Strong convergence for Gauss' law with edge elements

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Outline

- Time-harmonic Maxwell equations and discretization

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- Error estimates on the divergence of the fields

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- Error estimates on the divergence of the fields
- Stationary/Static Maxwell equations and discretizations
- Time-dependent Maxwell equations and discretization
- Numerical illustration

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Time-harmonic problem

- Let Ω be a Lipschitz, polyhedral domain with connected boundary $\partial\Omega$.
- Given $k > 0$ and source term $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ ($\operatorname{div} \boldsymbol{f} = 0$), solve:

$$\begin{cases} \textit{Find } \boldsymbol{E} \in \boldsymbol{L}^2(\Omega) \textit{ with } \mathbf{curl}\, \boldsymbol{E} \in \boldsymbol{L}^2(\Omega) \textit{ s.t.} & \\ \mathbf{curl}\, \left(\mu^{-1}\mathbf{curl}\, \boldsymbol{E}\right) - k^2\varepsilon\, \boldsymbol{E} = \boldsymbol{f} & \text{in } \Omega \,; \\ \operatorname{div} \varepsilon\, \boldsymbol{E} = 0 & \text{in } \Omega \,; \\ \boldsymbol{E} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

NB. With coefficients $\varepsilon, \mu > 0$ a.e. ; $\varepsilon, \varepsilon^{-1}, \mu, \mu^{-1} \in L^\infty(\Omega)$.

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- We assume that the problem is well-posed: $\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$, where

$$\boldsymbol{H}(\mathbf{curl}\,;\Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \,|\, \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}.$$

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- An equivalent variational formulation is:

$$(VF)\ \left\{\begin{array}{l} \textit{Find } \boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl}\,;\Omega) \textit{ s.t.} \\ \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}\,;\Omega),\quad (\mu^{-1}\mathbf{curl}\,\boldsymbol{E}|\mathbf{curl}\,\boldsymbol{v}) - k^2(\varepsilon\,\boldsymbol{E}|\boldsymbol{v}) = (\boldsymbol{f}|\boldsymbol{v}). \end{array}\right.$$

Above, $(\boldsymbol{v}|\boldsymbol{v}') := \int_\Omega \boldsymbol{v} \cdot \boldsymbol{v}'\, d\Omega$.

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Regularity of the fields

- $\boldsymbol{E} \in \mathcal{X}_N(\Omega, \varepsilon) := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}\,, \Omega) \mid \operatorname{div} \varepsilon \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}.$

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- `Theorem` [Costabel-Dauge-Nicaise'99]: Assume that $\varepsilon, \mu^{-1} \in W^{1,\infty}(\Omega)$.
  If Ω is *convex* then $\mathcal{X}_N(\Omega, \varepsilon) \subset \boldsymbol{H}^1(\Omega)$ and $\mathcal{X}_T(\Omega, \mu) \subset \boldsymbol{H}^1(\Omega)$.

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$$\mathcal{X}_N(\Omega, \varepsilon) \subset \boldsymbol{H}^\delta(\Omega),\ \forall \delta \in [0, \delta_{max}^{Dir}[\quad \text{and} \quad \mathcal{X}_T(\Omega, \mu) \subset \boldsymbol{H}^\delta(\Omega),\ \forall \delta \in [0, \delta_{max}^{Neu}[.$$

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  NB. If Ω is convex, then $\delta = 1$.

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Edge element discretization

- Let $(\mathcal{T}_h)_h$ be a shape regular family of tetrahedral meshes of Ω.
- Define $\mathcal{X}_h := \{\boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl}\,;\Omega) \,|\, \boldsymbol{v}_{h|K} = \mathbf{a}_K + \mathbf{b}_K \times \mathbf{x},\, \forall K \in \mathcal{T}_h\}$.

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$$(DVF)\left\{\begin{array}{l}\textit{Find } \boldsymbol{E}_h \in \mathcal{X}_h \textit{ s.t.}\\ \forall \boldsymbol{v}_h \in \mathcal{X}_h, \quad (\mu^{-1}\mathbf{curl}\,\boldsymbol{E}_h|\mathbf{curl}\,\boldsymbol{v}_h) - k^2(\varepsilon\,\boldsymbol{E}_h|\boldsymbol{v}_h) = (\boldsymbol{f}|\boldsymbol{v}_h).\end{array}\right.$$

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- Classically: $\exists h_0,\, \forall h < h_0,\; \|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim \inf_{\boldsymbol{v}_h \in \mathcal{X}_h} \|\boldsymbol{E} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)}$.

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- Edge element interpolation ($\delta \in]1/2, \delta_{max}[$), cf. [Alonso-Valli'99], [Jr-Zou'99]:

$$\forall h < h_0,\ \|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim h^\delta\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

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- `QUESTION`: *What of* $\|\mathrm{div}\,\varepsilon(\boldsymbol{E} - \boldsymbol{E}_h)\|$*?*

On the divergence

- Using $\boldsymbol{v} = \nabla q$ for $q \in H_0^1(\Omega)$ in (VF) yields

$$\langle \operatorname{div} \varepsilon\, \boldsymbol{E}, q \rangle = -(\varepsilon\, \boldsymbol{E} | \nabla q) = \frac{1}{k^2} (\boldsymbol{f} | \nabla q) = -\frac{1}{k^2} \langle \operatorname{div} \boldsymbol{f}, q \rangle = 0.$$

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  Conclusion: $\boldsymbol{E}_h \in \mathcal{V}_h := \{\boldsymbol{v}_h \in \mathcal{X}_h \,|\, (\varepsilon\, \boldsymbol{v}_h | \nabla q_h) = 0,\ \forall q_h \in Q_h\}$.

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- Theorem [$\mathcal{V}_h$]: Assume that $(\mathcal{T}_h)_h$ is quasi-uniform. Let $s \in]1/2, 1]$, then

$$\forall \boldsymbol{v}_h \in \mathcal{V}_h, \quad \|\operatorname{div} \varepsilon \boldsymbol{v}_h\|_{H^{-s}(\Omega)} \lesssim h^{s+\delta-1} \|\mathbf{curl}\, \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}.$$

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- `Corollary`: $\|\operatorname{div} \varepsilon(\boldsymbol{E} - \boldsymbol{E}_h)\|_{H^{-s}(\Omega)} \lesssim h^{s+\delta-1} \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}$.

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On the divergence – Proof

`Proof of the Theorem` $[\mathcal{V}_h]$

- Step 1: Let $\boldsymbol{v}_h \in \mathcal{V}_h$, $q \in H_0^s(\Omega)$, and $q_h \in Q_h$:

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$$\begin{aligned}
\langle \operatorname{div} \varepsilon\, \boldsymbol{v}_h , q \rangle &= -(\varepsilon\, \boldsymbol{v}_h | \nabla q) = -(\varepsilon\, \boldsymbol{v}_h | \nabla (q - q_h)) = -\sum_K \int_K \varepsilon\, \boldsymbol{v}_h \cdot \nabla (q - q_h)\, d\Omega \\
\text{ibp in } K\text{...} \quad &\lesssim \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \|q - q_h\|_{L^2(\Omega)} \\
&\quad + \Big(\sum_{f \in \mathcal{F}_h} \|[\varepsilon \boldsymbol{v}_h \cdot \boldsymbol{n}]\|^2_{L^2(f)} \Big)^{1/2} \Big(\sum_{f \in \mathcal{F}_h} \|q - q_h\|^2_{L^2(f)} \Big)^{1/2}
\end{aligned}$$

where $\mathcal{F}_h$ denotes the set of faces of $\mathcal{T}_h$ and $[\cdot]$ the jump across the faces.

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On the divergence – Proof

`Proof of the Theorem` $[\mathcal{V}_h]$

- Step 1: Let $\boldsymbol{v}_h \in \mathcal{V}_h$, $q \in H_0^s(\Omega)$, and $q_h \in Q_h$:

$$\begin{aligned}\langle \operatorname{div} \varepsilon \boldsymbol{v}_h, q \rangle \quad &\lesssim \quad \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \, \|q - q_h\|_{L^2(\Omega)} \\ &\quad + \Big(\sum_{f \in \mathcal{F}_h} \|[\varepsilon \boldsymbol{v}_h \cdot \boldsymbol{n}]\|^2_{L^2(f)} \Big)^{1/2} \Big(\sum_{f \in \mathcal{F}_h} \|q - q_h\|^2_{L^2(f)} \Big)^{1/2}.\end{aligned}$$

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- Step 2: Evaluate $\|q - q_h\|_{L^2(\Omega)}$ and $(\sum_{f\in\mathcal{F}_h} \|q - q_h\|^2_{L^2(f)})^{1/2}$ wrt $\|q\|_{H^s(\Omega)}$. (for some appropriate choice of q_h).

On the divergence – Proof

`Proof of the Theorem` $[\mathcal{V}_h]$

- Step 1: Let $\boldsymbol{v}_h \in \mathcal{V}_h$, $q \in H^s_0(\Omega)$, and $q_h \in Q_h$:

$$\begin{aligned}\langle \operatorname{div} \varepsilon \boldsymbol{v}_h, q\rangle \quad &\lesssim \quad \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \|q - q_h\|_{L^2(\Omega)} \\ &+ \Big(\sum_{f \in \mathcal{F}_h} \|[\varepsilon \boldsymbol{v}_h \cdot \boldsymbol{n}]\|^2_{L^2(f)} \Big)^{1/2} \Big(\sum_{f \in \mathcal{F}_h} \|q - q_h\|^2_{L^2(f)} \Big)^{1/2}.\end{aligned}$$

- Step 2: Evaluate $\|q - q_h\|_{L^2(\Omega)}$ and $(\sum_{f \in \mathcal{F}_h} \|q - q_h\|^2_{L^2(f)})^{1/2}$ wrt $\|q\|_{H^s(\Omega)}$.
  Inverse inequality:

$$\forall K \in \mathcal{T}_h, \forall q \in H^s(K), \quad \|q\|_{L^2(\partial K)} \lesssim h_K^{-1/2} \|q\|_{L^2(K)} + h_K^{s-1/2} |q|_{H^s(K)}.$$

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On the divergence – Proof

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Let $\Pi_h : H^s_0(\Omega) \to Q_h$ be the Scott-Zhang interpolation operator, then (cf. [Jr'13])

$$\forall q \in H^s_0(\Omega), \quad \|q - \Pi_h q\|_{H^s(\Omega)} \lesssim \|q\|_{H^s(\Omega)}, \; \|q - \Pi_h q\|_{L^2(\Omega)} \lesssim h^s \|q\|_{H^s(\Omega)}.$$

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On the divergence – Proof

Proof of the Theorem $[\mathcal{V}_h]$

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$$\forall q\in H^s_0(\Omega),\quad \|q-\Pi_h q\|_{H^s(\Omega)} \lesssim \|q\|_{H^s(\Omega)},\ \|q-\Pi_h q\|_{L^2(\Omega)} \lesssim h^s\|q\|_{H^s(\Omega)}.$$

Choose $q_h := \Pi_h q$: $(\sum_{f\in\mathcal{F}_h}\|q-q_h\|^2_{L^2(f)})^{1/2} \lesssim h^{s-1/2}\|q\|_{H^s(\Omega)}$.

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On the divergence – Proof

`Proof of the Theorem` $[\mathcal{V}_h]$

- Step 1: Let $\boldsymbol{v}_h \in \mathcal{V}_h$, $q \in H^s_0(\Omega)$, and $q_h \in Q_h$:

$$\langle \operatorname{div} \varepsilon\, \boldsymbol{v}_h, q\rangle \lesssim \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \|q - q_h\|_{L^2(\Omega)} + \Big(\sum_{f\in\mathcal{F}_h} \|[\varepsilon \boldsymbol{v}_h \cdot \boldsymbol{n}]\|^2_{L^2(f)}\Big)^{1/2} \Big(\sum_{f\in\mathcal{F}_h} \|q - q_h\|^2_{L^2(f)}\Big)^{1/2}.$$

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$$\forall q \in H^s_0(\Omega), \quad \|q - \Pi_h q\|_{H^s(\Omega)} \lesssim \|q\|_{H^s(\Omega)},\ \|q - \Pi_h q\|_{L^2(\Omega)} \lesssim h^s \|q\|_{H^s(\Omega)}.$$

Hence, $\|\operatorname{div} \varepsilon\, \boldsymbol{v}_h\|_{H^{-s}(\Omega)} \lesssim h^s \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} + h^{s-1/2} \Big(\sum_{f\in\mathcal{F}_h} \|[\varepsilon \boldsymbol{v}_h \cdot \boldsymbol{n}]\|^2_{L^2(f)}\Big)^{1/2}$.

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On the divergence – Proof

- Steps 1-2: For $\boldsymbol{v}_h \in \mathcal{V}_h$,

$$\|\operatorname{div} \varepsilon \, \boldsymbol{v}_h\|_{H^{-s}(\Omega)} \lesssim h^s \|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} + h^{s-1/2} \Big(\sum_{f \in \mathcal{F}_h} \|[\varepsilon \boldsymbol{v}_h \cdot \boldsymbol{n}]\|^2_{L^2(f)} \Big)^{1/2}.$$

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- Step 3: Evaluate $(\sum_{f \in \mathcal{F}_h} \|[\varepsilon \boldsymbol{v}_h \cdot \boldsymbol{n}]\|^2_{L^2(f)})^{1/2}$.

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  `Proposition` [Monk'03]: $\exists \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}\,; \Omega)$ s.t. $\mathbf{curl}\, \boldsymbol{v} = \mathbf{curl}\, \boldsymbol{v}_h$, $\operatorname{div} \varepsilon\, \boldsymbol{v} = 0$ in Ω,

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^\delta(\Omega)} \lesssim \|\mathbf{curl}\, \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}, \quad \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \lesssim h^\delta \|\boldsymbol{v}\|_{\boldsymbol{H}^\delta(\Omega)} + h \|\mathbf{curl}\, \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}.$$

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On the divergence – Proof

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  `Proposition` [Monk'03]: $\exists \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}\,;\Omega)$ s.t. $\mathbf{curl}\, \boldsymbol{v} = \mathbf{curl}\, \boldsymbol{v}_h$, $\operatorname{div} \varepsilon\, \boldsymbol{v} = 0$ in Ω,

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  By construction, $[\varepsilon \boldsymbol{v}_h \cdot \boldsymbol{n}] = [\varepsilon(\boldsymbol{v}_h - \boldsymbol{v}) \cdot \boldsymbol{n}]$ across all faces.

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On the divergence – Proof

- Steps 1-2: For $\boldsymbol{v}_h \in \mathcal{V}_h$,

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  By construction, $[\varepsilon\boldsymbol{v}_h\cdot\boldsymbol{n}] = [\varepsilon(\boldsymbol{v}_h-\boldsymbol{v})\cdot\boldsymbol{n}]$ across all faces.

  $+$ inverse inequality: $(\sum_{f\in\mathcal{F}_h} \|[\varepsilon\boldsymbol{v}_h\cdot\boldsymbol{n}]\|^2_{L^2(f)})^{1/2} \lesssim h^{\delta-1/2}\|\mathbf{curl}\,\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}$.

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On the divergence – Proof

- Steps 1-2: For $\boldsymbol{v}_h \in \mathcal{V}_h$,

$$\|\mathrm{div}\,\varepsilon\,\boldsymbol{v}_h\|_{H^{-s}(\Omega)} \lesssim h^s\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} + h^{s-1/2}\Big(\sum_{f\in\mathcal{F}_h}\|[\varepsilon\boldsymbol{v}_h\cdot\boldsymbol{n}]\|^2_{L^2(f)}\Big)^{1/2}.$$

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It follows that $\|\mathrm{div}\,\varepsilon\,\boldsymbol{v}_h\|_{H^{-s}(\Omega)} \lesssim h^s\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} + h^{s+\delta-1}\|\mathbf{curl}\,\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}$.

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On the divergence – Proof

- Steps 1-2: For $\boldsymbol{v}_h \in \mathcal{V}_h$,

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  With the help of the Proposition (and $\delta < 1$), one concludes that

$$\|\mathrm{div}\,\varepsilon\,\boldsymbol{v}_h\|_{H^{-s}(\Omega)} \lesssim h^{s+\delta-1}\|\mathbf{curl}\,\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}.$$

Time-harmonic problem (summary)

- Assumptions:
  - $\varepsilon, \mu^{-1} \in W^{1,\infty}(\Omega)$;
  - $(\mathcal{T}_h)_h$ is quasi-uniform.

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- If Ω is *convex*: let $s \in]1/2, 1]$, then

$$\forall h < h_0,\ h^{-1}\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} + h^{-s}\|\mathrm{div}\,\varepsilon(\boldsymbol{E} - \boldsymbol{E}_h)\|_{H^{-s}(\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

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- If Ω is *non-convex*: let $\delta \in]1/2, \delta_{max}[$ and $s \in]1/2, 1]$, then

$$\forall h < h_0,\ h^{-\delta}\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} + h^{1-s-\delta}\|\mathrm{div}\,\varepsilon(\boldsymbol{E} - \boldsymbol{E}_h)\|_{H^{-s}(\Omega)} \lesssim \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

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Stationary problem: mixed approach

- Given source terms $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ ($\mathrm{div}\,\boldsymbol{f} = 0$) and $g \in L^2(\Omega)$, solve:

$$\begin{cases} \textit{Find } \boldsymbol{E} \in \boldsymbol{L}^2(\Omega) \textit{ with } \mathbf{curl}\,\boldsymbol{E} \in \boldsymbol{L}^2(\Omega) \textit{ s.t.} & \\ \mathbf{curl}\,\left(\mu^{-1}\mathbf{curl}\,\boldsymbol{E}\right) = \boldsymbol{f} & \text{in } \Omega\ ; \\ \mathrm{div}\,\varepsilon\,\boldsymbol{E} = g & \text{in } \Omega\ ; \\ \boldsymbol{E} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

NB. With coefficients $\varepsilon, \mu^{-1} \in W^{1,\infty}(\Omega)$.

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- To take into account the condition on the divergence on the variational formulation, one uses classically an equivalent *mixed formulation* (with $p = 0$):

$$(MVF) \left\{ \begin{array}{l} \textit{Find } (\boldsymbol{E}, p) \in \boldsymbol{H}_0(\mathbf{curl}\,; \Omega) \times H_0^1(\Omega) \textit{ s.t.} \\ \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}\,; \Omega), \quad (\mu^{-1} \mathbf{curl}\, \boldsymbol{E} | \mathbf{curl}\, \boldsymbol{v}) + (\varepsilon\, \boldsymbol{v} | \nabla p) = (\boldsymbol{f} | \boldsymbol{v}) \\ \forall q \in H_0^1(\Omega), \quad (\varepsilon \boldsymbol{E} | \nabla q) = -(g|q). \end{array} \right.$$

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- The discrete mixed variational formulation uses edge elements for the field and P_1 elements for the multiplier (with $p_h = 0$):

$$(DMVF) \begin{cases} \textit{Find } (\boldsymbol{E}_h, p_h) \in \mathcal{X}_h \times Q_h \textit{ s.t.} \\ \forall \boldsymbol{v}_h \in \mathcal{X}_h, \quad (\mu^{-1}\mathbf{curl}\, \boldsymbol{E}_h | \mathbf{curl}\, \boldsymbol{v}_h) + (\varepsilon\, \boldsymbol{v}_h | \nabla p_h) = (\boldsymbol{f} | \boldsymbol{v}_h) \\ \forall q \in Q_h, \quad (\varepsilon \boldsymbol{E}_h | \nabla q_h) = -(g | q_h). \end{cases}$$

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- For $\delta \in]1/2, \delta_{max}[$, one obtains, cf. [Chen-Du-Zou'00]:

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim h^\delta \left\{ \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|g\|_{L^2(\Omega)} \right\}.$$

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NB. With coefficients $\varepsilon, \mu^{-1} \in W^{1,\infty}(\Omega)$.

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- To take into account the condition on the divergence on the variational formulation, we choose to add some *small perturbation* (below $\gamma(h) > 0$ is "small"), by introducing

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- If $g = 0$, we solve the discrete variational formulation

$$\textit{Find } \boldsymbol{E}_h \in \mathcal{X}_h \textit{ s.t. } \forall \boldsymbol{v}_h \in \mathcal{X}_h, \quad a_h(\boldsymbol{E}_h, \boldsymbol{v}_h) = (\boldsymbol{f}|\boldsymbol{v}_h).$$

By construction, $\boldsymbol{E}_h \in \mathcal{V}_h$.

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- If $g \neq 0$, we solve two discrete variational formulations

  1. *Find* $\phi_h \in Q_h$ *s.t.* $\forall q_h \in Q_h, \quad (\varepsilon \nabla \phi_h | \nabla q_h) = -(\varepsilon g | q_h)$.
  2. *Find* $\boldsymbol{E}_h \in \mathcal{X}_h$ *s.t.* $\forall \boldsymbol{v}_h \in \mathcal{X}_h, \quad a_h(\boldsymbol{E}_h, \boldsymbol{v}_h) = (\boldsymbol{f} | \boldsymbol{v}_h) + \gamma(h)(\varepsilon\, \nabla \phi_h | \boldsymbol{v}_h)$.

  By construction, $\boldsymbol{E}_h - \nabla \phi_h \in \mathcal{V}_h$.

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- `Theorem`: Assume that $(\mathcal{T}_h)_h$ is quasi-uniform, $0 < \gamma(h) \lesssim h^2$. Let $s \in]1/2, 1]$, then

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  $\boldsymbol{E}_h \in \mathcal{V}_h$, so the Theorem [$\mathcal{V}_h$] yields

$$\|\mathrm{div}\,\varepsilon(\boldsymbol{E} - \boldsymbol{E}_h)\|_{H^{-s}(\Omega)} \lesssim h^{s+\delta-1}\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

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Using an auxiliary problem, one can estimate $\|\boldsymbol{e}\|_{\boldsymbol{L}^2(\Omega)}$:

$$\left\{\begin{array}{ll} \textit{Find } \boldsymbol{z}\in\boldsymbol{H}_0(\mathbf{curl}\,;\Omega) \textit{ s.t.} & \\ \mathbf{curl}\,\left(\mu^{-1}\mathbf{curl}\,\boldsymbol{z}\right) = \boldsymbol{e} & \text{in } \Omega\,; \\ \mathrm{div}\,\boldsymbol{z} = 0 & \text{in } \Omega. \end{array}\right.$$

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We conclude that

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{L}^2(\Omega)} \lesssim h^\delta\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}.$$

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Time-dependent problem

- Given source terms $\boldsymbol{J}$ and ρ (charge conservation eq. $\rho_t + \operatorname{div} \boldsymbol{J} = 0$), solve:

$$\begin{cases} \textit{Find } \boldsymbol{E} \textit{ s.t.} & \\ \varepsilon \boldsymbol{E}_{tt} + \mathbf{curl}\, \left(\mu^{-1} \mathbf{curl}\, \boldsymbol{E}\right) = -\boldsymbol{J}_t & \text{in } \Omega \times]0,T[\ ; \\ \operatorname{div} \varepsilon\, \boldsymbol{E} = \rho & \text{in } \Omega \times]0,T[\ ; \\ \boldsymbol{E} \times \boldsymbol{n} = 0 & \text{on } \partial\Omega \times]0,T[\ ; \\ \boldsymbol{E}(0) = \boldsymbol{E}^0 \text{ and } \boldsymbol{E}_t(0) = \varepsilon^{-1}(-\boldsymbol{J}(0) + \mathbf{curl}\, \boldsymbol{H}^0) & \text{in } \Omega. \end{cases}$$

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$$\max_n \left(\|\operatorname{div}\varepsilon(\boldsymbol{E}_h^n - \boldsymbol{E}(n\tau))\|_{H^{-s}(\Omega)} \right) \lesssim \tau + h^{s+\delta-1}.$$

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  Consider that they are piecewise constant: there exists a *partition* $\mathcal{P} := \{\Omega_j\}_{j=1}^J$ of Ω into J polyhedral subdomains s.t. $\varepsilon_j := \varepsilon_{|\Omega_j}$, $\mu_j := \mu_{|\Omega_j}$ are constants for $j = 1, J$.
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- `Theorem` [Costabel-Dauge-Nicaise'99]: Assume that ε, μ are piecewise constant.
  $\exists \delta_{max}^{Dir}, \delta_{max}^{Neu} \in]0,1]$ s.t.

$$\mathcal{X}_N(\Omega,\varepsilon) \subset \boldsymbol{PH}^\delta(\Omega),\ \forall \delta \in [0, \delta_{max}^{Dir}[\quad \text{and} \quad \mathcal{X}_T(\Omega,\mu) \subset \boldsymbol{PH}^\delta(\Omega),\ \forall \delta \in [0, \delta_{max}^{Neu}[.$$

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$$\mathcal{X}_N(\Omega, \varepsilon) \subset \boldsymbol{PH}^{\delta}(\Omega),\ \forall \delta \in [0, \delta_{max}^{Dir}[\quad \text{and} \quad \mathcal{X}_T(\Omega, \mu) \subset \boldsymbol{PH}^{\delta}(\Omega),\ \forall \delta \in [0, \delta_{max}^{Neu}[.$$

- If $\delta_{max} := \min(\delta_{max}^{Dir}, \delta_{max}^{Neu}) > 1/2$ then one can use the previous results, since the edge element interpolation results hold element-by-element.

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Other configurations

- In the models, the coefficients ε and μ can be discontinuous.

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  For instance:
  - Ω is convex, and the maximal number of adjacent subdomains is equal to two ;
  - case of *separated inclusions*: $\exists j$ s.t. $\partial\Omega \subset \partial\Omega_j$, and the maximal number of adjacent subdomains is equal to two.

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Numerics

- `Numerical example`: stationary problem.
  $\varepsilon = \mu = 1$ in the unit cube Ω, with smooth solution ($\boldsymbol{f} \neq 0$, $g \neq 0$):

$$\boldsymbol{E}_{ex} = \begin{pmatrix} x_1 x_2 x_3 (1 - x_2)(1 - x_3) \\ x_1 x_2 x_3 (1 - x_3)(1 - x_1) \\ x_1 x_2 x_3 (1 - x_1)(1 - x_2) \end{pmatrix}.$$

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- One can choose $\delta = 1$ for the convergence rates. So, one expects

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim h$$

$$\|\mathrm{div}\,(\boldsymbol{E} - \boldsymbol{E}_h)\|_{H^{-s}(\Omega)} \lesssim h^s \text{ for } s \in]1/2, 1].$$

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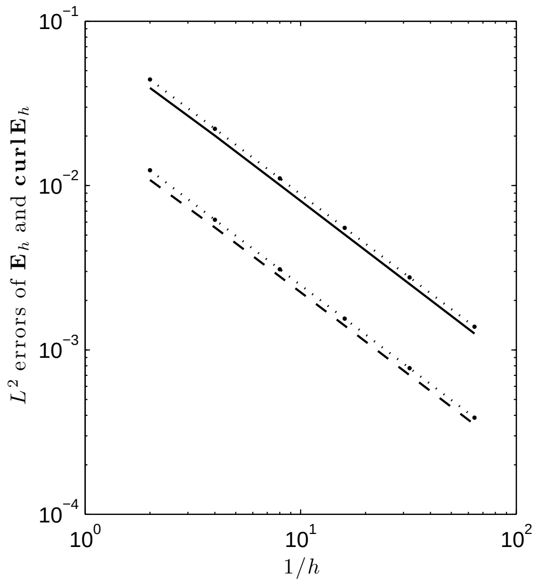
Numerics – results

- For the $\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)}$ error:



dashed line: $\|\boldsymbol{E}-\boldsymbol{E}_h\|_{\boldsymbol{L}^2(\Omega)}$;
solid line: $\|\mathbf{curl}\,(\boldsymbol{E}-\boldsymbol{E}_h)\|_{\boldsymbol{L}^2(\Omega)}$;
dotted lines: slope -1.

Numerics – results

- $\|\boldsymbol{E} - \boldsymbol{E}_h\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega)} \lesssim h$ is observed.
- For the $\|\mathrm{div}\,(\boldsymbol{E} - \boldsymbol{E}_h)\|_{H^{-s}(\Omega)}$ error, we recall that:

$$\|\mathrm{div}\,(\boldsymbol{E}-\boldsymbol{E}_h)\|_{H^{-s}(\Omega)} \lesssim h^s(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}+\|g\|_{L^2(\Omega)})+h^{s-1/2}\Big(\sum_{f\in\mathcal{F}_h}\|[\varepsilon\boldsymbol{E}_h\cdot\boldsymbol{n}]\|^2_{L^2(f)}\Big)^{1/2}.$$

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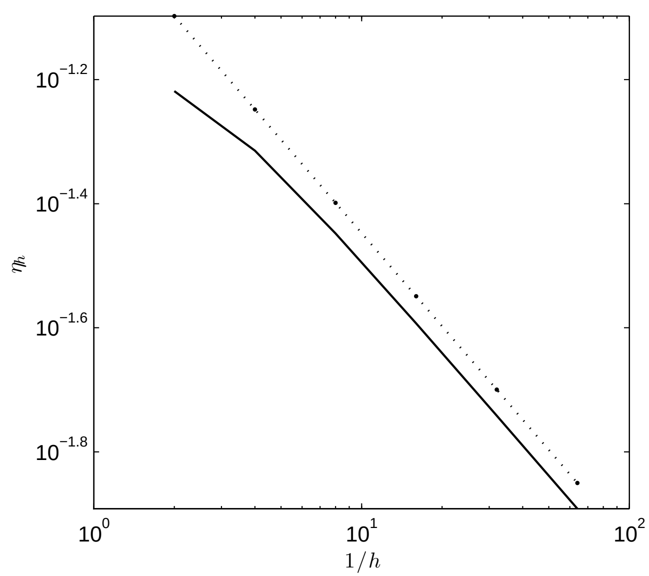

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Conclusions

- Discrete ε-divergence free elements have "small" $\|\mathrm{div}\,\varepsilon\cdot\|_{H^{-s}(\Omega)}$ ($s\in]1/2,1]$).

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- The time-harmonic, stationary/static and time-dependent Maxwell problems can be analyzed for "smooth", positive coefficients ε,μ^{-1}.
- For the stationary/static problem, there is no need to solve a mixed problem.
- The same results can be obtained in some configurations when the coefficients ε,μ are *piecewise constant*: in particular, the case of separated inclusions is covered.

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