Schémas de discrétisation en temps adaptatifs pour l'équation des ondes

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En collaboration avec

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24 mars 2010

Second Order Wave Equations :

- Acoustics :

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0$$

- Elastodynamics :

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathbf{div}\, \underline{\underline{C}}\, \underline{\underline{\varepsilon}}(\boldsymbol{u}) = 0$$

- Maxwell :

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} + c^2\, \mathbf{curl}\ \mathbf{curl}\ \boldsymbol{u} = 0$$

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After space discretization we obtain :

$$M \frac{d^2 U}{dt^2} + KU = 0$$

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We consider space discretization methods such that M and K are symmetric positive matrices and M is (block-)diagonal (FEM with mass lumping or DG methods).

Introduction — Space Discretization

$$M^{\frac{1}{2}}\frac{d^2U}{dt^2} + \underbrace{M^{-\frac{1}{2}}KM^{-\frac{1}{2}}}_{A}\,\underbrace{M^{\frac{1}{2}}U}_{Y} = 0$$

$$\frac{d^2Y}{dt^2} + AY = 0$$

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Classical Leap Frog Scheme :

$$\frac{Y(t+\Delta t) - 2Y(t) + Y(t-\Delta t)}{\Delta t^2} = \frac{d^2Y}{dt^2}(t) + O(\Delta t^2)$$

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Classical Leap Frog Scheme :

$$\frac{Y(t+\Delta t) - 2Y(t) + Y(t-\Delta t)}{\Delta t^2} = AY(t) + O(\Delta t^2)$$

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$$\frac{Y^{n+1} - 2Y^n + Y^{n-1}}{\Delta t^2} = -AY^n.$$

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Energy Conservation

$$E^{n+\frac{1}{2}} = \left\langle \frac{Y^{n+1} - Y^n}{\Delta t}, \frac{Y^{n+1} - Y^n}{\Delta t} \right\rangle + \langle AY^{n+1}, Y^n \rangle$$

$$\frac{d^2Y}{dt^2} + AY = 0$$

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Energy Conservation

$$E^{n+\frac{1}{2}} = \left\langle \left(I - \frac{\Delta t^2}{4}A\right)\frac{Y^{n+1} - Y^n}{\Delta t}, \frac{Y^{n+1} - Y^n}{\Delta t}\right\rangle$$

$$+ \left\langle A\frac{Y^{n+1} + Y^n}{2}, \frac{Y^{n+1} + Y^n}{2}\right\rangle$$

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CFL Condition

The scheme is stable if :

$$I - \frac{\Delta t^2}{4} A \text{ and } A \text{ are symmetric positive}$$

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$$0 \leq \lambda_A \leq \frac{4}{\Delta t^2}$$

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The scheme is stable under the CFL condition :

$$\Delta t \leq \alpha_{LF} h$$

Higher Order Schemes, Global Time Stepping

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Modified Equation Scheme :

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The scheme is stable if

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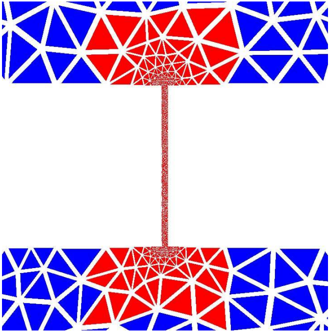
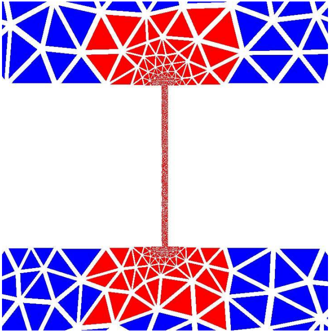


CFL Condition

We want the new scheme to satisfy

$$\Delta t^{\text{coarse}} \leq \alpha_{ME} h^{\text{coarse}} \text{ and } \Delta t^{\text{fine}} \leq \alpha_{ME} h^{\text{fine}}$$



Higher Order Schemes, Global Time Stepping

CFL Condition

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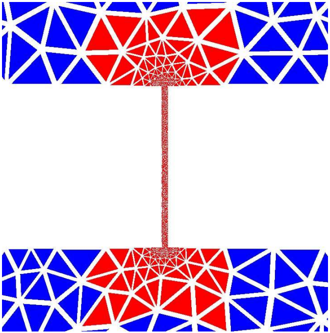
$$\Delta t^{\text{coarse}} \leq \alpha_{ME} h^{\text{coarse}} \text{ and } \Delta t^{\text{fine}} \leq \alpha_{ME} h^{\text{fine}} \approx \alpha_{ME} h^{\text{coarse}}/p$$



Bibliography

POems Team : Bécache, Collino, Fouquet, Joly, Rodríguez

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- Optimal stability condition
- Requires the introduction of a Lagrange Multiplier
- Implicit scheme on the interface

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- First-order Maxwell system
- Conservation of energy
- Optimal stability condition
- Explicit or implicit scheme on the interface

Bibliography

Hairer, Lubich and Wanner (2002), Leimkuhler and Reich (2004) : Local Time Stepping for ODE's (second order scheme)

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Hairer, Lubich and Wanner (2002), Leimkuhler and Reich (2004) : Local Time Stepping for ODE's (second order scheme)

Diaz, Grote (2009) : High-Order Local Time Stepping for the Wave Equation.

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Outline

1. Fourth order local time-stepping.

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3. Numerical Results

Auxiliary Function

At each time step n we define an auxiliary function

$$Q_n(\tau) = \frac{Y(n\Delta t - \tau) + Y(n\Delta t + \tau)}{2}$$

for $\tau \in [-\Delta t\,;\, \Delta t]$.

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This function is obviously even and satisfy :

$$\begin{cases} \dfrac{d^2 Q_n}{d\tau^2}(\tau) = -A Q_n(\tau), \\ Q_n(0) = Y(n\Delta t), \qquad \dfrac{dQ_n}{d\tau}(0) = 0, \end{cases}$$

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After having solved this equation, $Y((n+1)\Delta t)$ can be computed using $Y((n+1)\Delta t) = -Y((n-1)\Delta t) + 2Q_n(\Delta t)$

Two different ways to solve the auxiliary equation

First Way

Solve

$$\begin{cases} \dfrac{d^2 Q_n}{d\tau^2}(\tau) = -A Q_n(\tau), \\ Q_n(0) = Y(n\Delta t), \qquad \dfrac{dQ_n}{d\tau}(0) = 0, \end{cases}$$

by a fourth order modified equation scheme of time step $\Delta t/p$ and compute $Y((n+1)\Delta t) = -Y((n-1)\Delta t) + 2Q_n(\Delta t)$.

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by a fourth order modified equation scheme of time step $\Delta t/p$ and compute $Y((n+1)\Delta t) = -Y((n-1)\Delta t) + 2Q_n(\Delta t)$.

Remark

It is equivalent to solve the original equation by a fourth order modified equation scheme of time step $\Delta t/p$.

Two different ways to solve the auxiliary equation

Second Way

Use a fourth order approximation of $AQ_n(\tau)$:

$$AQ_n(\tau) \approx AQ_n(0) + \frac{\tau^2}{2} A \frac{d^2 Q_n(0)}{d\tau^2} = AY(n\Delta t) - \frac{\tau^2}{2} AAY(n\Delta t),$$

then solve

$$\begin{cases} \dfrac{d^2 Q_n}{d\tau^2}(\tau) = -AY(n\Delta t) + \dfrac{\tau^2}{2} AAY(n\Delta t), \\ Q_n(0) = Y(n\Delta t), \qquad \dfrac{dQ_n}{d\tau}(0) = 0, \end{cases}$$

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by a fourth order modified equation scheme of time step $\Delta t/p$ and compute $Y((n+1)\Delta t) = -Y((n-1)\Delta t) + 2Q_n(\Delta t)$.

Remark

Is is equivalent to solve the original equation by a fourth order modified equation scheme of time step Δt, whatever is p.

Local Time-Stepping

$$\frac{d^2 Q_n}{d\tau^2} + A Q_n = 0$$

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Let us now split Q_n in two parts :

$$Q_n = \begin{bmatrix} Q_n^{\text{coarse}} \\ Q_n^{\text{fine}} \end{bmatrix}$$

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$$Q_n = \begin{bmatrix} Q_n^{\text{coarse}} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ Q_n^{\text{fine}} \end{bmatrix} = (I - P)Q_n + PQ_n, \text{ with } P^2 = P$$

Local Time-Stepping

$$\frac{d^2 Q_n}{d\tau^2} + A(I-P)Q_n + APQ_n = 0$$

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Local Time-Stepping

$$\frac{d^2 Q_n}{d\tau^2} + A(I-P)Q_n + APQ_n = 0$$

Idea

Approximate only $A(I-P)Q_n(\tau)$ by

$$A(I-P)Q_n(\tau) \approx A(I-P)Q_n(0) + \frac{\tau^2}{2} A(I-P)\frac{d^2 Q_n(0)}{d\tau^2}$$

Local Time-Stepping

$$\frac{d^2 Q_n}{d\tau^2} + A(I-P)Q_n + APQ_n = 0$$

Idea

Approximate only $A(I-P)Q_n(\tau)$ by

$$A(I-P)Q_n(\tau) \approx A(I-P)Y(n\Delta t) - \frac{\tau^2}{2}A(I-P)AY(n\Delta t)$$

$$\frac{d^2 Q_n}{d\tau^2} + A(I-P)Q_n + APQ_n = 0$$

Idea

Approximate only $A(I-P)Q_n(\tau)$ by

$$A(I-P)Q_n(\tau) \approx A(I-P)Y(n\Delta t) - \frac{\tau^2}{2}A(I-P)AY(n\Delta t)$$

So that Q_n is the solution to

$$\left| \begin{array}{l} \dfrac{d^2 Q_n(\tau)}{d\tau^2} + A(I-P)Y(t) - \dfrac{\tau^2}{2}A(I-P)AY(n\Delta t) + APQ_n(\tau) = 0 \\ Q_n(0) = Y(t) \\ Q_n'(0) = 0 \end{array} \right.$$

Algorithm

Computation of $Q(\Delta t)$

We solve
$$\left|\begin{array}{l} \dfrac{d^2}{d\tau^2}Q_n(\tau) + A(I-P)Y^n - \dfrac{\tau^2}{2}A(I-P)AY^n + APQ_n(\tau) = 0 \\ Q_n(0) = Y^n \\ Q'_n(0) = 0 \end{array}\right.$$
from $\tau = 0$ to $\tau = \Delta t$, using a fourth order Modified Equation Scheme with a time step $\dfrac{\Delta t}{p}$.

Algorithm

Computation of $Q(\Delta t)$

$$Q_n^0 = Y^n$$

$$V_1 = -A(I-P)Y^n - APQ_n^0 = -AY^n$$

$$V_2 = A(I-P)AY^n - APV_1$$

$$Q_n^{\frac{1}{p}} = Q_n^0 + \frac{\Delta t^2}{2p^2}V_1 + \frac{\Delta t^4}{24p^4}V_2$$

For $i = 1..p-1$

$$V_1 = -A(I-P)Y^n + \frac{1}{2}\left(\frac{i\Delta t}{p}\right)^2 A(I-P)AY^n - APQ_n^{\frac{i}{p}}$$

$$V_2 = A(I-P)AY^n - APV_1$$

$$Q_n^{\frac{i+1}{p}} = 2Q_n^{\frac{i}{p}} - Q_n^{\frac{i-1}{p}} + \frac{\Delta t^2}{p^2}V_1 + \frac{\Delta t^4}{12p^4}V_2$$

Endfor

Algorithm

Computation of Y^{n+1}

$$Y^{n+1} = 2Q_n^1 - Y^{n-1}$$

Algorithm

Computation of Y^{n+1}

$$Y^{n+1} = 2Q_n^1 - Y^{n-1}$$

This algorithm requires only one multiplication by $A(I-P)$ and p multiplications by AP.

Energy Conservation

This algorithm can be rewritten as

$$\frac{Y^{n+1} - 2Y^n + Y^{n-1}}{\Delta t^2} = -A_p Y^n$$

where A_p is defined by

$$A_p = A - \frac{\Delta t^2}{12} A^2 - \frac{2}{p^2} \sum_{j=1}^{2(p-1)} \left(\frac{\Delta t}{p}\right)^{2(j+1)} \beta_j^p (AP)^j A^2.$$

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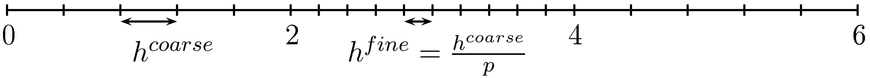
Property

AA_p is symmetric and the following energy is conserved

$$E^{n+\frac{1}{2}} = \frac{1}{2} \left[\left\langle \left(A - \frac{\Delta t^2}{4} AA_p \right) \frac{Y_{n+1} - Y_n}{\Delta t}, \frac{Y_{n+1} - Y_n}{\Delta t} \right\rangle + \left\langle AA_p \frac{Y_{n+1} + Y_n}{2}, \frac{Y_{n+1} + Y_n}{2} \right\rangle \right].$$

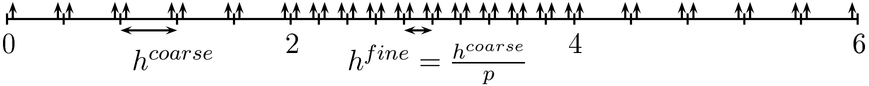
CFL Condition, Experimental Results



CFL Condition, Experimental Results

h^{coarse}

$h^{fine} = \frac{h^{coarse}}{p}$



$$\alpha = 10, \Delta t \leq 0.265h \Longrightarrow \Delta t_{opt} = 0.265h^{coarse}$$

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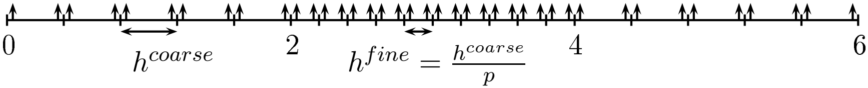

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$$p = 2, h^{coarse} = 0.2$$

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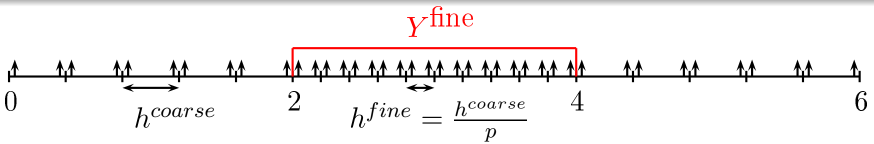

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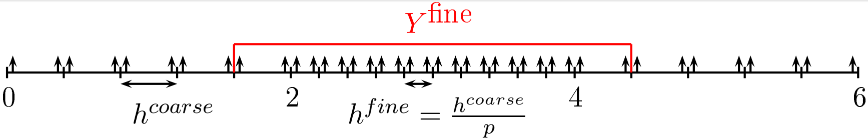

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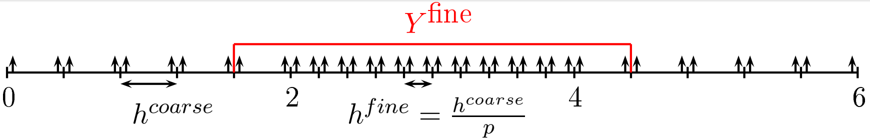

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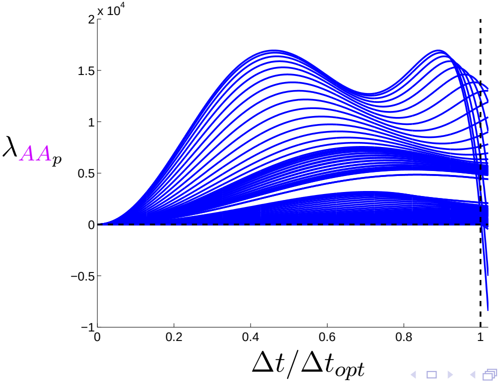
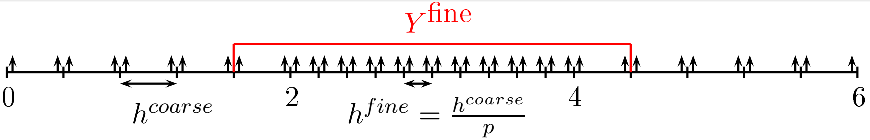
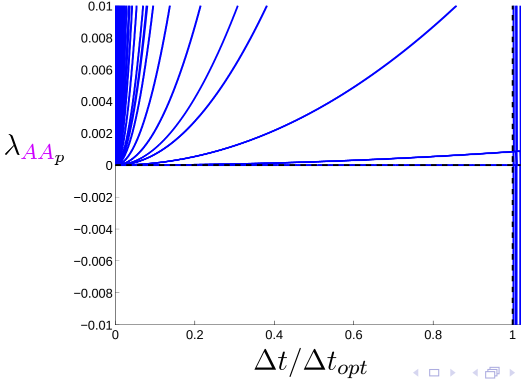


CFL Condition, Experimental Results

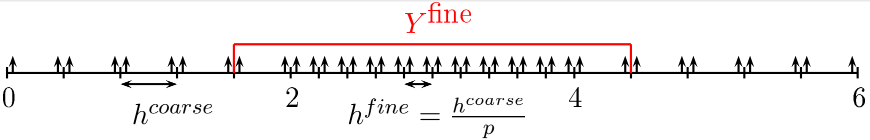


$$\alpha = 10, \Delta t \leq 0.265h \Longrightarrow \Delta t_{opt} = 0.265h^{coarse}$$

$$p = 2, h^{coarse} = 0.2$$



CFL Condition, Experimental Results



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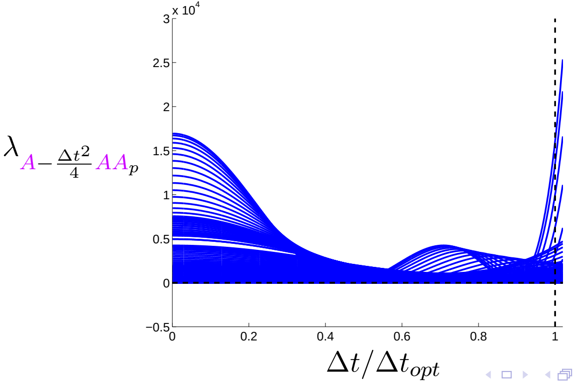
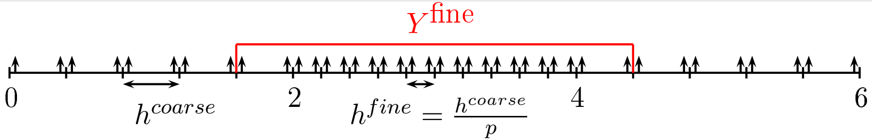


CFL Condition, Experimental Results



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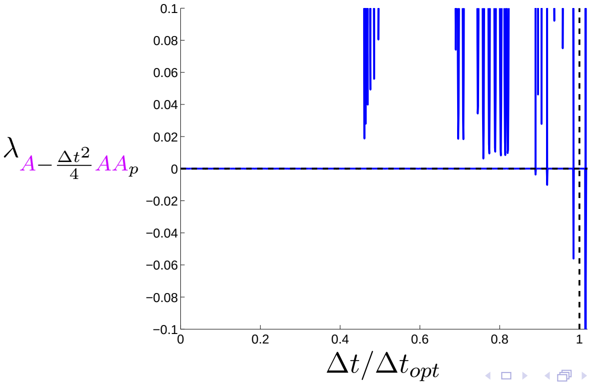
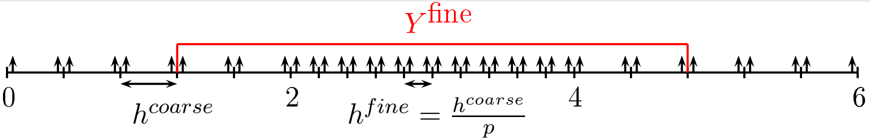


CFL Condition, Experimental Results



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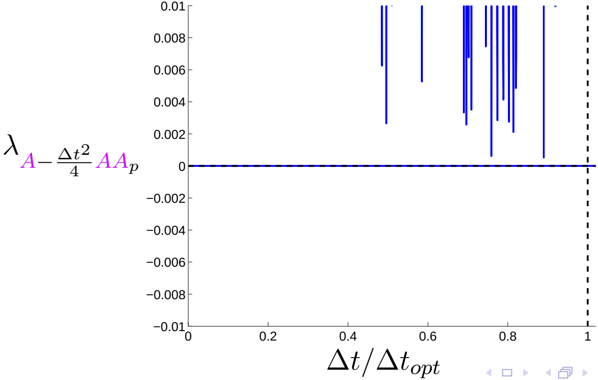
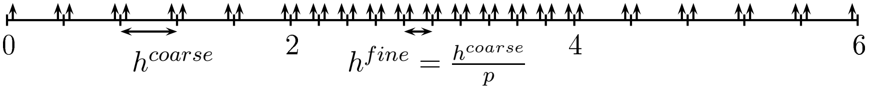


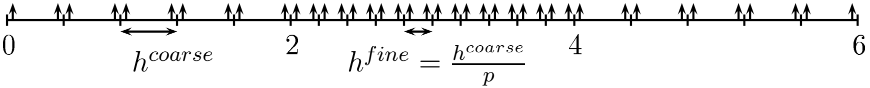


$$\alpha = 10, \Delta t \leq 0.265h \Longrightarrow \Delta t_{opt} = 0.265h^{coarse}$$

CFL		
	Overlap	
h^{coarse}	1	2
0.5	0.85	1
0.2	0.84	1
0.1	0.86	1

Ratio $\Delta t_p / \Delta t_{opt}$ for $p = 2$

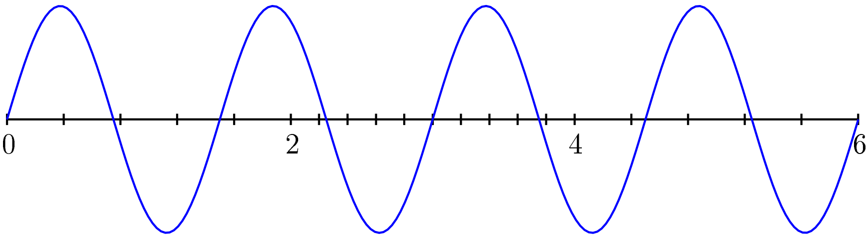
CFL Condition, Experimental Results



$$\alpha = 10, \Delta t \leq 0.265h \Longrightarrow \Delta t_{opt} = 0.265h^{coarse}$$

CFL		
	Overlap	
p	1	2
2	0.85	1
3	0.603	1
4	0.4	1
6	0.3	1
7	0.4	1

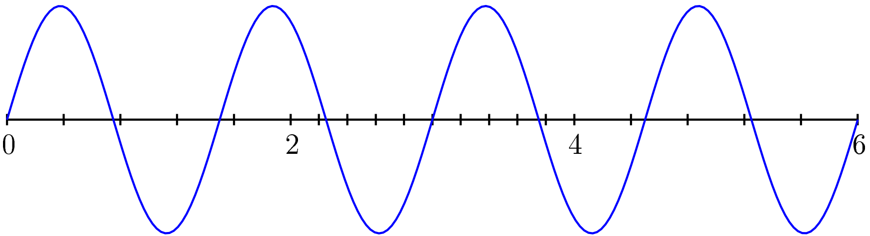
Ratio $\Delta t_p / \Delta t_{opt}$ for $h^{\text{coarse}} = 0.2$



Initial Condition

$$U(0,x) = \sin\left(\frac{8\pi}{6}x\right), \quad U'(0,x) = -\frac{8\pi}{6}\cos\left(\frac{8\pi}{6}x\right)$$

$$\Longrightarrow U_{ex}(t,x) = \sin\left(\frac{8\pi}{6}(x-t)\right)$$



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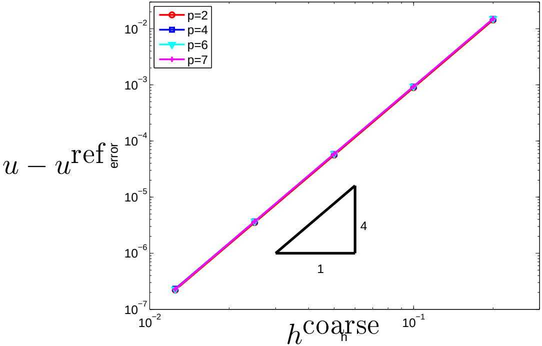
Space Discretization

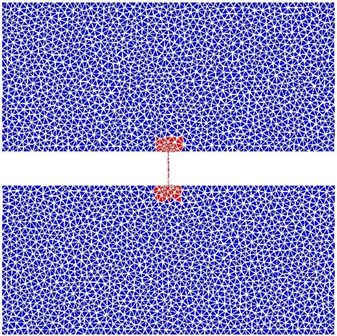
IPDG, $\mathcal{P}^3$ – elements with $\alpha = 7$.

Numerical Experiments in 1D

Order of convergence
p=2, 4, 6, 7



Numerical Experiments in 2D

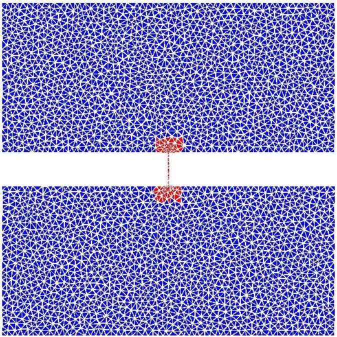


Computational Domain

$\Omega = [-0.5\,; 0.5] \times [-0.5\,; 0.5]$ with Neumann Condition

Width of the slot : 0.004

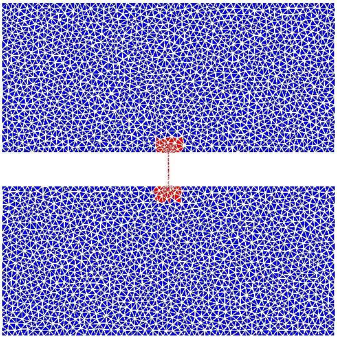
Numerical Experiments in 2D



Space Discretization

IPDG, $\mathcal{P}^3$-elements with $\alpha = 11$, $\Delta t \leq 0.14h$

Numerical Experiments in 2D



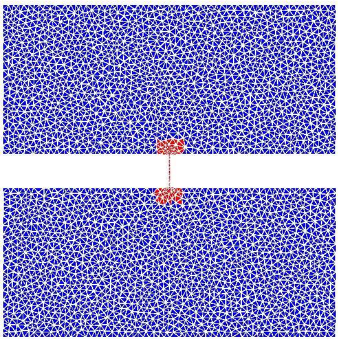
Space Discretization

IPDG, $\mathcal{P}^3$-elements with $\alpha = 11$, $\Delta t \leq 0.14h$

Time Discretization

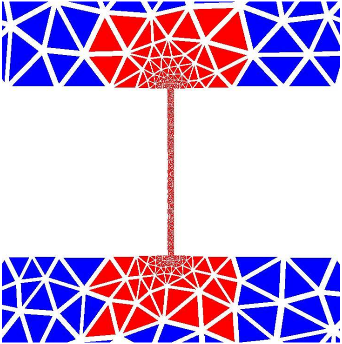
Order 4 in time, using the modified equation technique

Mesh



$$h^{\text{coarse}} = 0.0125,\ h^{\text{fine}} = 7.62.10^{-4} \approx h^{\text{coarse}}/16.44$$
$$\Delta t = 0.14 h^{\text{coarse}},\ p = 17$$

Mesh



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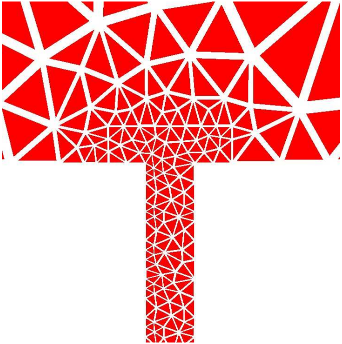

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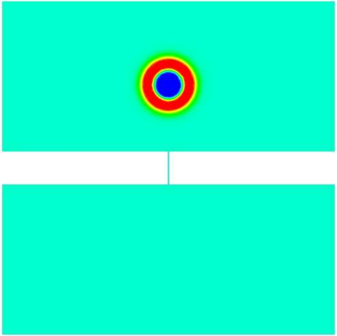
Solution



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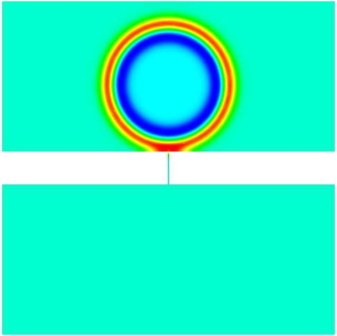

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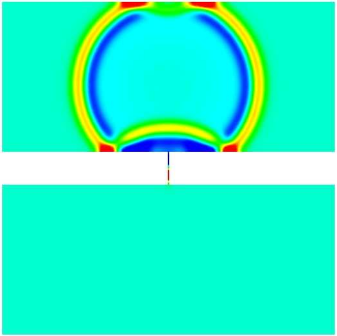

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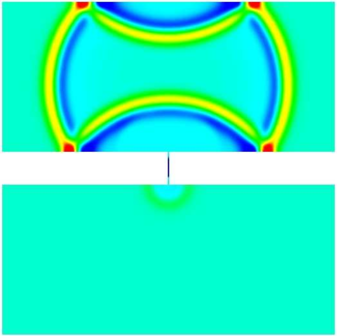

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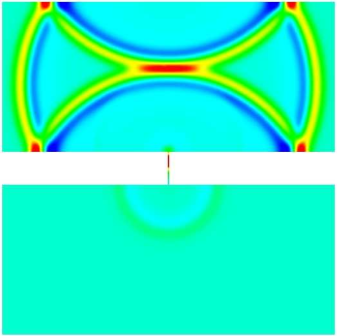

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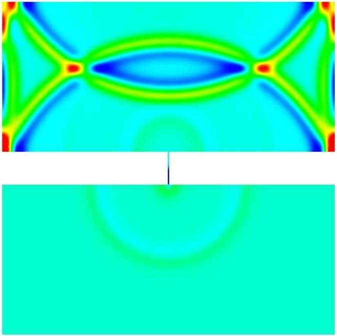

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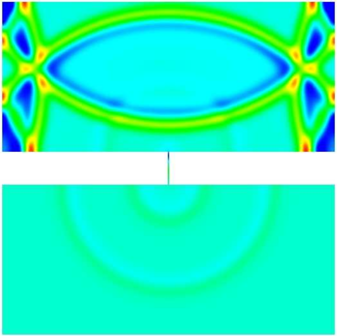

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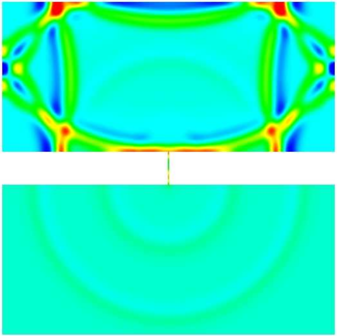

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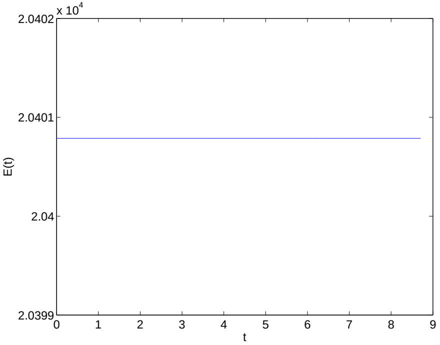
Energy



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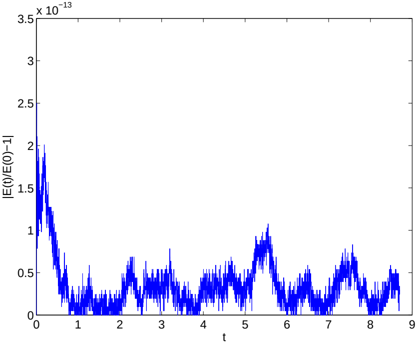

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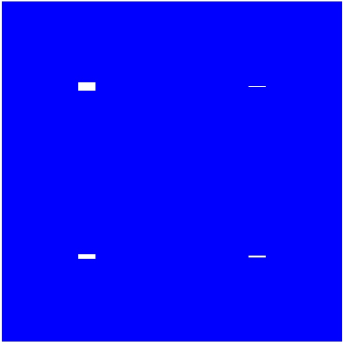
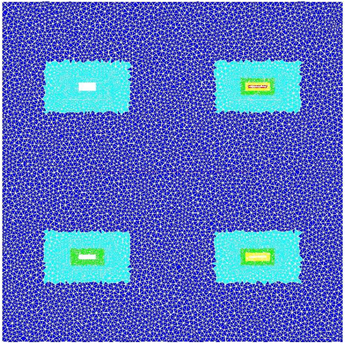


Multilevel Local Time Stepping



Solve

$$\left|\begin{array}{l} \dfrac{d^2 Q_n(\tau)}{d\tau^2} + A(I-P)Y(t) - \dfrac{\tau^2}{2}A(I-P)AY(n\Delta t) + APQ_n(\tau) = 0 \\ Q_n(0) = Y(t) \\ Q_n'(0) = 0 \end{array}\right.$$

for $\tau \in [0\,;\,\Delta t]$ with a time-step $\Delta\tau = \Delta t/p$ in the fine grid and $\Delta\tau/q$ in the very fine grid.

Multilevel Local Time Stepping

Solve

$$\left|\begin{array}{l} \dfrac{d^2Q_n(\tau)}{d\tau^2} + A(I-P)Y(t) - \dfrac{\tau^2}{2}A(I-P)AY(n\Delta t) + APQ_n(\tau) = 0 \\ Q_n(0) = Y(t) \\ Q_n'(0) = 0 \end{array}\right.$$

for $\tau \in [0\,;\,\Delta t]$ with a time-step $\Delta\tau = \Delta t/p$ in the fine grid and $\Delta\tau/q$ in the very fine grid.

Auxiliary Function

At each small time step m we define an auxiliary function

$$R_m(\theta) = \frac{Q_n(m\Delta\tau - \theta) + Q_n(m\Delta\tau + \theta)}{2}$$

for $\theta \in [-\Delta\tau\,;\,\Delta\tau]$.

Multilevel Local Time Stepping

Auxiliary Function

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$$R_m(\theta) = \frac{Q_n(m\Delta\tau - \theta) + Q_n(m\Delta\tau + \theta)}{2}$$

for $\theta \in [-\Delta\tau\,;\,\Delta\tau]$.

This function is obviously even and satisfy :

$$\begin{cases} \dfrac{d^2 R_m}{d\theta^2}(\theta) &= A(I-P)Y(n\Delta t) \\ &- \dfrac{(\tau-\theta)^2+(\tau+\theta)^2}{4} A(I-P)AY(n\Delta t) + APR_m(\theta), \\ R_m(0) = Q_n(m\Delta\tau), & \dfrac{dR_m}{d\tau}(0) = 0, \end{cases}$$

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$$\begin{cases} \dfrac{d^2 R_m}{d\theta^2}(\theta) & = \quad A(I-P)Y(n\Delta t) \\ & - \quad \dfrac{(\tau-\theta)^2+(\tau+\theta)^2}{4} A(I-P)AY(n\Delta t)+APR_m(\theta), \\ R_m(0) = Q_n(m\Delta\tau), & \dfrac{dR_m}{d\tau}(0) = 0, \end{cases}$$

After having solved this equation, $Q_n((m+1)\Delta\tau)$ can be computed using $Q_n((m+1)\Delta\tau) = -Q_n((m-1)\Delta\tau) + 2R_m(\Delta\tau)$

$$\frac{d^2 R_m}{d\theta^2}(\theta) = A(I-P)Y(n\Delta t) - \frac{(\tau-\theta)^2+(\tau+\theta)^2}{4} A(I-P)AY(n\Delta t) + APR_m(\theta)$$

Multilevel Local Time Stepping

$$\frac{d^2 R_m}{d\theta^2}(\theta) = A(I-P)Y(n\Delta t) - \frac{(\tau-\theta)^2+(\tau+\theta)^2}{4}A(I-P)AY(n\Delta t)$$
$$+ A(P-P_1)R_m(\theta) + AP_1R_m(\theta)$$

Multilevel Local Time Stepping

$$\frac{d^2 R_m}{d\theta^2}(\theta) = A(I-P)Y(n\Delta t) - \frac{(\tau-\theta)^2+(\tau+\theta)^2}{4}A(I-P)AY(n\Delta t)$$
$$+ A(P-P_1)R_m(\theta) + AP_1R_m(\theta)$$

Idea

Approximate only $A(P-P_1)R_m(\theta)$ by

$$A(P-P_1)R_m(\theta) \approx A(P-P_1)R_m(0) + \frac{\theta^2}{2}A(P-P_1)\frac{d^2 R_m(0)}{d\theta^2},$$

$$\frac{d^2 R_m}{d\theta^2}(\theta) = A(I-P)Y(n\Delta t) - \frac{(\tau-\theta)^2+(\tau+\theta)^2}{4}A(I-P)AY(n\Delta t)$$
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Idea

Approximate only $A(P-P_1)R_m(\theta)$ by

$$A(P-P_1)R_m(\theta) \approx A(P-P_1)Q_n(m\Delta\tau) - \frac{\theta^2}{2}A(P-P_1)\frac{d^2 R_m(0)}{d\theta^2},$$

Multilevel Local Time Stepping

$$\frac{d^2 R_m}{d\theta^2}(\theta) = A(I-P)Y(n\Delta t) - \frac{(\tau-\theta)^2+(\tau+\theta)^2}{4}A(I-P)AY(n\Delta t) + A(P-P_1)R_m(\theta) + AP_1R_m(\theta)$$

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with

$$\frac{d^2 R_m(0)}{d\theta^2} = A(I-P)Y(n\Delta t) + APQ_n(m\Delta\tau) - \frac{\tau^2}{2}A(I-P)AY(n\Delta t).$$

Multilevel Local Time Stepping

$$\frac{d^2 R_m}{d\theta^2}(\theta) = A(I-P)Y(n\Delta t) - \frac{(\tau-\theta)^2+(\tau+\theta)^2}{4} A(I-P)AY(n\Delta t)$$
$$+ A(P-P_1)Q_n(m\Delta\tau) - \frac{\theta^2}{2} A(P-P_1)\frac{d^2 R_m(0)}{d\theta^2} + AP_1 R_m(\theta)$$

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Algorithm for the computation of Y^{n+1}

- Computation of $Q_n^{1/p}$:
  - Compute R_m^1 by a fourth order Modified Equation Scheme ;
  - Compute $Q_n^{1/p} = R_m^1$;
- For $i = 1..p-1$
  - Computation of $Q_n^{(i+1)/p}$:
    - Compute R_m^1 by a fourth order Modified Equation Scheme ;
    - Compute $Q_n^{(i+1)/p} = Q_n^{(i-1)/p} + 2R_m^1$;
- Compute $Y^{(n+1)/p} = Y^{(n-1)/p} + 2Q_n^1$.

Energy Conservation

This algorithm can be rewritten as

$$\frac{Y^{n+1} - 2Y^n + Y^{n-1}}{\Delta t^2} = -A_{p,q} Y^n$$

where $A_{p,q}$ is such that $AA_{p,q}$ is symmetric.

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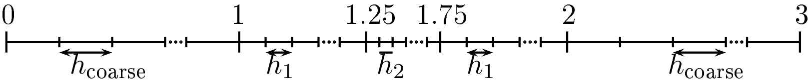
where $A_{p,q}$ is such that $AA_{p,q}$ is symmetric.

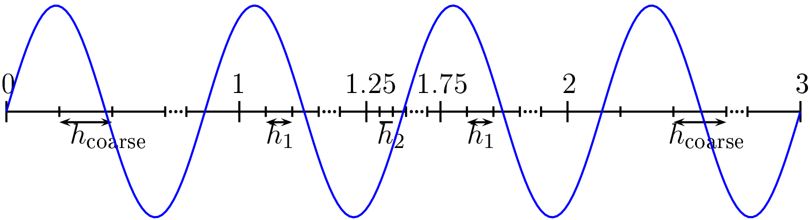
Corollary

The following energy is conserved

$$E^{n+\frac{1}{2}} = \frac{1}{2}\left[\left\langle\left(A - \frac{\Delta t^2}{4} AA_{p,q}\right)\frac{Y_{n+1} - Y_n}{\Delta t}, \frac{Y_{n+1} - Y_n}{\Delta t}\right\rangle + \left\langle AA_{p,q}\frac{Y_{n+1} + Y_n}{2}, \frac{Y_{n+1} + Y_n}{2}\right\rangle\right].$$

Numerical Experiments in 1D





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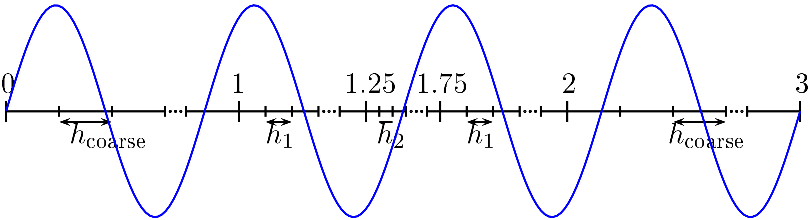

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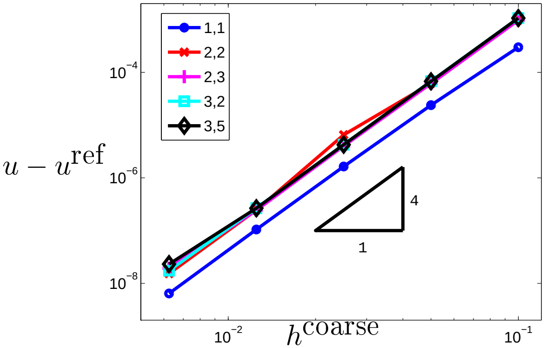
Space Discretization

IPDG, $\mathcal{P}^3$ – elements with $\alpha = 7$.

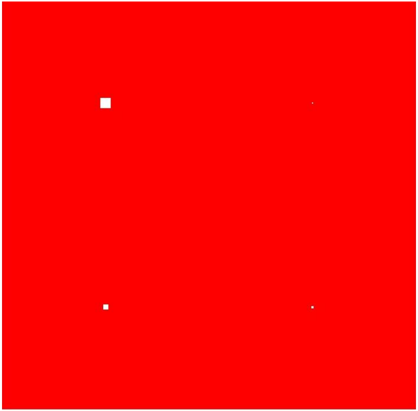
Numerical Experiments in 1D

Order of convergence

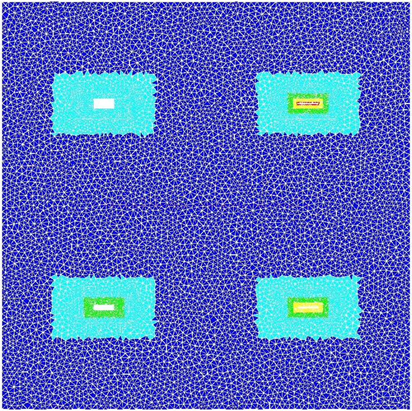
(p,q)= (1,1), (2,2), (2,3), (3,2), (3,5)



Computational Domain

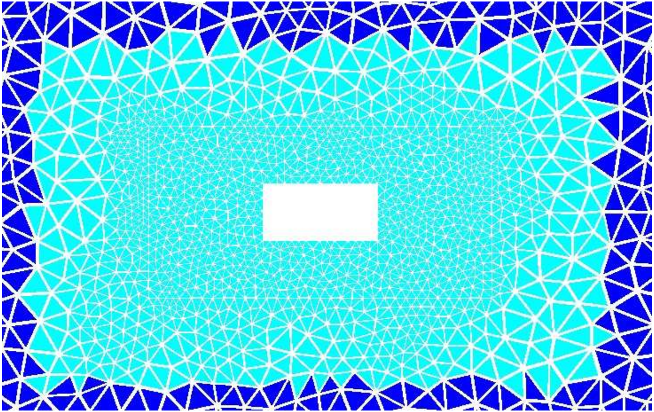


Mesh



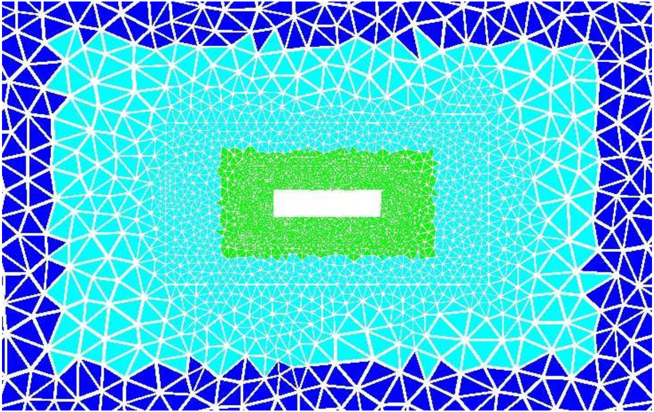
Top left corner

$$p = 4$$



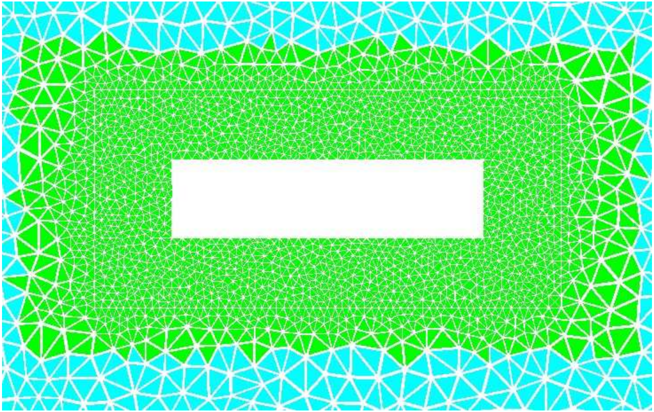
Bottom left corner

$$p_1 = 3, p_2 = 3$$



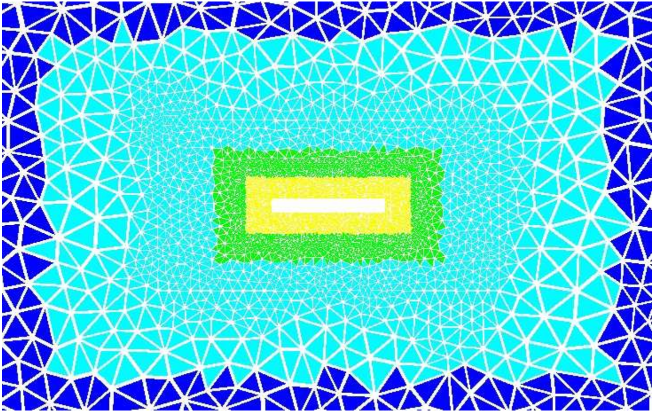
Bottom left corner

$$p_1 = 3, p_2 = 3$$



Bottom right corner

$$p_1 = 5, p_2 = 2, p_3 = 3$$



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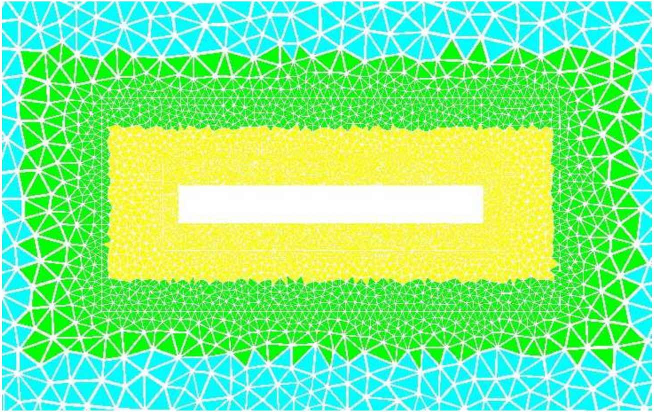
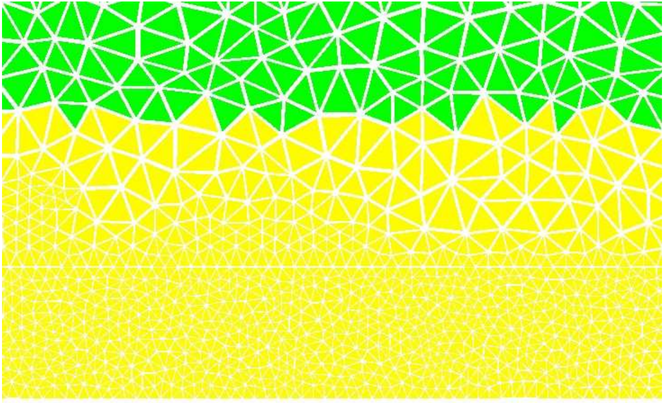
$$p_1 = 5, p_2 = 2, p_3 = 3$$



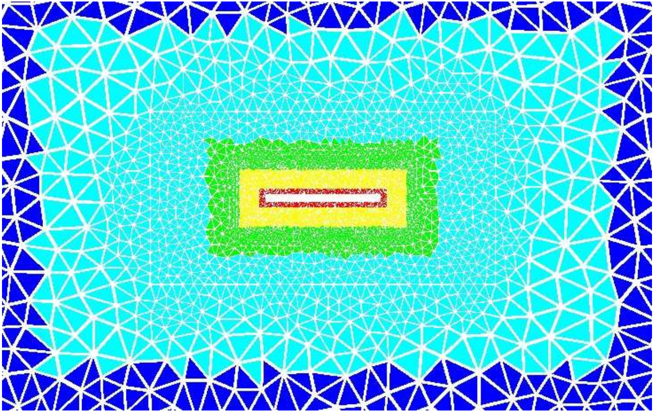
Bottom right corner

$$p_1 = 5, p_2 = 2, p_3 = 3$$



Top right corner

$$p_1 = 4, p_2 = 2, p_3 = 5, p_4 = 4$$



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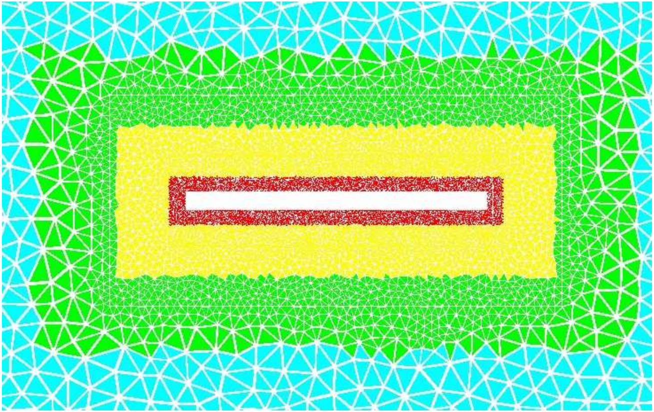
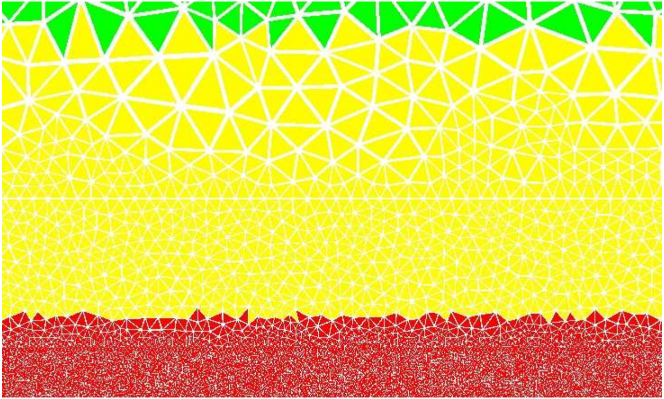
$$p_1 = 4, p_2 = 2, p_3 = 5, p_4 = 4$$



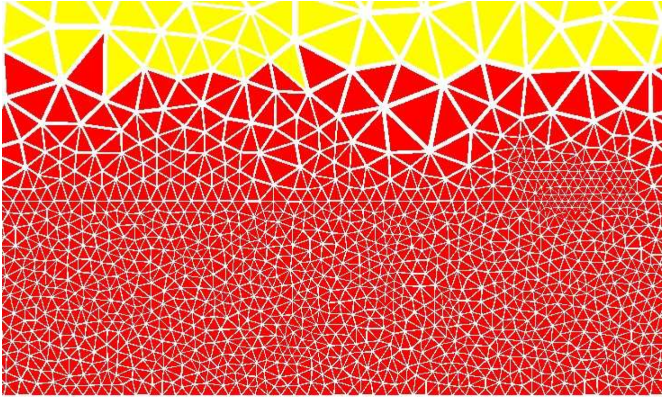
Top right corner

$$p_1 = 4, p_2 = 2, p_3 = 5, p_4 = 4$$



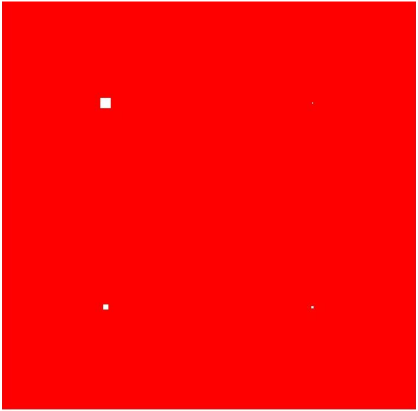
Top right corner

$$p_1 = 4, p_2 = 2, p_3 = 5, p_4 = 4$$

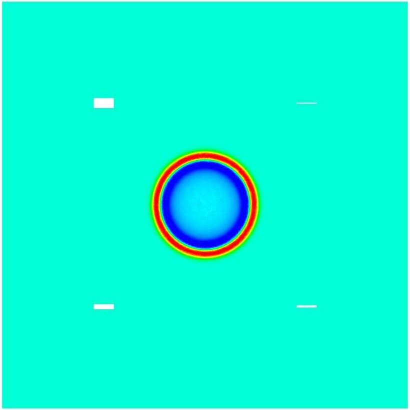


Numerical Experiments in 2D

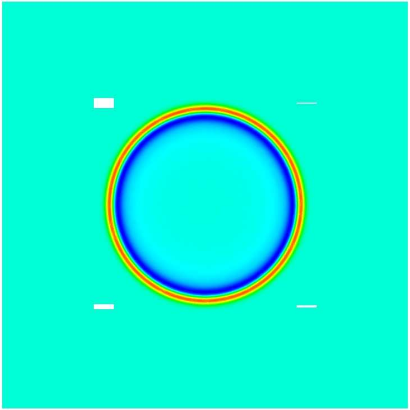
$$\Delta t = \Delta t_{opt}$$



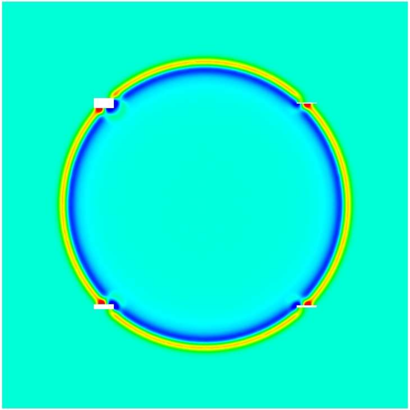
Solution



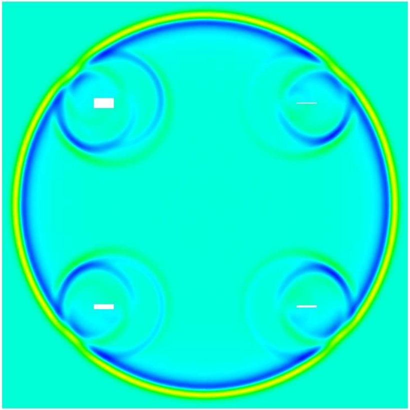
Solution



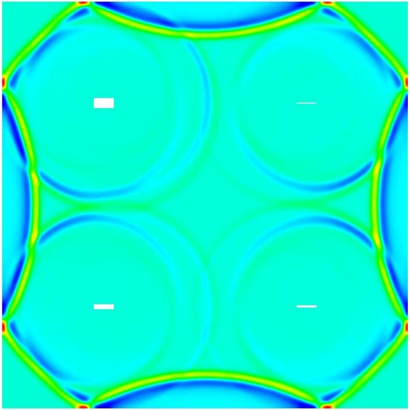
Solution



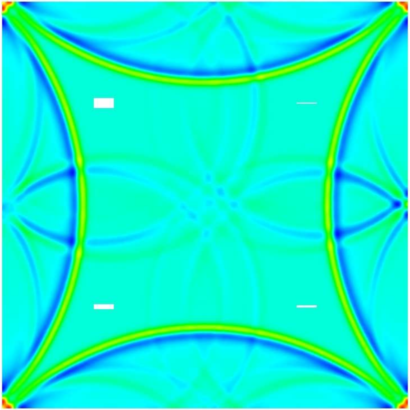
Solution



Solution



Solution



Algorithm of a 4/2 order local time-stepping scheme

New method : we consider the equation

$$\frac{d^2 Q_n}{d\tau^2}(\tau) + A(I-P)Y^n - \frac{\tau^2}{2}A(I-P)AY^n + APQ_n(\tau) = 0$$

and we discretize

$$(I-P)\frac{d^2}{d\tau^2}Q_n(\tau)$$

by a fourth order Modified equation scheme and

$$P\frac{d^2}{d\tau^2}Q_n(\tau)$$

by a second order Leap-Frog scheme

Algorithm of a 4/2 order local time-stepping scheme

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$$\frac{d^2 Q_n}{d\tau^2}(\tau) + A(I-P)Y^n - \frac{\tau^2}{2}A(I-P)AY^n + APQ_n(\tau) = 0$$

so that

$$(I-P)\frac{Q_n^{\frac{i+1}{p}} - 2Q_n^{\frac{i}{p}} + Q_n^{\frac{i-1}{p}}}{\Delta\tau^2} = (I-P)\frac{d^2 Q_n}{d\tau^2}(\tau) + \frac{\Delta\tau^2}{12}(I-P)\frac{d^4 Q_n}{d\tau^4}(\tau)$$

and

$$P\frac{Q_n^{\frac{i+1}{p}} - 2Q_n^{\frac{i}{p}} + Q_n^{\frac{i-1}{p}}}{\Delta\tau^2} = P\frac{d^2 Q_n}{d\tau^2}(\tau)$$

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so that

$$\frac{Q_n^{\frac{i+1}{p}} - 2Q_n^{\frac{i}{p}} + Q_n^{\frac{i-1}{p}}}{\Delta\tau^2} = -A(I-P)Y^n + \frac{\tau^2}{2}A(I-P)AY^n - APQ_n^{\frac{i}{p}}$$

$$+\frac{\Delta\tau^2}{12}(I-P)\left(A(I-P)AY^n - AP\frac{d^2 Q_n}{d\tau^2}(\tau)\right)$$

Algorithm of a 4/2 order local time-stepping scheme

New method : we consider the equation

$$\frac{d^2 Q_n}{d\tau^2}(\tau) + A(I-P)Y^n - \frac{\tau^2}{2}A(I-P)AY^n + APQ_n(\tau) = 0$$

so that

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$$+\frac{\Delta\tau^2}{12}(I-P)A(I-P)AY^n - \frac{\Delta\tau^2}{12}(I-P)AP\frac{d^2 Q_n}{d\tau^2}(\tau)$$

Algorithm of a 4/2 order local time-stepping scheme

New method : we consider the equation

$$\frac{d^2 Q_n}{d\tau^2}(\tau) + A(I-P)Y^n - \frac{\tau^2}{2}A(I-P)AY^n + APQ_n(\tau) = 0$$

Finally

$$\frac{Q_n^{\frac{i+1}{p}} - 2Q_n^{\frac{i}{p}} + Q_n^{\frac{i-1}{p}}}{\Delta\tau^2} = -A(I-P)Y^n + \frac{\tau^2}{2}A(I-P)AY^n - APQ_n^{\frac{i}{p}} + \frac{\Delta\tau^2}{12}(I-P)A(I-P)AY^n$$

Energy Conservation

This algorithm can be rewritten as

$$\frac{Y^{n+1} - 2Y^n + Y^{n-1}}{\Delta t^2} = -A_p Y^n$$

where A_p is defined by

$$A_p = A - \frac{\Delta t^2}{12} A(I-P)A - \frac{2}{p^2} \sum_{j=1}^{2(p-1)} \left(\frac{\Delta t}{p}\right)^{2(j+1)} \beta_j^p (AP)^j A^2.$$

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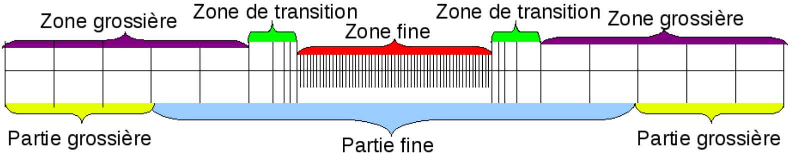
Property

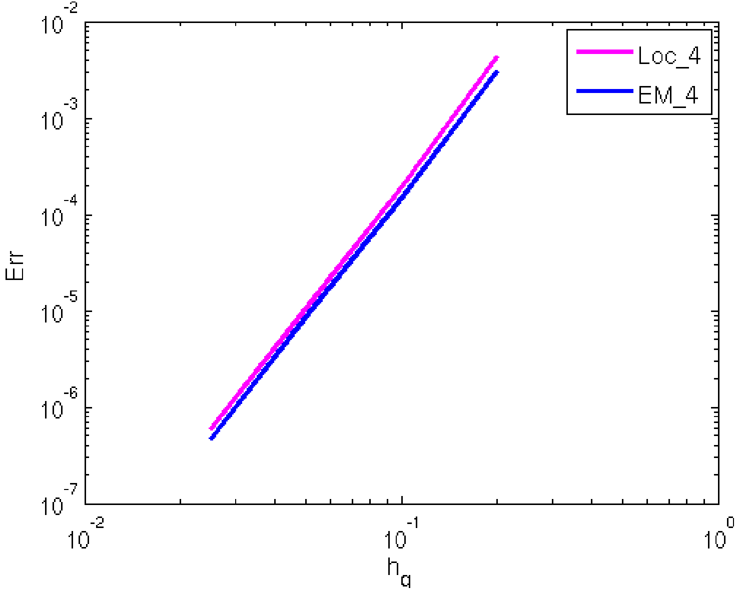
AA_p is symmetric and the following energy is conserved

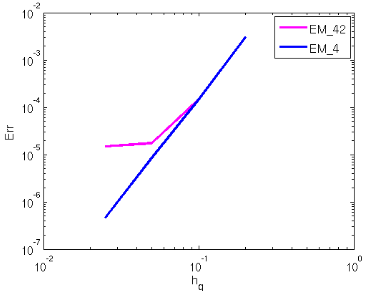
$$E^{n+\frac{1}{2}} = \frac{1}{2}\Big[\Big\langle\Big(A - \frac{\Delta t^2}{4} AA_p\Big)\frac{Y_{n+1} - Y_n}{\Delta t}, \frac{Y_{n+1} - Y_n}{\Delta t}\Big\rangle + \Big\langle AA_p \frac{Y_{n+1} + Y_n}{2}, \frac{Y_{n+1} + Y_n}{2}\Big\rangle\Big].$$



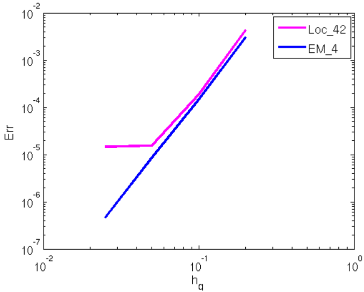
- Fine zone : $h_f = 0.1/160$, meshed with $P1$ or $P3$ elements ;
- Transition zone : meshed with $P1$ or $P3$ elements ;
- Coarse zone : $h_c = 0.1, 0.05, 0.025, 0.0125, 0.006125$, meshed with $P3$ elements.

Fourth order time scheme, $P3$ Polynomials



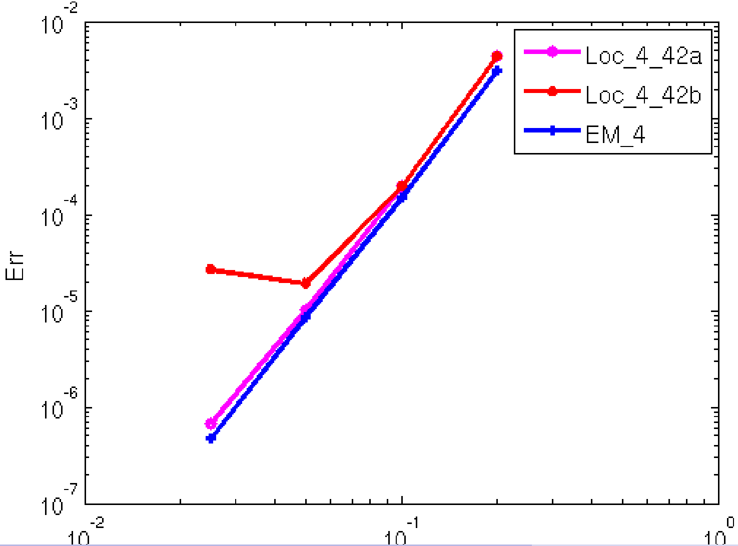


Without local time stepping

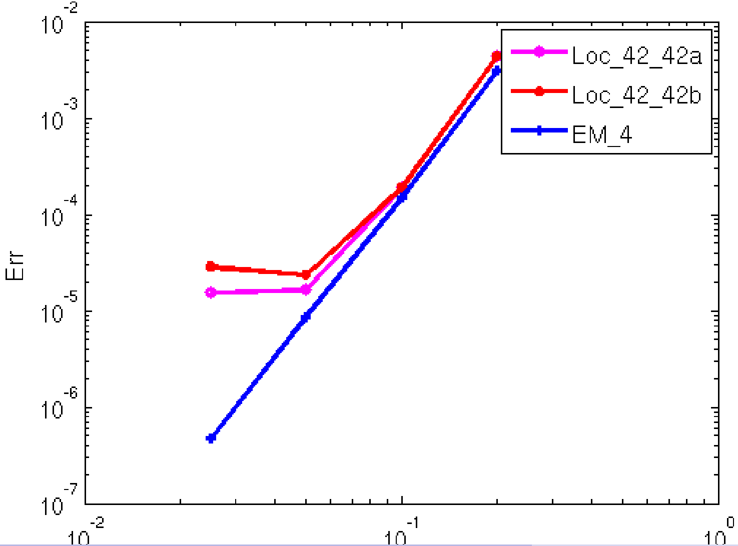


With local time stepping

4/2 order time scheme, $P3$ Polynomials, local time stepping



4/2 order time scheme, $P3/P1$ Polynomials, local time stepping



Concluding Remarks

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- This method is fully explicit.
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- The CFL is condition is “optimal”.
- It can be extended to any even order in time.
- K, M must be symmetric positive and M must be (block-)diagonal
  $\Rightarrow$ computing M^{-1} must be cheap.