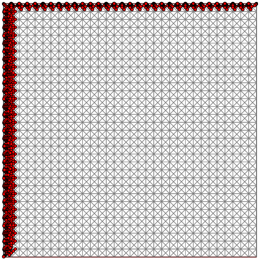
A Posteriori Estimator
$\epsilon = 1e-4$ and $\sigma = 10$

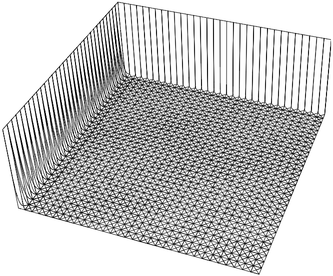




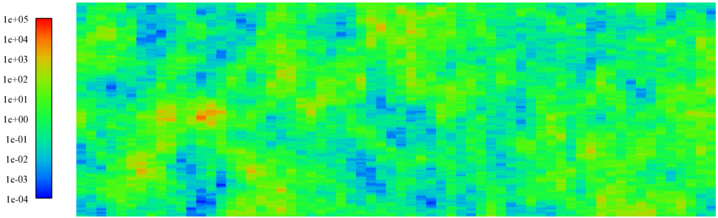
A Posteriori Estimator

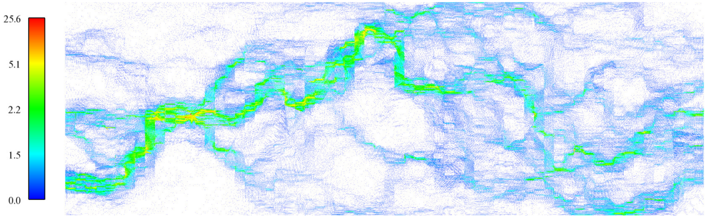
$\epsilon = 1e-4$ and $\sigma = 100$



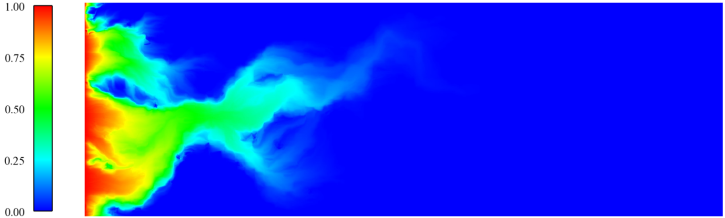


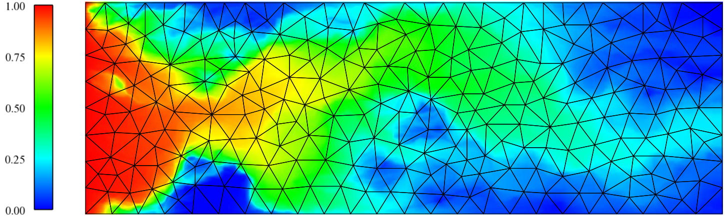
Flow in a Heterogeneous Medium





Flow in a Heterogeneous Medium





Conclusion

- The MHM framework builds on Hybrid formulation of problems.
- The MHM methods consist of local solves and a global solve.
    - The local solves are easily parallelized;
    - Local solves capture local information.
- Dual variables may be approximated.
- An edge-based a posteriori estimator;
    - refine on edges of a given mesh.

Thank You Slide

Merci Beaucoup!