Automatically finding stable pairs of spaces in DPG schemes

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Outline

1. What?
   - CG, DG, PG, CPG, . . . , DPG
2. Why?
   - 1D 1-element example
   - 2D pure transport
   - Laplace's equation
   - Elasticity
   - Wave propagation
3. How?
   - Optimal test functions
   - Deriving an ultraweak formulation
   - Theory leading to quasioptimality.

"Petrov-Galerkin" schemes (PG)

PG schemes are distinguished by different trial and test spaces.

The problem: P.D.E.+ boundary conditions.

$\downarrow$

Variational form: Find u in a trial space U satisfying

$$b(u, v) = l(v)$$

for all v in a test space V.

$\downarrow$

Discretization: Find u_n in a discrete trial space $U_n \subset U$ satisfying

$$b(u_n, v_n) = l(v_n)$$

for all v_n in a discrete test space $V_n \subset V$.

For PG schemes, $U_n \neq V_n$ in general.

Elements of theory

- *Variational formulation:*

$$\left[\begin{array}{c}\text{Exact inf-sup condition}\\ C\|u\|_U \le \sup_{v\in V} \dfrac{|b(u,v)|}{\|v\|_V}\end{array}\right] + \left[\begin{array}{c}\text{a uniqueness}\\ \text{condition}\end{array}\right] \implies \text{wellposedness}$$

- *Babuška's theorem:*

$$\left[\begin{array}{c}\text{Discrete inf-sup condition}\\ C\|u_n\|_U \le \sup_{v_n\in V_n} \dfrac{|b(u_n,v_n)|}{\|v_n\|_V}\end{array}\right] \implies \|u-u_n\|_U \le C \inf_{w_n\in U_n} \|u-w_n\|_U.$$

- *Difficulty:* Exact inf-sup condition $\not\Longrightarrow$ Discrete inf-sup condition
- Is there a way to find a stable test space for *any* given trial space (thus giving a stable method automatically)?

The DPG method

Pick *any* $U_h \subseteq U$. The DPG method finds $u_h \in U_h$ such that

$$b(u_h, v_h) = l(v_h), \qquad \forall v_h \in V_h \equiv T(U_h),$$

where $T : U \mapsto V$ is defined by

$$(Tw, v)_V = b(w, v), \qquad \forall w \in U,\ v \in V.$$

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Motivation:

- *Q:* Which function v maximizes $\dfrac{|b(u,v)|}{\|v\|_V}$ for any given u ?
- *A:* $v = Tu$ is the maximizer. $\leftarrow$ *The optimal test function.*

DPG Idea: If the discrete test space contains the *optimal test functions*, then stability of the discrete scheme is inherited from the wellposedness.

The DPG method

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But . . . can we really compute Tu?

- For a few problems, Tu can be calculated in closed form.
- For the remaining problems. . . this idea is applicable if an *ultraweak variational formulation can be found* in a space V with a "broken" innerproduct $(\cdot, \cdot)_V$. Then, Tu can be *locally approximated*.

Next

- Examples where T can be calculated in closed form:
  - Example 1: A 1D example
  - Example 2: 2D transport
- Examples where new ultraweak formulations need to be derived:
  - Example 3: Laplace's equation
  - Example 4: Elasticity
  - Example 5: Wave propagation

Example 1: An ODE

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1D transport: $\left[\begin{array}{ll} u' = f & \text{in } (0,1), \\ u(0) = u_0 & \text{(inflow b.c.)} \end{array}\right.$

L^2 variational form:

Find $u \in L^2$, and a number $\hat{u}_1 \in \mathbb{R}$, satisfying

$$\underbrace{-\int_0^1 u v' + \hat{u}_1 v(1)}_{b(\,(u,\hat{u}_1),\, v)} = \underbrace{\int_0^1 f v + u_0 v(0)}_{l(v)}, \quad \forall v \in H^1.$$

Trial space: $U = L^2 \times \mathbb{R}$, Test space: $V = H^1$.

Example 1: An ODE

1D transport: $$\begin{cases} u' = f & \text{in } (0,1), \\ u(0) = u_0 & \text{(inflow b.c.)} \end{cases}$$

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Spectral DPG: $$\begin{cases} \text{Find } (u_p, \hat{u}_1) \in U_m \equiv P_p \times \mathbb{R}, \text{ satisfying} \\ b(\,(u_p, \hat{u}_1), v) = l(v), \qquad \forall v \in V_m = T(U_m) =? \end{cases}$$

Optimal test functions

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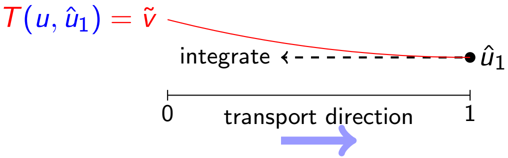
$$b(\,(u,\hat u_1),\ v) = -\int_0^1 u v' + \hat u_1 v(1), \qquad U = L^2 \times \mathbb{R}, \qquad V = H^1$$

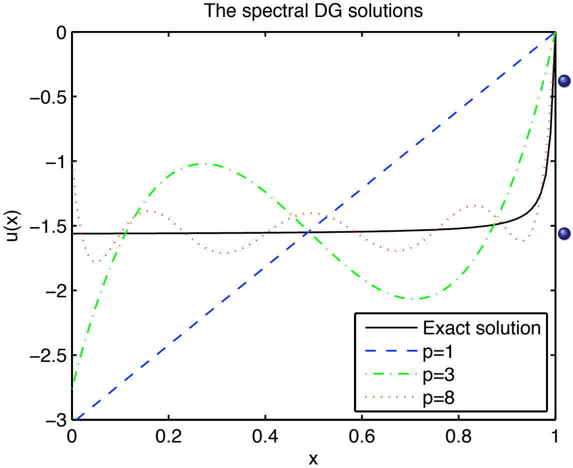
Q: Which function achieves $\displaystyle\sup_{v\in H^1}\left(\frac{b(\,(u,\hat u_1),\ v)^2}{\|v'\|_{L^2}^2 + |v(1)|^2}\right)$?

A: The maximizer is $\displaystyle\tilde v = \hat u_1 + \int_x^1 u(s)\,ds.$ $\leftarrow$ *Optimal test function*



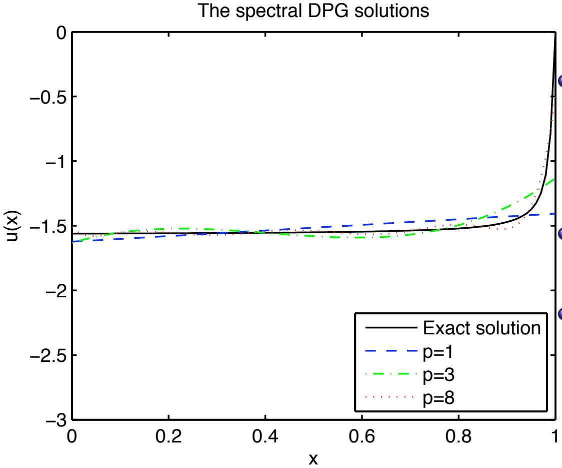
Q: If $U_m \equiv P_p \times \mathbb{R}$, what is V_m?
A: By the above formula for T, we conclude that $V_m = T(U_m) = P_{p+1}$.

What have we gained?



- Experiment: Solve 1D transport equation using DG and DPG on one element.
- Exact solution has a sharp layer at $x = 1$.

What have we gained?



- Experiment: Solve 1D transport equation using DG and DPG on one element.
- Exact solution has a sharp layer at $x = 1$.
- *DPG is more stable.* (Solution oscillates an order of magnitude less.)

Example 2: 2D transport

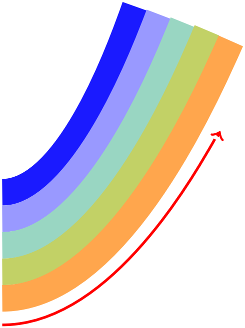
- The same ideas can be applied to multidimensional transport:

$$\vec{\beta} \cdot \vec{\nabla} u = f, \qquad \text{on } \Omega,$$
$$u = g, \qquad \text{on } \partial_{in}\Omega \text{ (inflow boundary).}$$

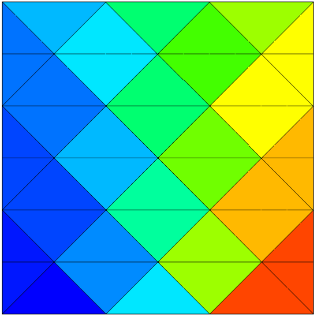
- The optimal test functions can have lines of discontiuity within mesh elements.

- Optimal h and p convergence rates can be proved for the resulting DPG method [Demkowicz+G,'10].

Crosswind diffusion in 2D transport



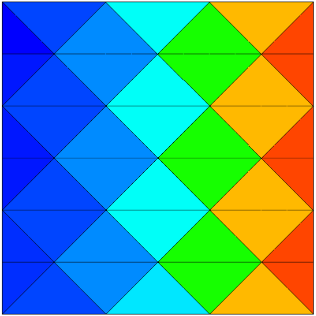
Pure transport should not diffuse materials crosswind.



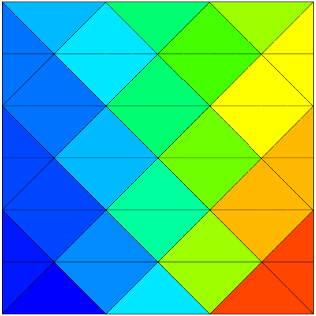
But most numerical methods do.

Experiment: Use DG and DPG for simulating vertically upward transport of linearly varying density from the bottom of the unit square.

Crosswind diffusion in 2D transport



DPG doesn't.



DG has crosswind diffusion.

Experiment: Use DG and DPG for simulating vertically upward transport of linearly varying density from the bottom of the unit square.

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Example 3: Laplace equation

$\vec{\sigma} + \vec{\nabla} u = 0 \implies$

$$\int_K \vec{\sigma} \cdot \vec{\tau} - \int_K u \ \nabla \cdot \vec{\tau} + \int_{\partial K} u \ (\vec{\tau} \cdot \vec{n}) = 0$$

$\nabla \cdot \vec{\sigma} = f \implies$

Example 3: Laplace equation

$\vec{\sigma} + \vec{\nabla} u = 0 \implies$

$$\int_K \vec{\sigma}_h \cdot \vec{\tau} - \int_K u_h \nabla \cdot \vec{\tau} + \int_{\partial K \setminus \partial\Omega} \hat{u}_h (\vec{\tau} \cdot \vec{n}) \qquad = \quad 0$$

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$\nabla \cdot \vec{\sigma} = f \implies$

$$-\int_K \vec{\nabla} v \cdot \vec{\sigma} + \int_{\partial K} v \, \vec{\sigma} \cdot \vec{n} \quad = \quad \int_K f \, v$$

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- *Traditionally,* various DG methods are obtained by setting various expressions for the *numerical trace* $\hat{u}_h$ and *numerical flux* $\hat{\sigma}_h$.

Example 3: Laplace equation

$\vec{\sigma} + \vec{\nabla} u = 0 \implies$

$$\int_K \vec{\sigma}_h \cdot \vec{\tau} - \int_K u_h \nabla \cdot \vec{\tau} + \int_{\partial K \setminus \partial\Omega} \hat{u}_h (\vec{\tau} \cdot \vec{n}) \quad = \quad 0$$

$\nabla \cdot \vec{\sigma} = f \implies$

$$-\int_K \vec{\nabla} v \cdot \vec{\sigma}_h + \int_{\partial K} v \, \hat{\sigma}_h \cdot \vec{n} \quad = \quad \int_K f \, v$$

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- *DPG methods* set both $\hat{u}_h$ and $\hat{\sigma}_h$ as unknowns.

Example 3: Laplace equation

$$\sum_K \Big[\int_K \vec{\sigma}_h \cdot \vec{\tau} - \int_K u_h \, \nabla \cdot \vec{\tau} + \int_{\partial K \setminus \partial \Omega} \hat{u}_h (\vec{\tau} \cdot \vec{n})$$

$$- \int_K \vec{\nabla} v \cdot \vec{\sigma}_h + \int_{\partial K} v \, \hat{\sigma}_h \cdot \vec{n} \Big] = \sum_K \int_K f \, v$$

$$\boxed{b(\, (\vec{\sigma}_h, u_h, \hat{u}_h, \hat{\sigma}_h), \ (\vec{\tau}, v) \,) \quad = \quad l(\vec{\tau}, v)}$$

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- *DPG methods* set both $\hat{u}_h$ and $\hat{\sigma}_h$ as unknowns.

DPG theory

Form: *Trial functions* $(\vec{\sigma}, u, \hat{u}, \hat{\sigma}_n) \in U$, *Test functions* $(\vec{\tau}, v) \in V$,

$$b(\ (\vec{\sigma}, u, \hat{u}, \hat{\sigma}_n),\ (\vec{\tau}, v)\ ) = \sum_K \Big[\int_K \vec{\sigma} \cdot \vec{\tau} - u\, \nabla \cdot \vec{\tau} + \int_{\partial K} \hat{u}\, \vec{\tau} \cdot \vec{n} - \int_K \vec{\sigma} \cdot \vec{\nabla} v + \int_{\partial K} \hat{\sigma}_n v \Big].$$

Spaces:

$\vec{\sigma} \in L^2(\Omega)^N$

$u \in L^2(\Omega)$

$\hat{u} \in$ the set of traces of $H_0^1(\Omega)$-functions on $\bigcup_K \partial K$

$\hat{\sigma}_n \in$ the set of normal traces of $H(div)$-functions on $\bigcup_K \partial K$

$\vec{\tau} \in$ "broken" $H(div)$ (i.e., $\vec{\tau}|_K \in H(div, K),\ \forall K$)

$v \in$ "broken" H^1 (i.e., $v|_K \in H^1(K),\ \forall K$)

Form:

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Discrete Spaces:

Given any $U_m \subseteq U$, we set $V_m \equiv T(U_m)$. Recall the definition of T :

$$(T\mathscr{W}, \mathscr{V})_V = b(\mathscr{W}, \mathscr{V}), \qquad \forall \mathscr{W} \in U,\ \mathscr{V} \in V.$$

For this application, the V-innerproduct is

$$((\vec{\tau}_1, v_1), (\vec{\tau}_2, v_2))_V = \sum_K \int_K \vec{\tau}_1\cdot\vec{\tau}_2 + (\nabla\cdot\vec{\tau}_1)(\nabla\cdot\vec{\tau}_2) + v_1 v_2 + \vec{\nabla} v_1\cdot\vec{\nabla} v_2.$$

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Form:

Trial functions $(\vec{\sigma}, u, \hat{u}, \hat{\sigma}_n) \in U$, *Test functions* $(\vec{\tau}, v) \in V$,

$$b(\,(\vec{\sigma}, u, \hat{u}, \hat{\sigma}_n),\,(\vec{\tau}, v)\,) = \sum_K \Big[\int_K \vec{\sigma}\cdot\vec{\tau} - u\,\nabla\cdot\vec{\tau} + \int_{\partial K} \hat{u}\,\vec{\tau}\cdot\vec{n} - \int_K \vec{\sigma}\cdot\vec{\nabla} v + \int_{\partial K} \hat{\sigma}_n v \Big].$$

Theorem

1. *This ultraweak formulation is wellposed.* [Demkowicz+G, 2011]
2. *For any $U_m \subset U$, the DPG solution $(\vec{\sigma}, u, \hat{u}, \hat{\sigma}_n)_m \in U_m$ satisfies*

$$\|(\vec{\sigma}, u, \hat{u}, \hat{\sigma}_n) - (\vec{\sigma}, u, \hat{u}, \hat{\sigma}_n)_m\|_U \le C \inf_{w_n \in U_m} \|(\vec{\sigma}, u, \hat{u}, \hat{\sigma}_n) - w_n\|_U$$

3. *This implies optimal h and p error estimates if U_m is an hp space.*

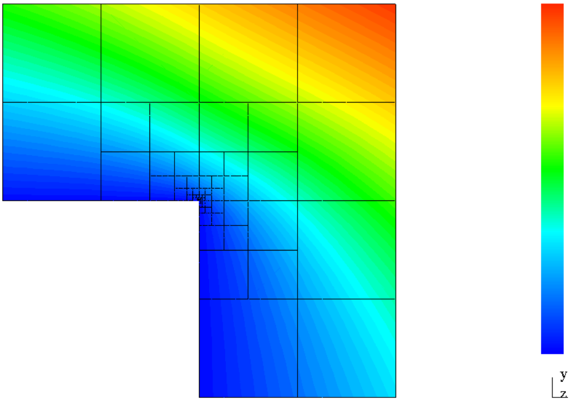
Example 4: Linear elasticity

- We can develop an ultraweak formulation in a similar fashion for

$$\begin{aligned} A\sigma - \varepsilon(u) &= 0 \\ \nabla \cdot \sigma &= f \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \sigma &= \text{stress}, \\ u &= \text{displacement}. \end{aligned}$$

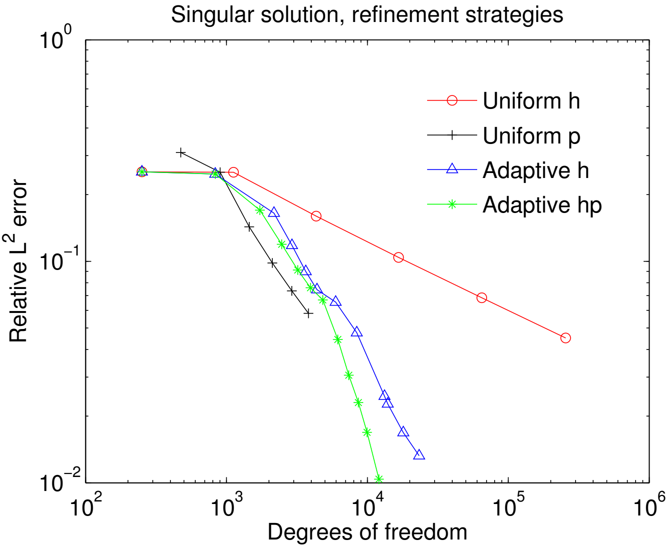
- *Traditional difficulty:*
  - Discrete stress spaces are complex due to competing requirements:
    - ⋆ σ must be a *symmetric* matrix valued function,
    - ⋆ Interelement forces must be in *equilibrium*, i.e., $\sigma \in H(div)$.
    - ⋆ Stress space must be chosen in relation to displacement space for *discrete stability*. [Arnold+Awanou+Winther, '08]
- *Advantages of DPG:*
  - Discrete stability is automatic. (Choose your favorite trial space for σ!)
  - We get *hp* optimal results if *hp* trial spaces are chosen.
  - DPG outperforms the mixed method.
  - There is no locking. [Bramwell+Demkowicz+G+Qiu,'11]

L-shaped steel



The x-component of the computed displacement (u_x).

L-shaped steel



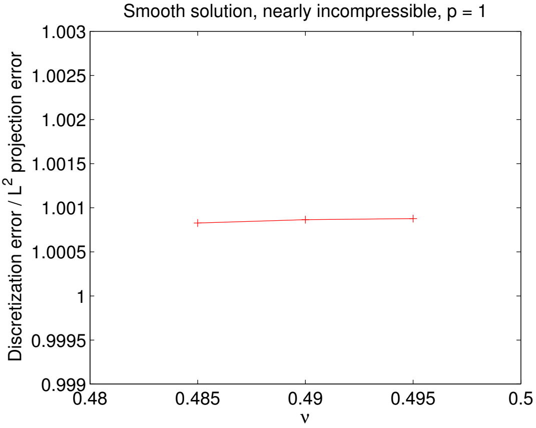
A measure of robustness

- The best any method can do is to deliver the *Best Approximation* from its trial space.
- For any method, the *ratio*

$$\frac{\text{Discretization Error}}{\text{Best Approximation Error}} \geq 1$$

  and its optimal value is 1.
- We investigate how this ratio changes as we approach incompressibility.

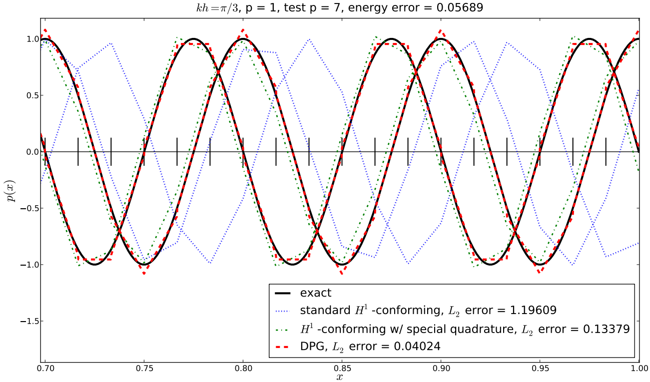
A measure of robustness



Experiment: Solve the linear elasticity system by the DPG method using DG spaces for stresses and displacements. [Bramwell+Demkowicz+G+Qiu,'11]

Example 5: Wave propagation

- Standard FEM exhibits pollution (manifested as *phase errors*).
- DPG shows smaller phase errors.



1D case: [Calo+Demkowicz+G+Muga+Pardo+Zitelli,'10]

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- Pollution errors arise because the ratio [Babuška+Sauter,'97]

$$\left(\frac{\text{Discretization Error}}{\text{Best Approximation Error}}\right) \quad \text{depends on the frequency } \omega.$$

- For standard FEM in 1D, [Ihlenburg,'98]

$$\frac{\|u-u_h\|_{L^2}}{\|u\|_{L^2}} \le C(\omega) \inf_{w_h \in U_n} \frac{\|u-w_h\|_{L^2}}{\|u\|_{L^2}}, \qquad \text{with } C(\omega) = C_1 + C_2\omega.$$

- For DPG in 1D, [Calo+Demkowicz+G+Muga+Pardo+Zitelli,'10]

$$C(\omega) \le C \qquad \text{(independent of } \omega\text{)}.$$

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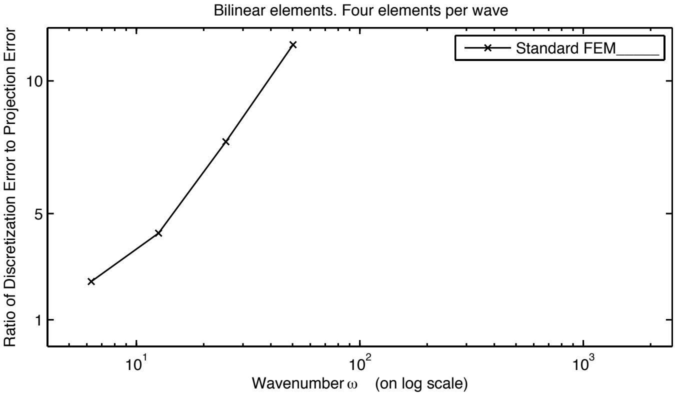

2D case: Work in progress

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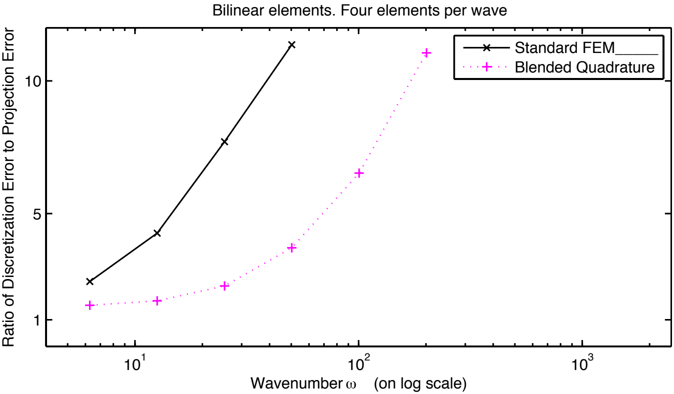

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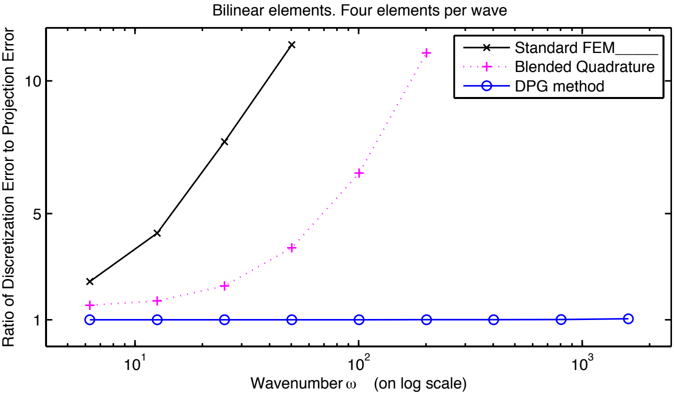
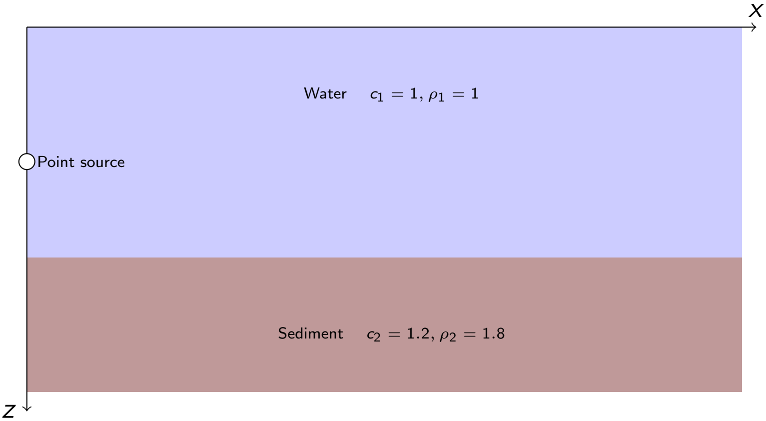


2D case: Work in progress

Pekeris waveguide

A shallow-water acoustic waveguide. (Just four elements per wavelength.)



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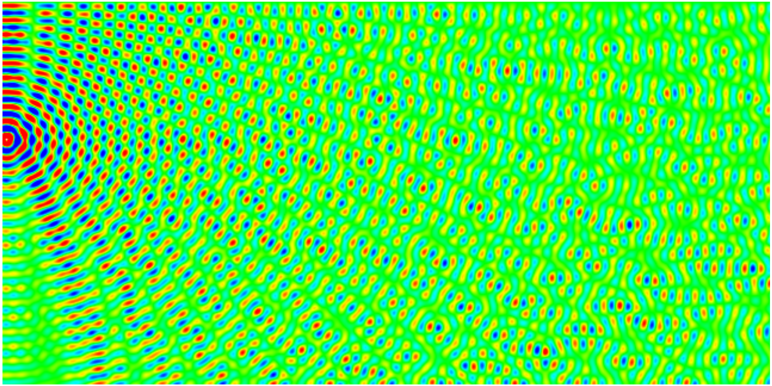
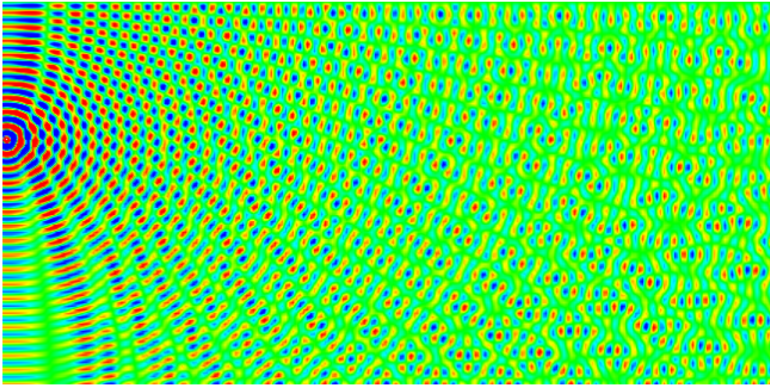


Discrete solution

Biquadratic FEM with blended quadrature

Pekeris waveguide

A shallow-water acoustic waveguide. (Just four elements per wavelength.)



Discrete solution
Bilinear DPG method

Pekeris waveguide

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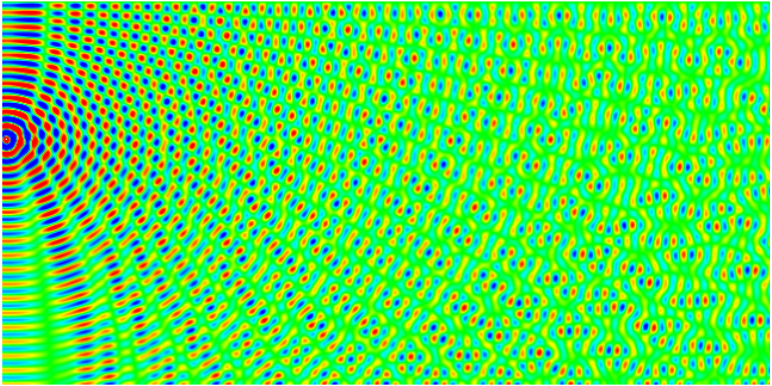

The exact solution

Conclusions

- Wellposedness implies discrete stability through the concept of optimal test functions.
- The DPG method often outperforms DG and other standard methods.
- We can prove optimal hp convergence estimates.
- The DPG methods exhibit extraordinary stability with respect to variations in h, p, and singular parameters.