The Hybridized Eigenproblem

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Portland State University

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INRIA Sophia Antipolis

Thanks: NSF

Collaborators: B. Cockburn, F. Li, N.C. Nguyen, J. Peraire

Outline

- A domain decomposition perspective
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- Hybridized methods
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- Eigenvalue problems
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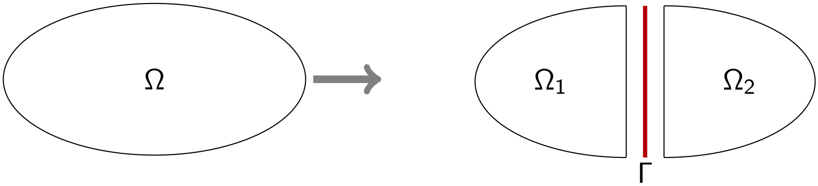
Divide & Conquer

Problem:

$$\begin{aligned} -\Delta u &= f && \text{on } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

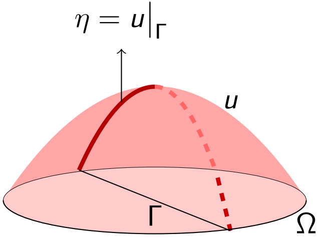


Split Ω into Ω_1 and Ω_2.

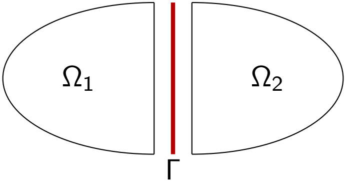
Interface $= \Gamma$.

Problem:

$$-\Delta u = f \qquad \text{on } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega.$$



If we know η, then the problem decouples into two problems, one on Ω_1, and another on Ω_2.

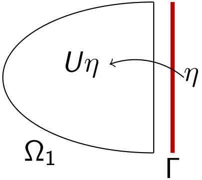


Interface = Γ.

The decoupling

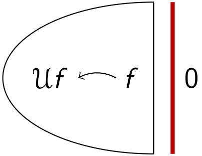
If we know the solution η on the interface Γ, then:

1. Compute $U\eta \equiv$ Harmonic extension of η into Ω_1:



$$\begin{cases} -\Delta(U\eta) = 0 & \text{on } \Omega_1 \\ U\eta = \eta & \text{on } \Gamma \\ U\eta = 0 & \text{on } \partial\Omega_1 \setminus \Gamma. \end{cases}$$

2. Compute $\mathcal{U}f$ on Ω_1:



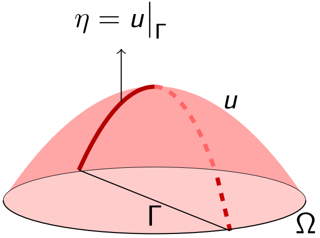
$$\begin{cases} -\Delta(\mathcal{U}f) = f & \text{on } \Omega_1 \\ \mathcal{U}f = 0 & \text{on } \partial\Omega_1. \end{cases}$$

3. Linear superposition $\implies$ $u = U\eta + \mathcal{U}f$ on Ω_1.
Same on Ω_2.

Divide & Conquer. . . ?

Problem:

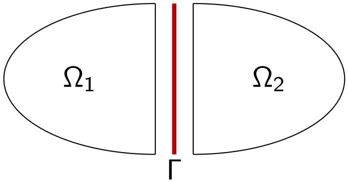
$$\begin{aligned} -\Delta u &= f && \text{on } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$



- If we know η, then

$$u = U\eta + \mathcal{U}f.$$

- But, can we find η on Γ . . . ?



Interface = Γ.

Solve the "interface problem"

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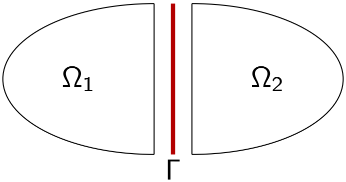
1. Classical theorem: η is the unique function in $\mathring{H}^{1/2}(\Gamma)$ satisfying

$$a(\,\eta\,,\,\mu\,) = b(\,\mu\,), \qquad \forall \mu \in \mathring{H}^{1/2}(\Gamma)$$

where

$$a(\eta, \mu) = \int_\Omega \vec{\nabla}(U\ \eta) \cdot \vec{\nabla}(U\ \mu),$$
$$b(\mu) = \int_\Omega (U\ \mu)\ f.$$



2. Recover solution by $u = U\ \eta + \mathcal{U}\ f$.

Dimensional reduction: The interface problem is 1D!

Solve the "interface problem"

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1. Classical theorem: η_h is the unique function in M_h satisfying

$$a(\eta_h, \mu) = b(\mu), \qquad \forall \mu \in M_h \subset \mathring{H}^{1/2}(\Gamma)$$

where

$$a(\eta, \mu) = \int_\Omega \vec{\nabla}(U_h \eta) \cdot \vec{\nabla}(U_h \mu),$$
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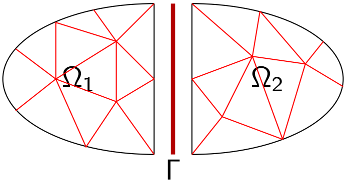

2. Recover solution by $u_h = U_h \eta_h + \mathcal{U}_h f$.

[Bramble+Pasciak+Schatz, 1986]: *The same statements hold for the Lagrange finite element approximation of u, provided U and $\mathcal{U}$ are replaced by their discrete analogues U_h and $\mathcal{U}_h$.*

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- A domain decomposition perspective ✓
  - The interface function η_h $\quad a(\eta_h, \mu) = b(\mu)$.
  - Recovery of solution u_h $\quad u_h = U_h \eta_h + \mathcal{U}_h f$.
- Hybridized methods ←
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- Eigenvalue problems
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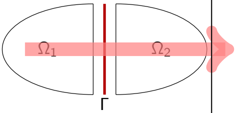
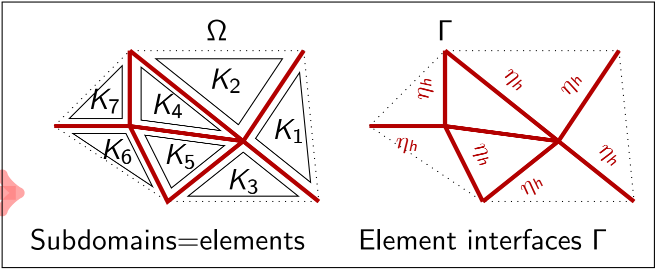
Let subdomains be elements

"Hybridized methods" are obtained by applying domain decomposition where subdomains are elements.





As we transition from the simple two-domain splitting to the case

$$\textit{subdomains } \Omega_i = \textit{elements } K_i,$$

we continue to have $a(\eta_h, \mu) = b(\mu),$ and $u_h = U_h\eta_h + \mathcal{U}_h f$.

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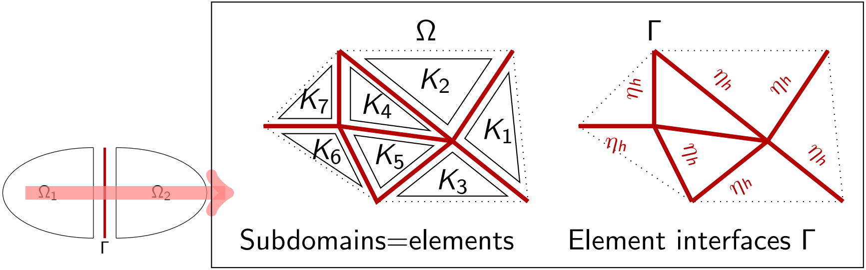


As we transition from the simple two-domain splitting to the case

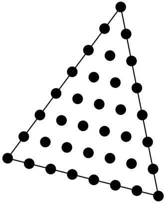
$$\textit{subdomains } \Omega_i = \textit{elements } K_i,$$

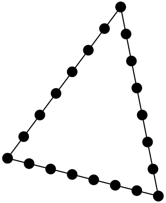
we continue to have $a(\eta_h, \mu) = b(\mu),$ and $u_h = U_h \eta_h + \mathcal{U}_h f$.

This is the "statically condensed" system.

Dimensional reduction

Static condensation is good for high order finite elements:





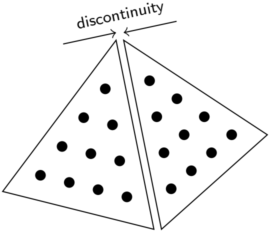
A degree $p = 7$ element $\rightarrow$ statically condensed

If $p =$ polynomial degree of FEM, then for 2D problems,

original system size		reduced system size
$O(p^2)$	$\rightarrow$	$O(p)$.

What about DG methods?

In DG (discontinuous Galerkin) methods, approximations can be discontinuous across interfaces.



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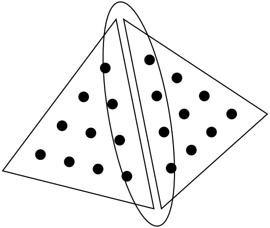

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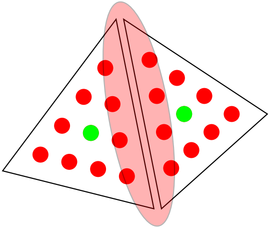

Nodes that can be condensed out (●). Remaining coupled nodes (●).

HDG methods improve the situation . . .

What is HDG?

- Many HDG methods were discovered and presented together in [Cockburn+G+Lazarov,'09] ("Unified hybridization of DG, mixed, and CG methods ...", SINUM).

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  Discontinuous Galerkin methods

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- “HDG” methods:

  Hybridized
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- Uses approximating functions with no interelement continuity.

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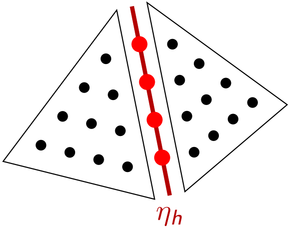
- "HDG" methods:

  Hybridized
  Discontinuous Galerkin methods

- Uses approximating functions with no interelement continuity.
- Elements are coupled through interelement traces (a separate unknown of the method).

HDG methods

In *HDG methods*, coupling is achieved through new interface variables η_h, which are called *numerical traces* (indicated by "●" below).



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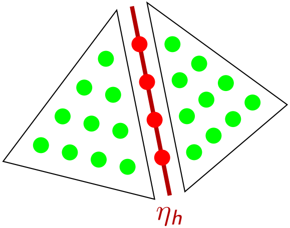


Nodes that can be condensed out (●). Remaining coupled nodes (●).

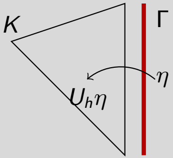
$\Longrightarrow$ *More nodes can be condensed out in HDG methods!*

Common elements of all HDG methods

1. An interface function η_h satisfying $a(\eta_h, \mu) = b(\mu)$.
2. Recovery of interior solution u_h by $u_h = U_h \eta_h + \mathcal{U}_h f$.

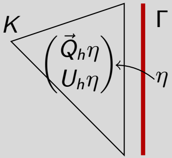
Standard condensed FEM

$$a(\eta, \mu) = \int_\Omega \vec{\nabla}(U_h \eta) \cdot \vec{\nabla}(U_h \mu)$$



HDG method

$$a(\eta, \mu) = \int_\Omega \vec{Q}_h \eta \cdot \vec{Q}_h \mu$$



$$U_h \eta \approx U\eta : \begin{cases} -\Delta(U\eta) = 0 & \text{on } K \\ U\eta = \eta & \text{on } \Gamma \\ U\eta = 0 & \text{on } \partial K \setminus \Gamma. \end{cases}$$

For HDG, use DG flux approx:

$$\vec{Q}_h \eta \approx -\vec{\nabla}(U\eta).$$

A popular HDG method

$\vec{q} + \vec{\nabla} u = 0 \implies$

$$\int_K \vec{q} \ \cdot \vec{v} - \int_K u \ \nabla \cdot \vec{v} = - \int_{\partial K} u \ (\vec{v} \cdot \vec{n})$$

$\nabla \cdot \vec{q} = f \implies$

A popular HDG method

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$\nabla \cdot \vec{q} = f \implies$

$$-\int_K \vec{\nabla} w \cdot \vec{q} + \int_{\partial K} w \, \vec{q} \cdot \vec{n} = \int_K f \, w$$

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$\nabla \cdot \vec{q} = f \implies$

$$-\int_K \vec{\nabla} w \cdot \vec{q}_h + \int_{\partial K} w \, \widehat{q}_h \cdot \vec{n} = \int_K f \, w$$

- Set $\boxed{\widehat{q}_h = \vec{q}_h + \tau(u_h - \eta)}$ to obtain a stable method for any $\tau > 0$.
- Spaces: $\vec{q}_h, u_h$ are polynomials of degree at most k.
- $\vec{Q}_h \eta = \vec{q}_h$ and $U_h \eta = u_h$ when $f = 0$.

Extension to other problems

- We used $\boxed{\widehat{q}_h = \vec{q}_h + \tau(u_h - \hat{u}_h)}$ for the Dirichlet problem.
- Such numerical flux prescriptions can be made for many problems.

Example of Euler equations, *courtesy of* Jaime Peraire (MIT):

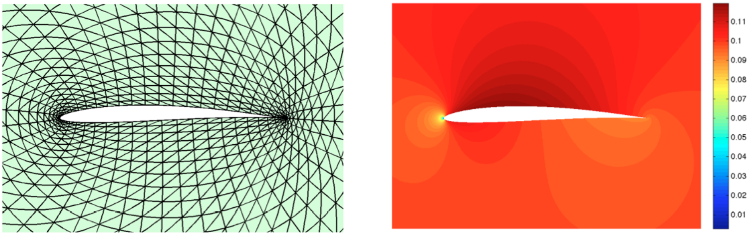

Figure 1. Inviscid flow over a Kármán-Trefftz airfoil: $M_\infty = 0.1$, $\alpha = 0$. **Detail of the mesh employed (left) and Mach number contours of the solution using fourth order polynomial approximations (right).**

$$\nabla \cdot \vec{F}(\vec{u}) = 0$$

$$-(\vec{F}(\vec{u}_h), \nabla \vec{w})_K + \langle \hat{F}_h \cdot \vec{n}, \vec{w} \rangle_{\partial K} = 0$$

$$\boxed{\hat{F}_h \cdot \vec{n} = \vec{F}(\hat{u}_h) \cdot \vec{n} + \mathcal{T}_{\hat{u}_h, \vec{u}_h}(\vec{u}_h - \hat{u}_h)}.$$

Why HDG?

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- HDG methods yield matrices of the same *size* and *sparsity* as mixed methods (finally overcoming the criticism that "all DG methods are bloated with too many unknowns").

- Stability is guaranteed for *any* positive stabilization parameter. (It does *not* have to be "sufficiently large".)

- Mixed methods require carefully crafted spaces for stability, while HDG methods offer greater *flexibility* in the choice of spaces.

- Unlike most older DG methods, HDG methods yield (provably) *optimal* error estimates for flux (and the other unknowns).

- *Coupling* methods, even across non-matching mesh interfaces, is easy.

Next

- A domain decomposition perspective ✓
  - The interface function η_h $\quad a(\eta_h, \mu) = b(\mu)$.
  - Recovery of solution u_h $\quad u_h = U_h \eta_h + \mathcal{U}_h f$.
- Hybridized methods ✓
  - Static condensation
  - HDG methods $\quad \boxed{\widehat{q}_h = \vec{q}_h + \tau(u_h - \eta_h)}$
- Eigenvalue problems ⟵
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  -
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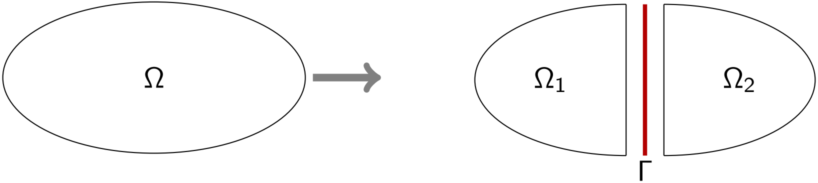
Divide the eigenproblem?

Problem:

$$\begin{aligned} -\Delta u &= \lambda u && \text{on } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$



Split Ω into Ω_1 and Ω_2.

Interface $= \Gamma$.

Condensation/Hybridization

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Source Problem

- Condensed problem at interface: Find $\eta_h \in M_h$ satisfying

$$a(\eta_h, \mu) = b(\mu) \quad \forall \mu \in M_h,$$

where

$$a(\eta, \mu) = \int_\Omega \vec{Q}_h \eta \cdot \vec{Q}_h \mu$$

$$b(\mu) = \int_\Omega (U_h \mu) f$$

Eigenproblem, by analogy...

- Could we not condense the eigenproblem to interfaces? Guess:

$$a(\eta_h, \mu) = \lambda_h \langle \eta_h, \mu \rangle \quad \forall \mu \in M_h$$

where

$$\langle \eta, \mu \rangle = \int_\Omega (U_h \eta)\,(U_h \mu).$$

Condensation/Hybridization

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Really? *Spectrum reduced!*

– Which eigenvalues disappeared?

– Condensed λ_h = Actual λ_h?

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Eigenproblem, by analogy...

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$$a(\eta_h, \mu) = \tilde{\lambda}_h \langle \eta_h, \mu \rangle \quad \forall \mu \in M_h$$

where

$$\langle \eta, \mu \rangle = \int_\Omega (U_h \eta)\,(U_h \mu).$$

Really? *Spectrum reduced!*

– Which eigenvalues disappeared?

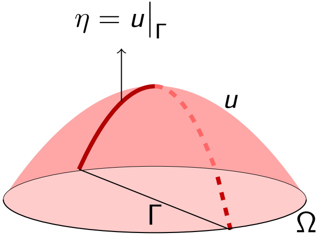
– Condensed $\tilde{\lambda}_h$ = Actual λ_h?

Recall

Returning to the simple 2-domain case temporarily, recall:

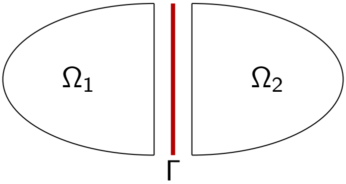
Source Problem:

$$-\Delta u = f \qquad \text{on } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega.$$



- If we know η, then

$$u = U\eta + \mathcal{U}f.$$



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Eigenproblem:

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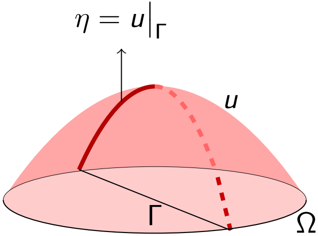
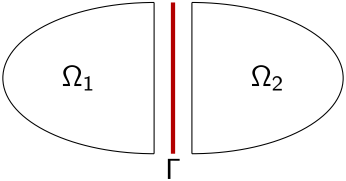


- If we know η, then

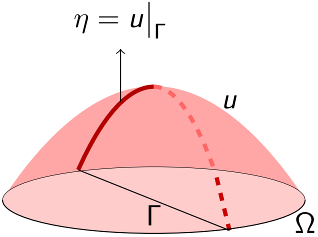
$$u = U\eta + \mathcal{U}(\lambda u).$$



Deriving the condensed eigenproblem

$$u = U\eta + \mathcal{U}(\lambda u)$$

$$-\Delta u = \lambda u \quad \text{on } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$



Deriving the condensed eigenproblem

$$u = U\eta + \mathcal{U}(\lambda u)$$
$$= U\eta + \lambda \mathcal{U}\left(U\eta + \mathcal{U}(\lambda u) \right)$$

$$-\Delta u = \lambda u \quad \text{on } \Omega,$$
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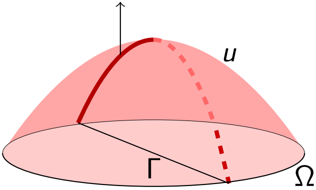

Deriving the condensed eigenproblem

$$
\begin{aligned}
u &= \boxed{U\eta + \mathcal{U}(\lambda u)} \\
&= U\eta + \lambda \mathcal{U}\left(\boxed{U\eta + \mathcal{U}(\lambda u)} \right) \\
&= \cdots [\text{recursively repeat}] \cdots \\
&= \left(I + \lambda\mathcal{U} + (\lambda\mathcal{U})^2 + \cdots\right) U\eta \\
&= (I - \lambda\mathcal{U})^{-1} U\eta,
\end{aligned}
$$

provided the series converges.

$$
\begin{aligned}
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u &= 0 \qquad \text{on } \partial\Omega.
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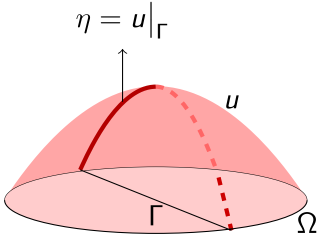

Recall definition of $\mathcal{U}f$:

$$
\begin{cases}
-\Delta(\mathcal{U}f) = f, & \text{on subdom.,} \\
\mathcal{U}f = 0, & \text{on } \partial(\text{subdom}).
\end{cases}
$$

Deriving the condensed eigenproblem

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$$
\begin{aligned}
u &= U\eta + \mathcal{U}(\lambda u) \\
&= U\eta + \lambda \mathcal{U}\left(U\eta + \mathcal{U}(\lambda u) \right) \\
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&= (I - \lambda\mathcal{U})^{-1} U\eta,
\end{aligned}
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provided the series converges.

- Series converges if subdomains small.

$$
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-\Delta u &= \lambda u && \text{on } \Omega, \\
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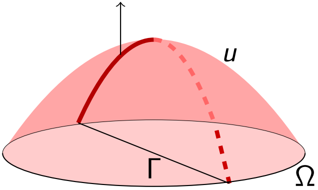
\end{aligned}
$$

$\eta = u\big|_\Gamma$



Recall definition of $\mathcal{U}f$:

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\end{cases}
$$

Deriving the condensed eigenproblem

Portland State University

$$
\begin{aligned}
u &= U\eta + \mathcal{U}(\lambda u) \\
&= U\eta + \lambda \mathcal{U}\left(U\eta + \mathcal{U}(\lambda u) \right) \\
&= \cdots \text{[recursively repeat]} \cdots \\
&= \left(I + \lambda \mathcal{U} + (\lambda \mathcal{U})^2 + \cdots \right) U\eta \\
&= (I - \lambda \mathcal{U})^{-1} U\eta,
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provided the series converges.

- Series converges if subdomains small.
- Then u can be recovered from η.

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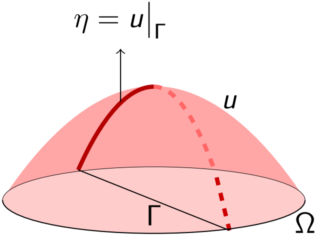

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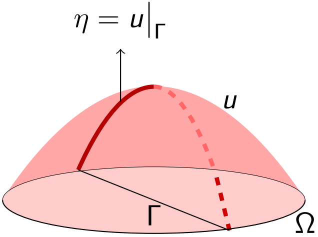

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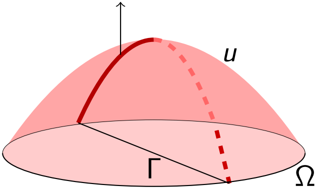

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Deriving the condensed eigenproblem

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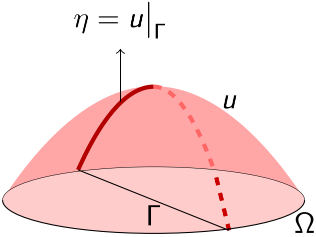

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Nonlinear eigenproblem

The preceding arguments indicate:

- It should be possible to "hybridize" or "condense" the eigenproblem to element interfaces when meshsize is small enough.

- Upon condensation, we should expect a *linear* eigenproblem to become a *nonlinear eigenproblem* of the form:

$$\text{Find } \eta : \qquad a(\eta,\mu) = \int_\Omega (U\mu) \lambda (I - \lambda \mathcal{U})^{-1} U\eta, \quad \forall \mu.$$

- The first guess that λ may solve

$$\text{Find } \eta : \qquad a(\eta,\mu) = \lambda \langle \eta, \mu \rangle, \qquad \forall \mu,$$

where $\langle \eta, \mu \rangle = \int_\Omega (U\eta)\,(U\mu)$ is *not* correct.

Condensed HDG eigenproblem

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Theorem

There is a constant $C > 0$ such that for any $\lambda_h < C/h$, the operator $I - \lambda_h \mathcal{U}$ is invertible, and moreover, λ_h satisfies

$$a(\eta_h, \mu) = \int_\Omega \lambda_h (I - \lambda_h \mathcal{U})^{-1} U\eta_h \, U\mu \qquad \forall \mu \in M_h$$

with some $\eta_h \not\equiv 0$ in M_h, if and only if the number λ_h and the functions

$$\eta_h, \quad u_h = (I - \lambda_h \mathcal{U})^{-1} U\eta_h$$

together solve the HDG eigenproblem.

$\implies$ *Condensed HDG eigenproblem does not lose lower eigenmodes.*

Perturbed interface eigenproblem

The condensed interface eigenproblem: Find $\lambda_h \in \mathbb{R}$ and $\eta_h \not\equiv 0$ satisfying

$$a(\eta_h, \mu) = \int_\Omega \lambda_h \,(I - \lambda_h \mathcal{U})^{-1} (U\eta_h)\,(U\mu) \quad \forall \mu \in M_h.$$

Perturbed interface eigenproblem: Find $\lambda_h \in \mathbb{R}$ and $\tilde{\eta}_h \not\equiv 0$ satisfying

$$a(\tilde{\eta}_h, \mu) = \tilde{\lambda}_h \int_\Omega (U\tilde{\eta}_h)\,(U\mu) \quad \forall \mu \in M_h.$$

Theorem

For any HDG eigenvalue $\lambda_h < C/h$, there is an eigenvalue $\tilde{\lambda}_h$ of the perturbed eigenproblem satisfying

$$\frac{|\lambda_h - \tilde{\lambda}_h|}{\lambda_h} \le C\,\lambda_h \tilde{\lambda}_h\, h$$

for sufficiently small h.

Perturbed interface eigenproblem

The condensed interface eigenproblem: Find $\lambda_h \in \mathbb{R}$ and $\eta_h \not\equiv 0$ satisfying

$$a(\eta_h, \mu) = \int_\Omega \lambda_h \, (I - \lambda_h \mathcal{U})^{-1} (U\eta_h) \, (U\mu) \qquad \forall \mu \in M_h.$$

Perturbed interface eigenproblem: Find $\lambda_h \in \mathbb{R}$ and $\tilde{\eta}_h \not\equiv 0$ satisfying

$$a(\tilde{\eta}_h, \mu) = \tilde{\lambda}_h \int_\Omega (U\tilde{\eta}_h) \, (U\mu) \qquad \forall \mu \in M_h.$$

Theorem

$\Longrightarrow$ *We can use the solution of the perturbed eigenproblem as initial iterates in a nonlinear solver for* λ_h*!*

Discretization errors

Eigenproblem	
$-\Delta u = \lambda u$	
Eigenfunction:	u
Eigenvalue:	λ

→

HDG discretization	
Eigenfunction:	u_h
Interface fn:	η_h
Eigenvalue:	λ_h

→

Condensation	
Interface fn:	η_h
Eigenvalue:	λ_h

Theorem

If the exact eigenfunction is smooth, then

$$|\lambda - \lambda_h| \le Ch^{2k+1}$$

for the HDG discretization using polynomials of degree at most k. The $L^2(\Omega)$- "gap" between the discrete and exact eigenspaces is $O(h^{k+1})$.

Conclusion

- A domain decomposition perspective
  - The interface function η_h $a(\eta_h, \mu) = b(\mu)$.
  - Recovery of solution u_h $u_h = U_h \eta_h + \mathcal{U}_h f$.
- Hybridized methods
  - Static condensation
  - HDG methods $\widehat{q}_h = \vec{q}_h + \tau(u_h - \eta_h)$
- Eigenvalue problems
  - HDG eigenproblem & its condensation
  - HDG eigenvalue convergence rates $O(h^{2k+1})$
  - Perturbed interface eigenproblem $a(\eta_h, \mu) = \tilde{\lambda}_h \int_\Omega U\eta_h \, U\mu$
  - Nonlinear eigenproblem $a(\eta_h, \mu) = \int_\Omega \lambda_h \, (I - \lambda_h \mathcal{U})^{-1} U\eta_h \, U\mu$