

Solving Multiphysics PDE Problems with Space-Time Adaptive *hp*-FEM on Dynamical Meshes

P. Solin et al.

University of Nevada, Reno, USA

Institute of Thermomechanics, Prague, Czech Republic

<http://hpfem.org>

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Acknowledgement

Team members:

- David Andrs
- Ondrej Certik
- Jakub Cerveny
- Lenka Dubcova
- Pavel Kus

Collaborators:

- I. Dolezel, P. Karban, B. Ulrych (electrical engineering)
- J. Kruis (civil engineering)
- G. Hansen, J. Ragusa (nuclear engineering)
- D. Kuzmin, V. Kumar (computational fluid dynamics)
- B. Granger, P. Winkler (quantum chemistry)
- D. Koracin, I. Zaliapin (atmospherical sciences)
- G. Bebis, K. Bekris (computer vision, robotics)

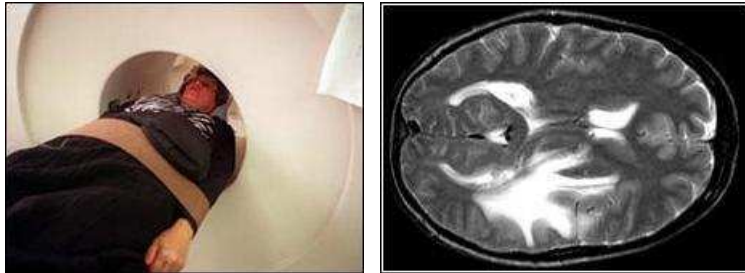


Figure: Magnetic resonance imaging (MRI) – Maxwell's equations.

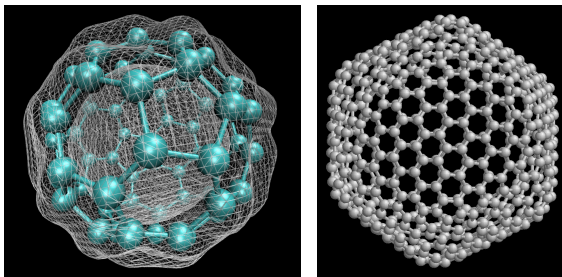


Figure: Fullerene molecules C_{60} and C_{540} – Schrödinger equation.

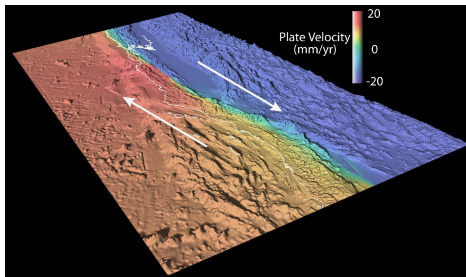


Figure: Contact problems in geodynamics – viscoplastic flow.



Figure: Tornado (nonlinear PDE system based on Navier-Stokes equations).



Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

S ... price of underlying stock,

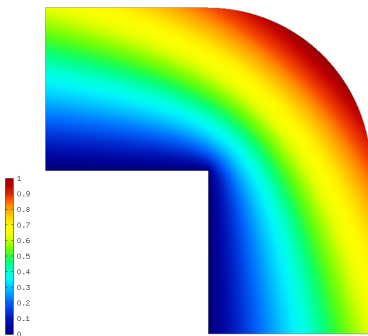
V ... option on S at time t , $V = V(S, t)$.

Our goal

Adaptive, higher-order computational methods that are PDE-independent.

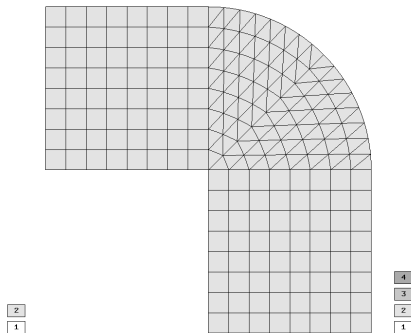
Higher-order elements

(\neq higher-order method)

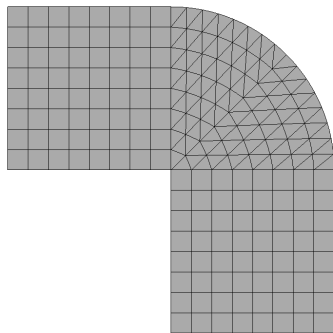


Example problem.

Higher-order elements (example 1)

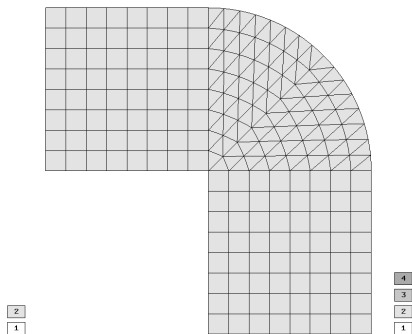


$err = 2.7\%$



$err = 1.2\%$

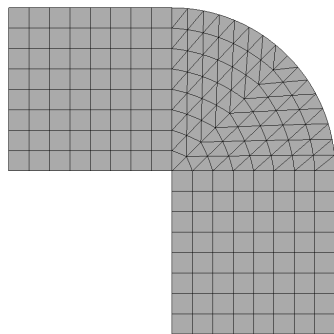
Higher-order elements (example 1)



$err = 2.7 \%$

$ndof = 705$

$err \cdot ndof = 1903$

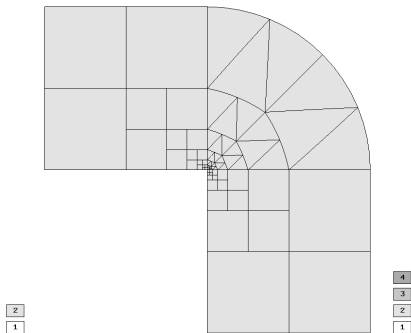


$err = 1.2 \%$

$ndof = 2900$

$err \cdot ndof = 3480$

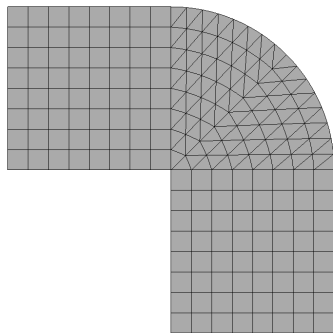
Higher-order elements (example 2)



$err = 0.92 \%$

$ndof = 173$

$err \cdot ndof = 159$

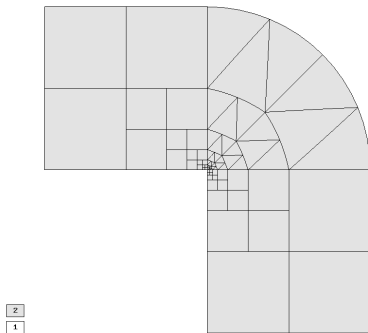


$err = 1.2 \%$

$ndof = 2900$

$err \cdot ndof = 3480$

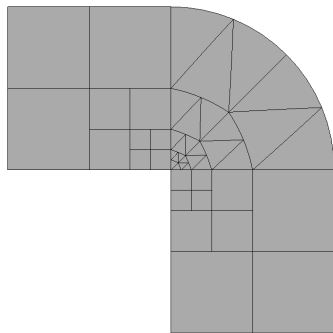
Higher-order elements (example 3)



$err = 0.92 \%$

$ndof = 173$

$err \cdot ndof = 159$



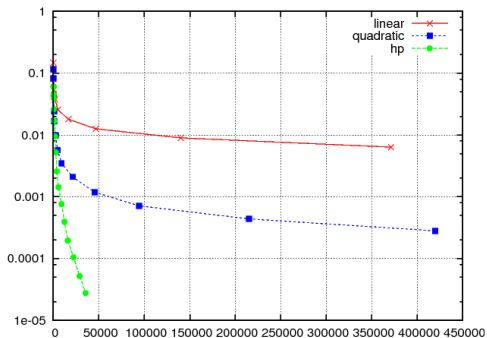
$err = 1.02 \%$

$ndof = 457$

$err \cdot ndof = 466$

Higher-order elements

<i>err</i>	$p = 1$	$p = 2$	$p = 4$	<i>hp</i>
1 %	934	173	457	154
0.1 %	> 50000	1415	1185	622
0.01 %		18800	2317	1556



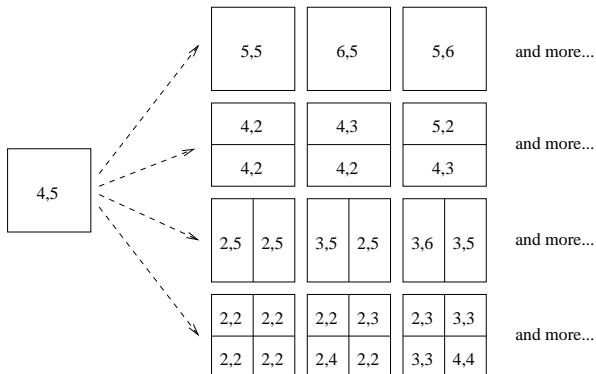
Higher-order method = higher-order elements + *good automatic adaptivity*

	Affine elements	<i>hp</i> elements
DOF	259393	6331
Error	1.617%	1.521%
Iterations	228	60
CPU time	34 min	11.58 sec

	Whitney edge elements	<i>hp</i> edge elements
DOF	2586540	4324
Error	0.6445%	0.6211%
CPU time	21.2 min	2.49 sec

Adaptive hp -FEM

To refine a single quad element:

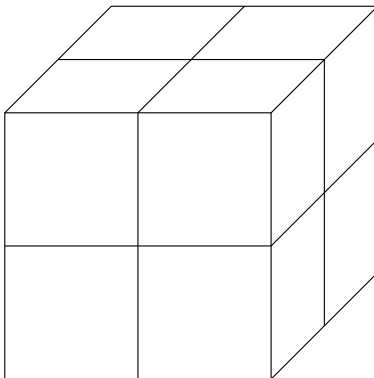


Variation of p, q by 2 in subelements \rightarrow approx. 7000 candidates.

Variation of p, q by 1 in subelements \rightarrow approx. 300 candidates.

Adaptive hp -FEM

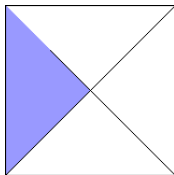
To refine a single brick element:



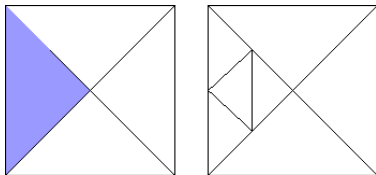
Variation of p, q by 2 in subelements \rightarrow 282 *billion* candidates.

Variation of p, q by 1 in subelements \rightarrow 17 *million* candidates.

- Regular mesh

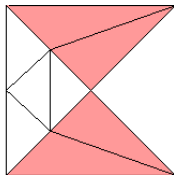
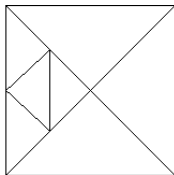
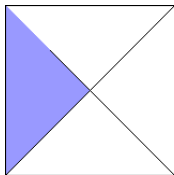


- Regular mesh



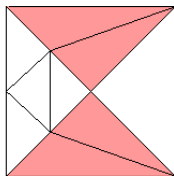
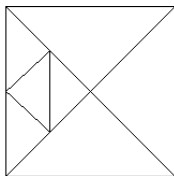
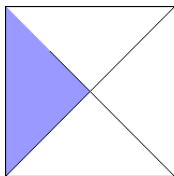
Hanging nodes

- Regular mesh

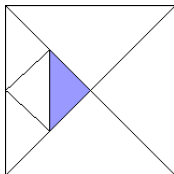


Hanging nodes

- Regular mesh

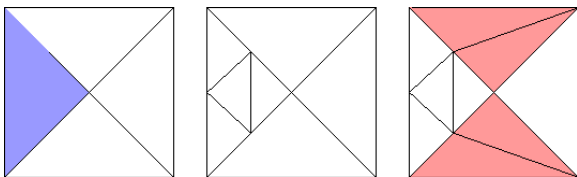


- One-level hanging nodes (1-irregular mesh)

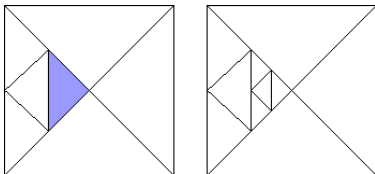


Hanging nodes

- Regular mesh

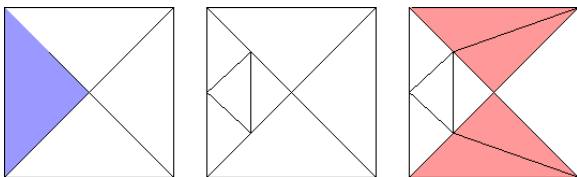


- One-level hanging nodes (1-irregular mesh)

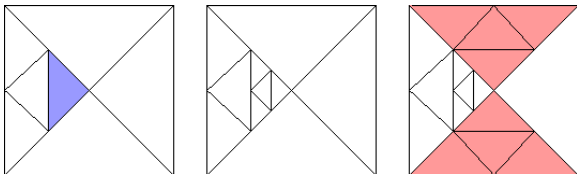


Hanging nodes

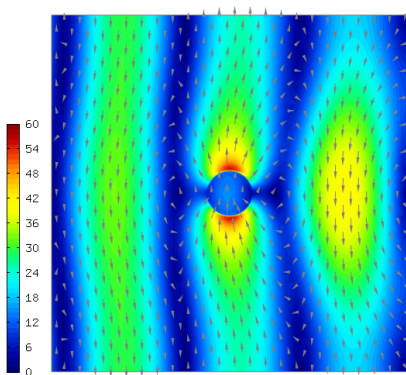
- Regular mesh



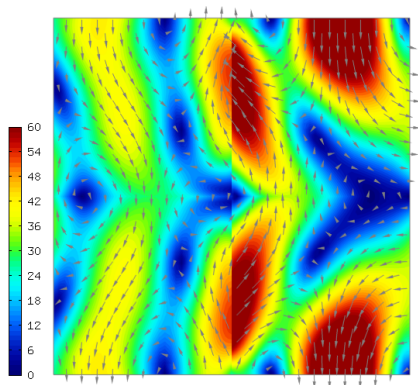
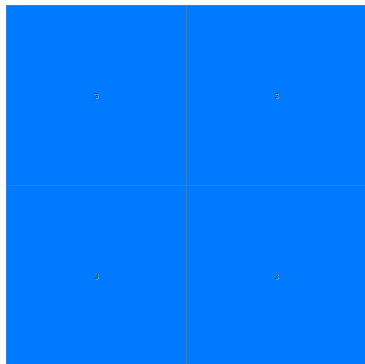
- One-level hanging nodes (1-irregular mesh)



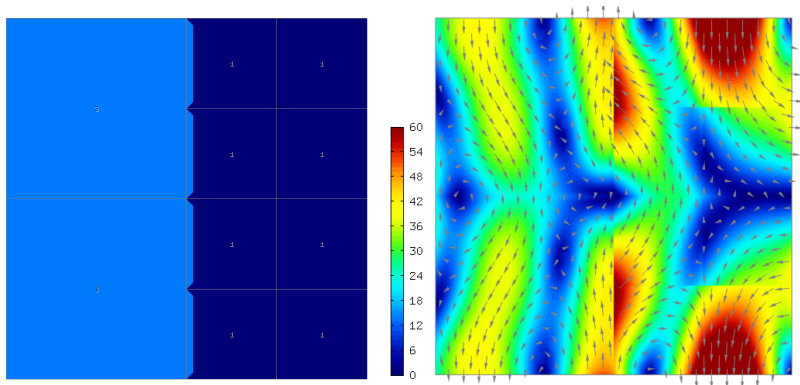
Waveguide problem



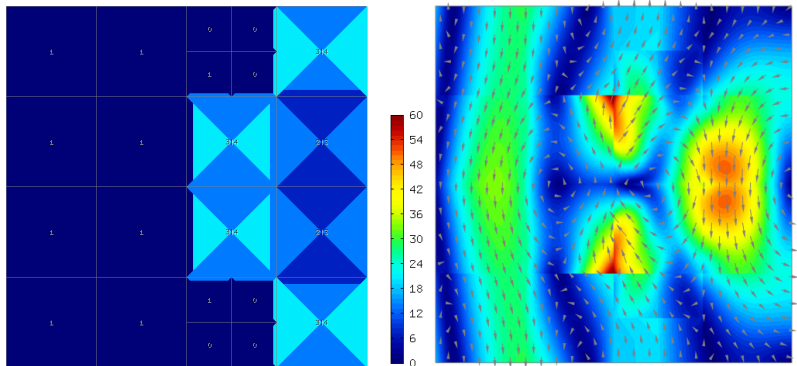
Waveguide problem (step 1)



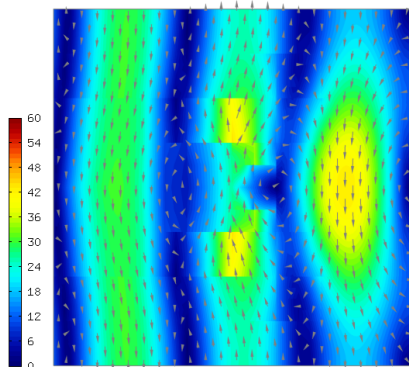
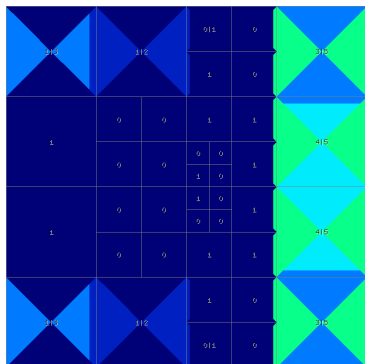
Waveguide problem (step 2)



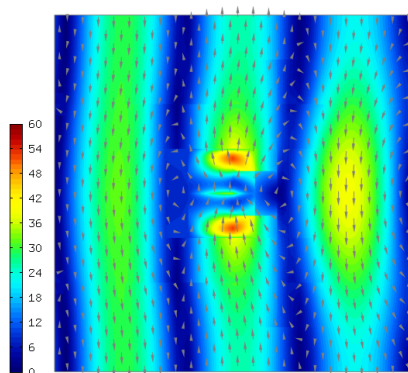
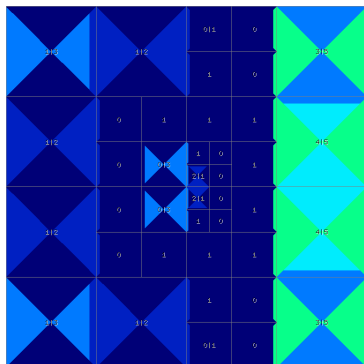
Waveguide problem (step 3)



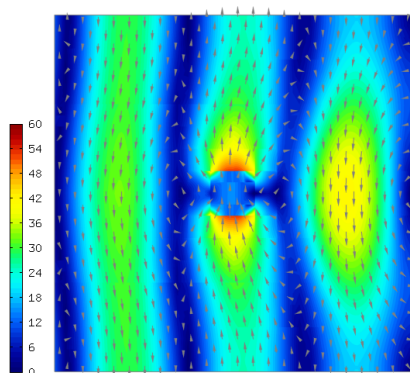
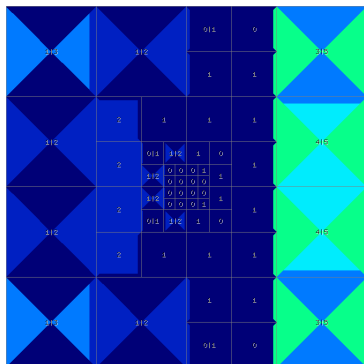
Waveguide problem (step 4)



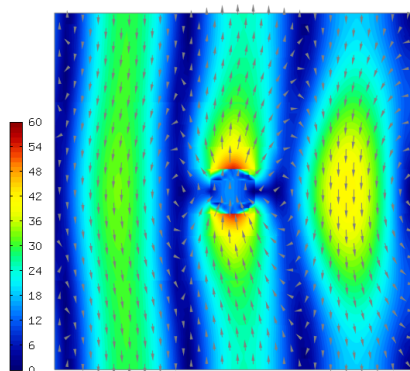
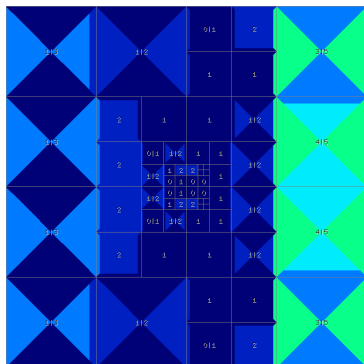
Waveguide problem (step 5)



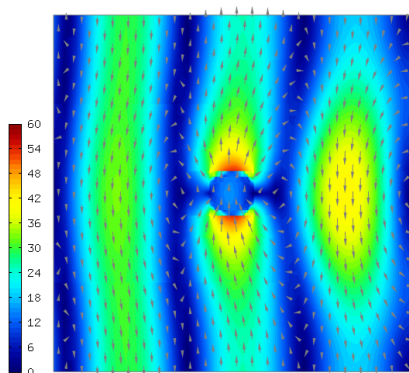
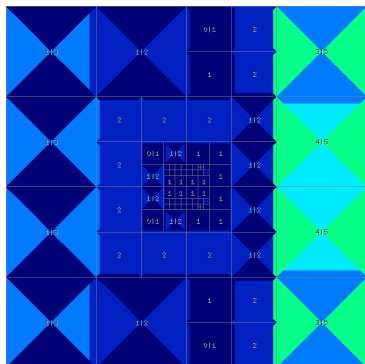
Waveguide problem (step 6)



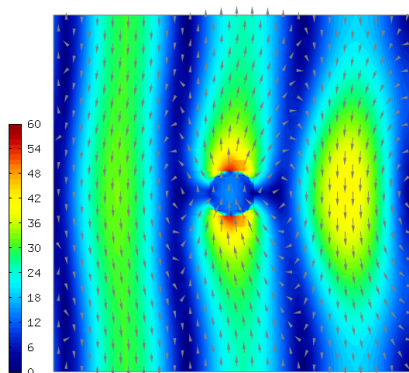
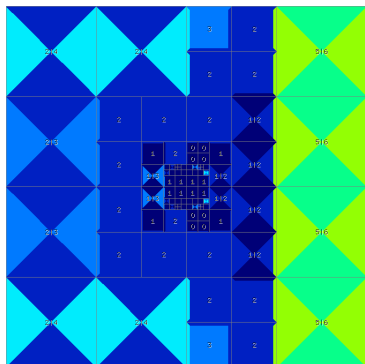
Waveguide problem (step 7)



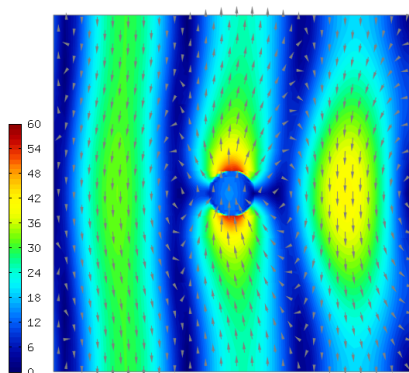
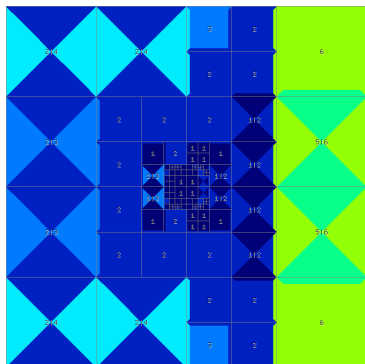
Waveguide problem (step 8)



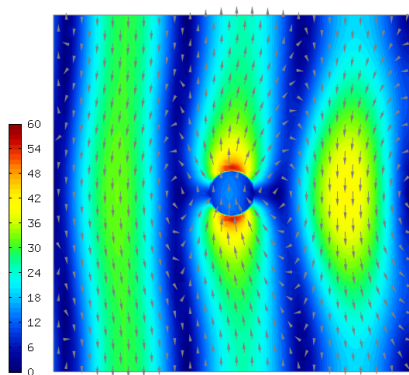
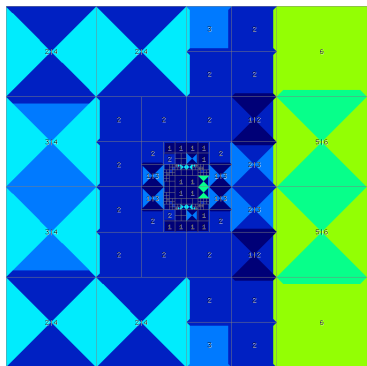
Waveguide problem (step 9)



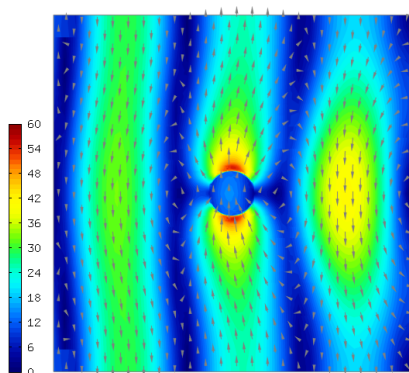
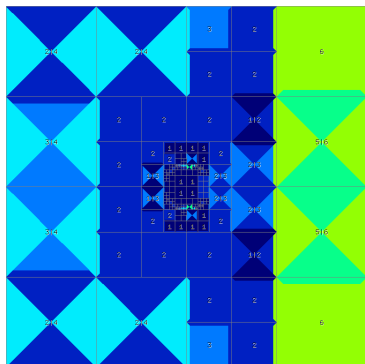
Waveguide problem (step 10)



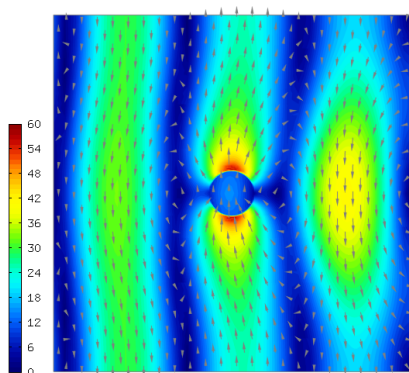
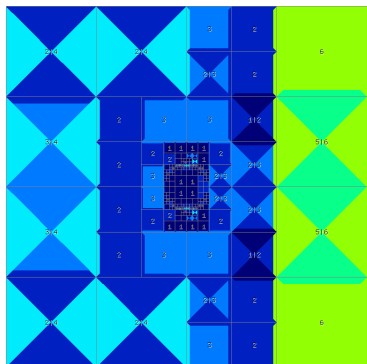
Waveguide problem (step 11)



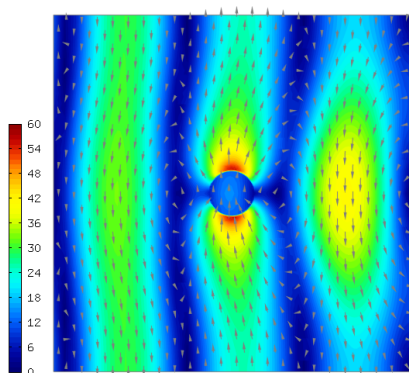
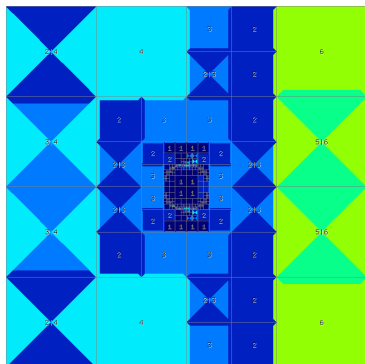
Waveguide problem (step 12)



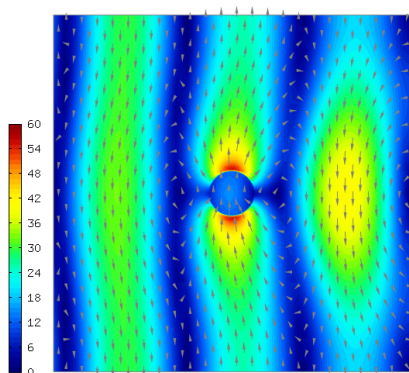
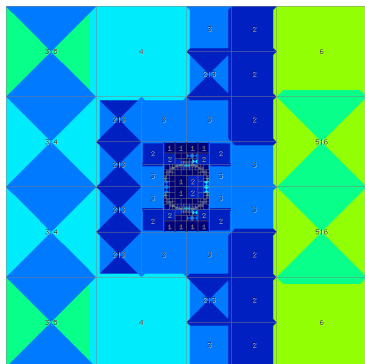
Waveguide problem (step 14)



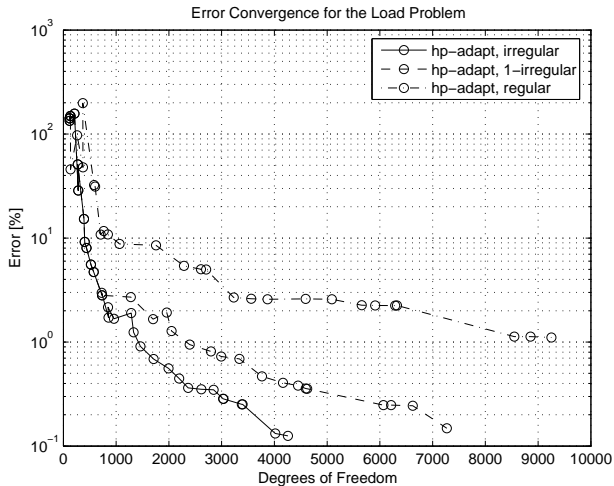
Waveguide problem (step 15)



Waveguide problem (step 16)



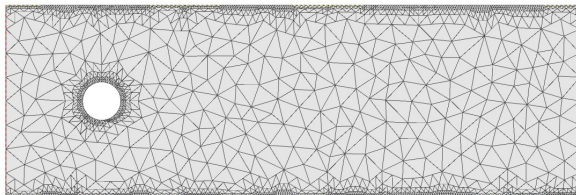
Mesh regularity vs. performance



Examples

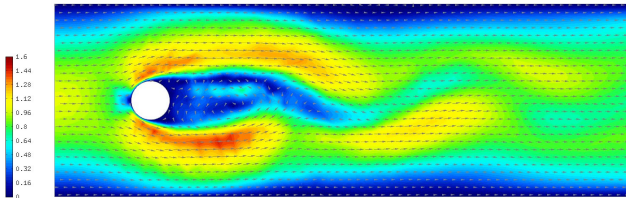
Credibility of results

Navier-Stokes equations, $Re = 200$



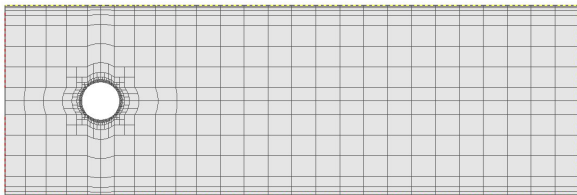
Credibility of results

Navier-Stokes equations, $Re = 200$



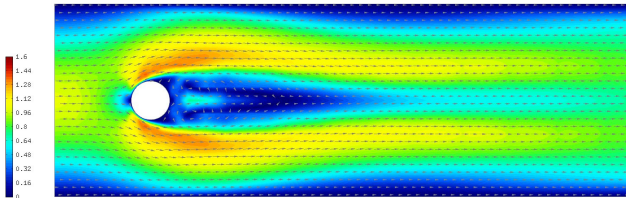
Credibility of results

Re = 200, symmetric mesh



Credibility of results

Re = 200, symmetric mesh

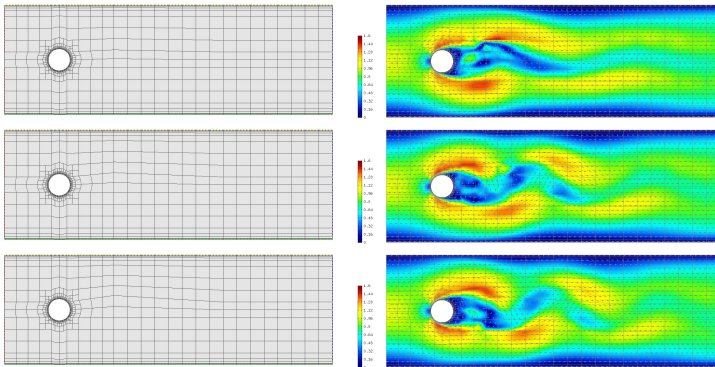


Credibility of results

It is easy to show that the exact solution **has to be symmetric**.

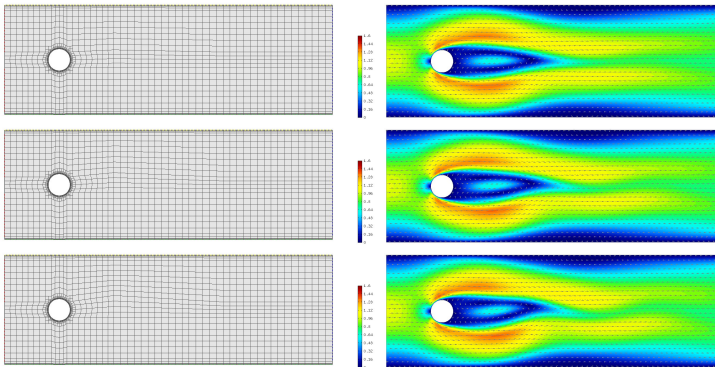
Credibility of results

Re = 200, time = 20 s, small mesh perturbation



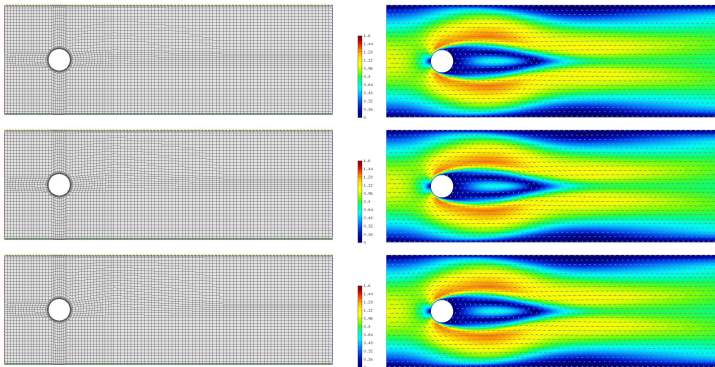
Credibility of results

Re = 200, time = 20 s, small mesh perturbation, uniform refinement



Credibility of results

Re = 200, time = 20 s, small mesh perturbation, 2x uniform refinement

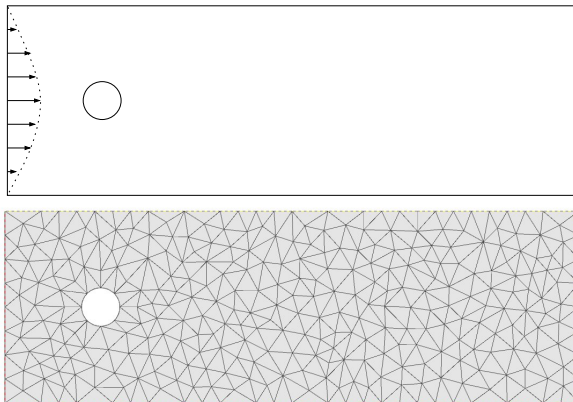


Credibility of results

Uniform refinement helps, but usually **one cannot afford it.**

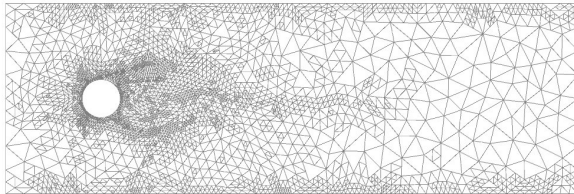
Credibility of results

Re = 200, dynamical meshes



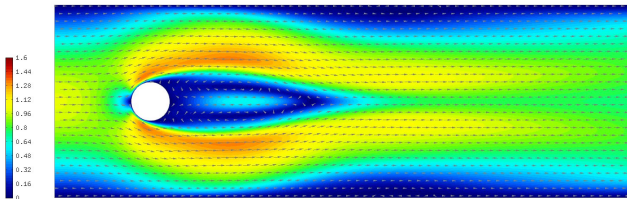
Credibility of results

Re = 200, dynamical meshes



Credibility of results

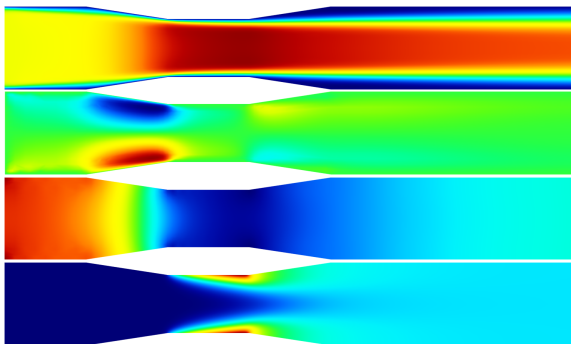
Re = 200, dynamical meshes



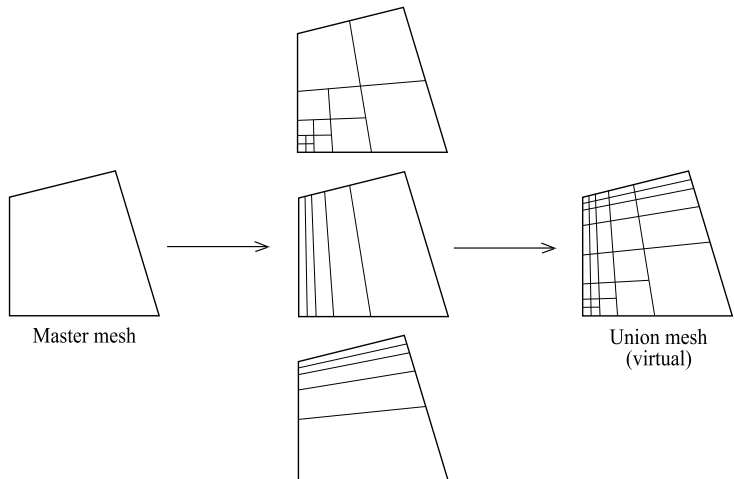
Adaptive hp -FEM on dynamical meshes

- Multimesh assembling
- Rothe's method

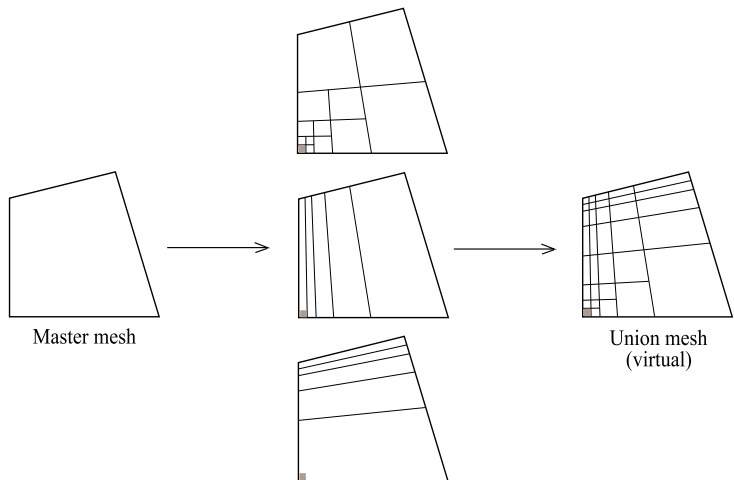
Multimesh assembling



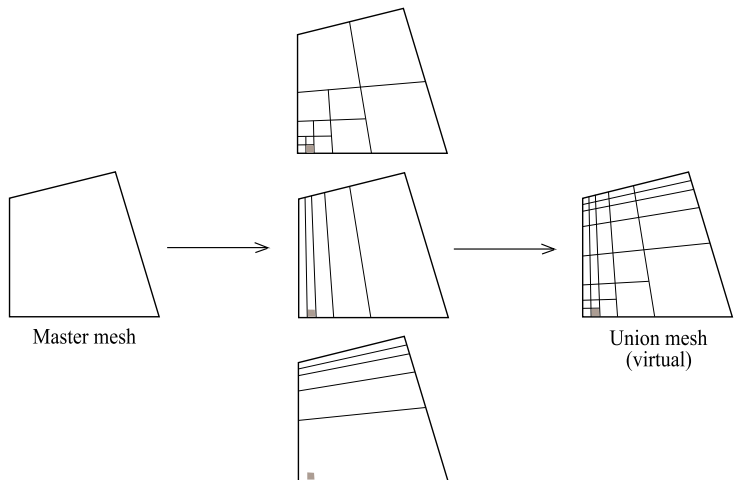
Multimesh assembling



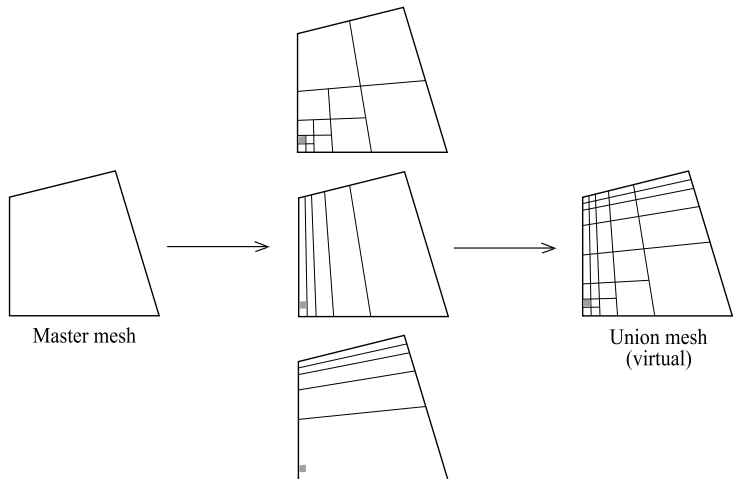
Multimesh assembling



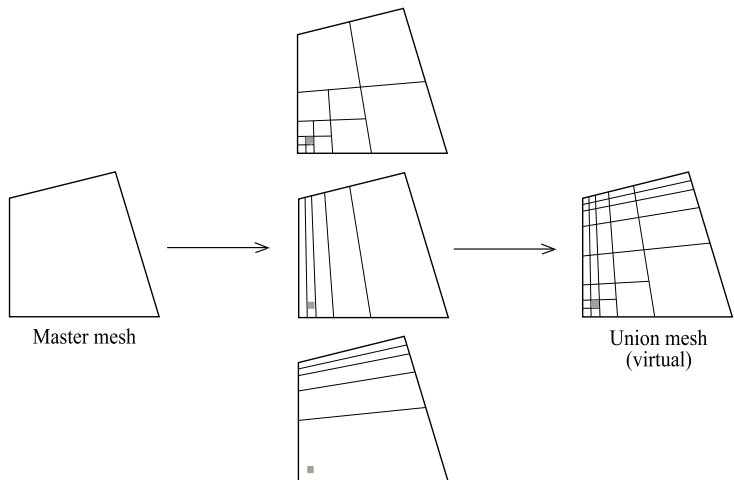
Multimesh assembling



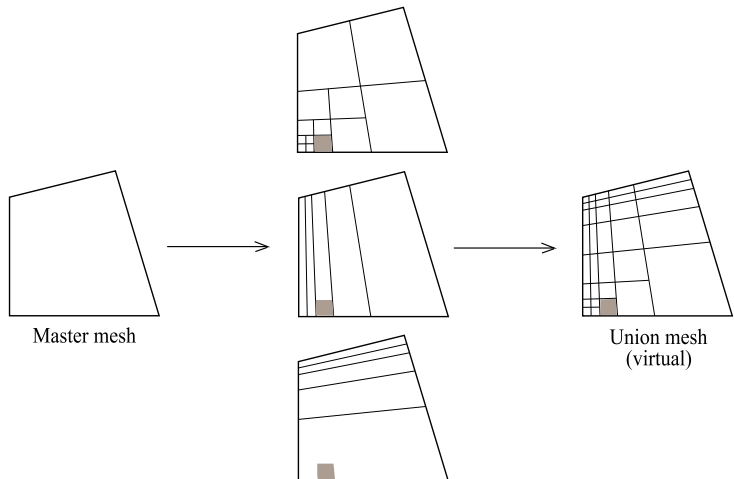
Multimesh assembling



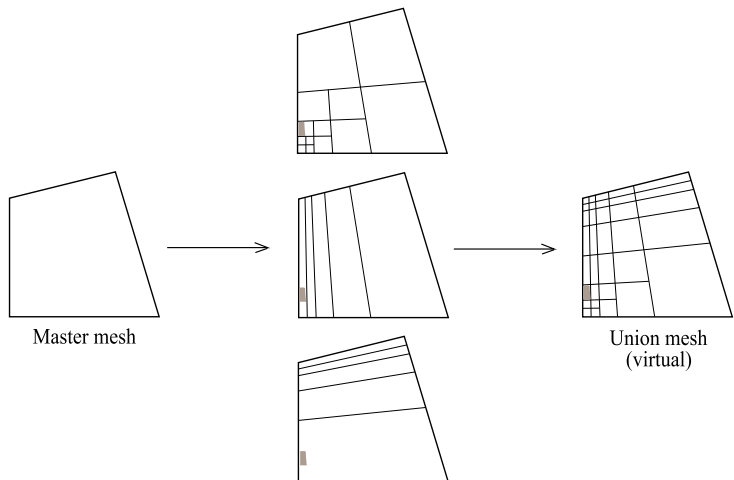
Multimesh assembling



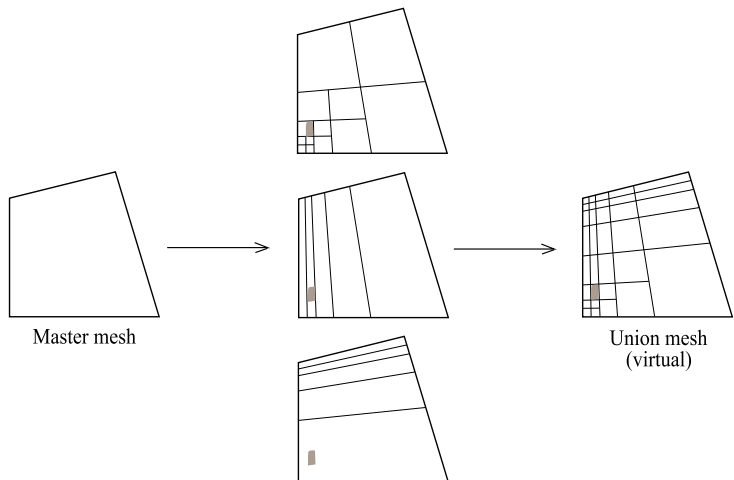
Multimesh assembling



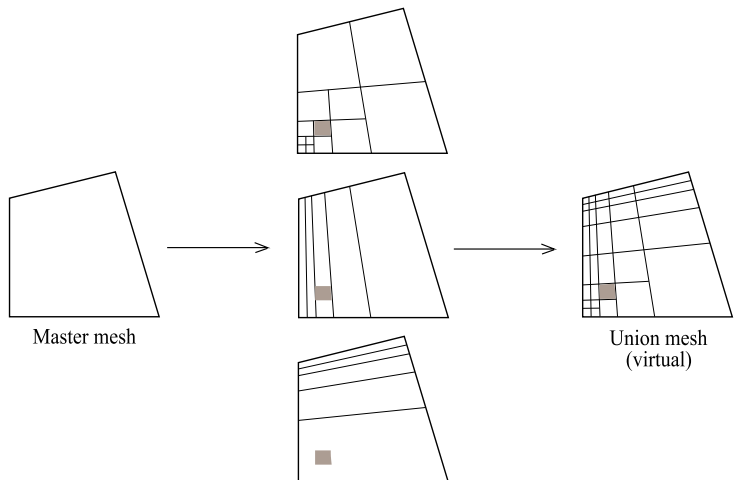
Multimesh assembling



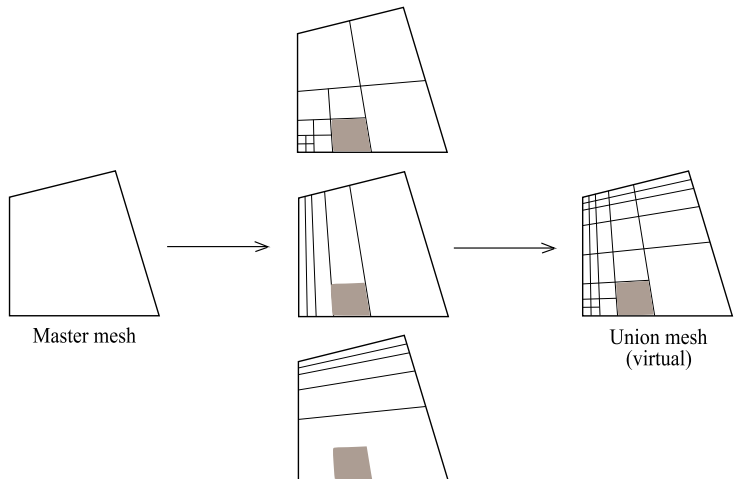
Multimesh assembling



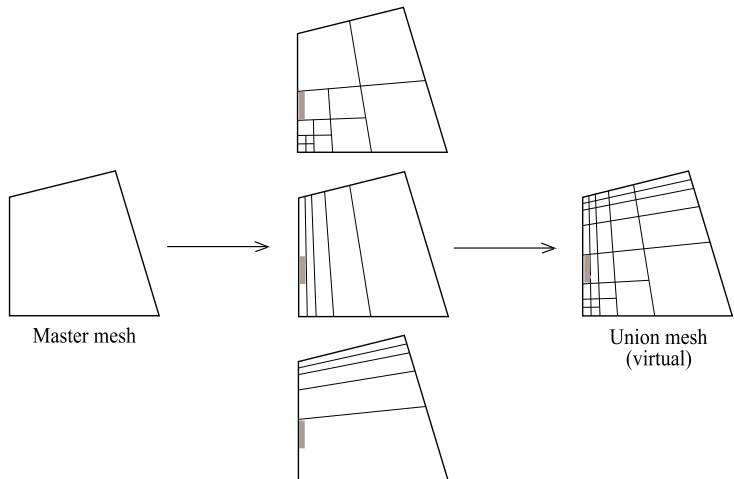
Multimesh assembling



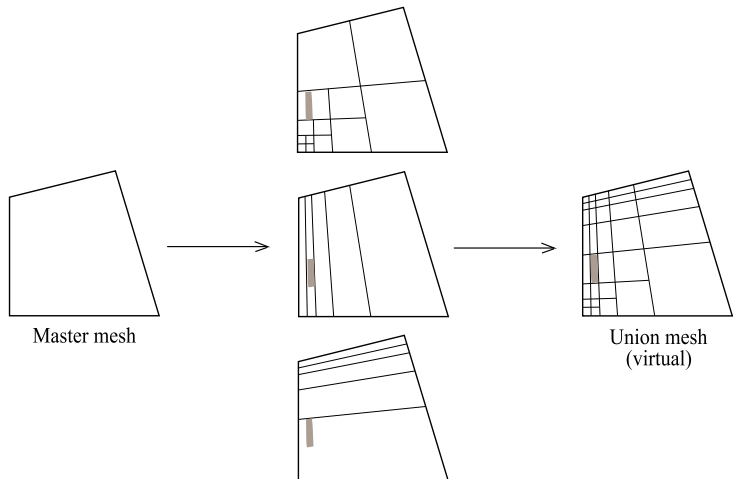
Multimesh assembling



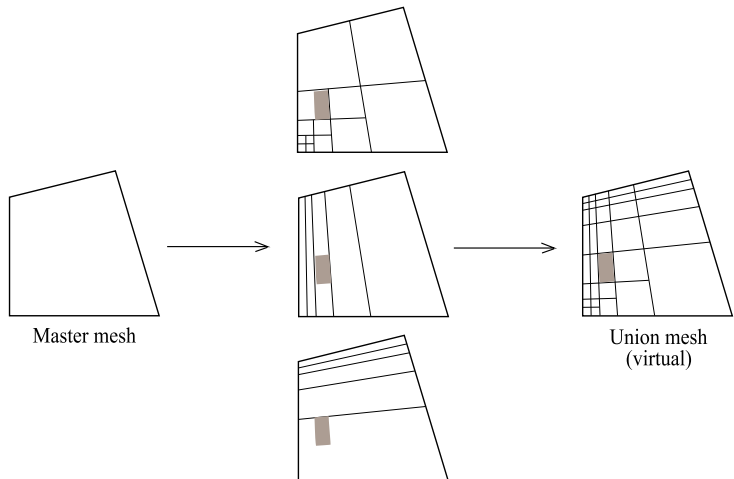
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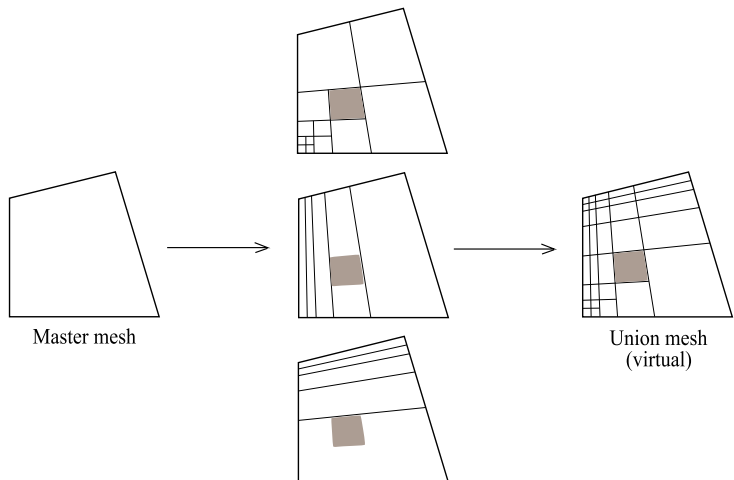
Multimesh assembling



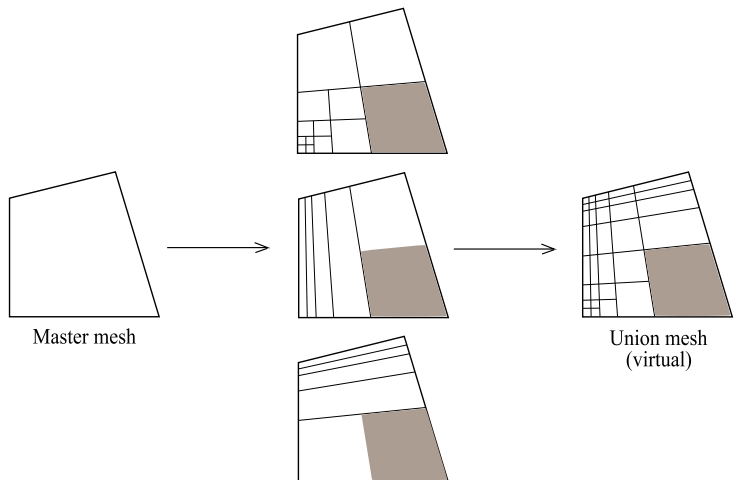
Multimesh assembling



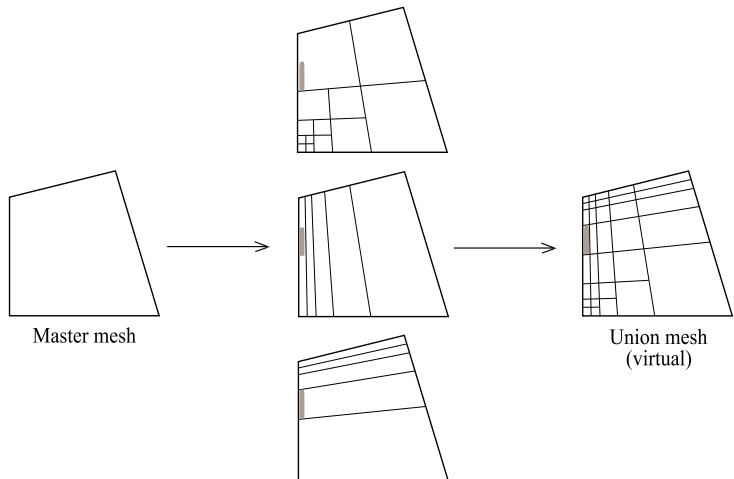
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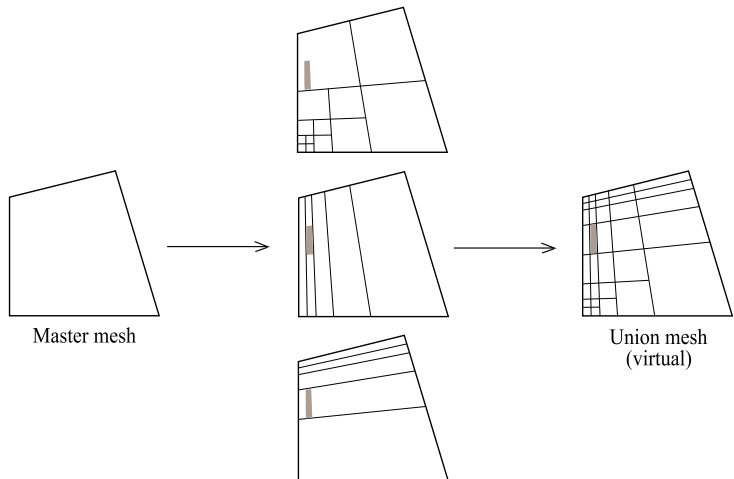
Multimesh assembling



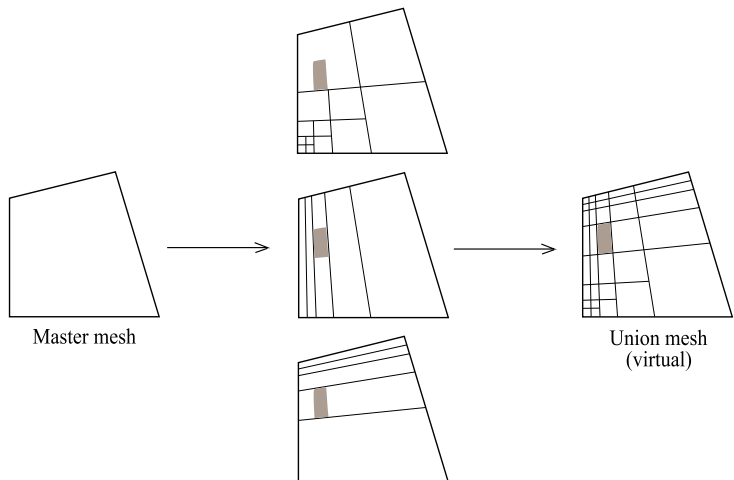
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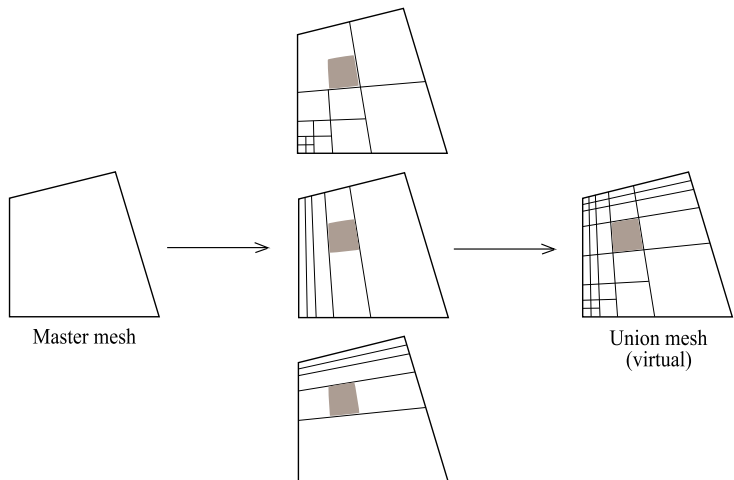
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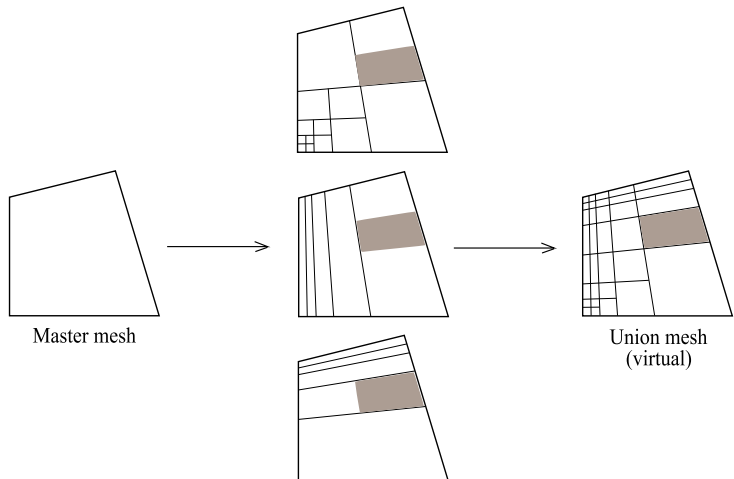
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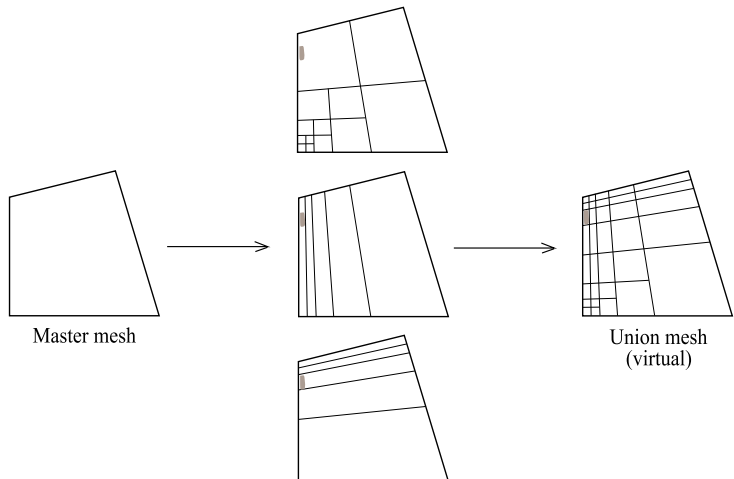
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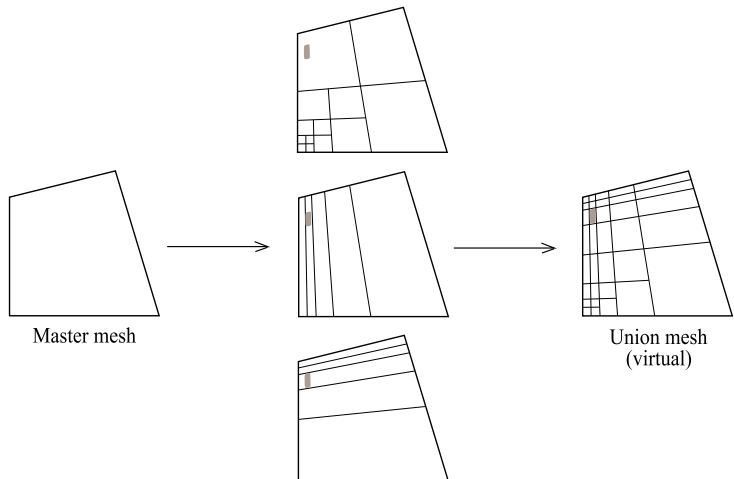
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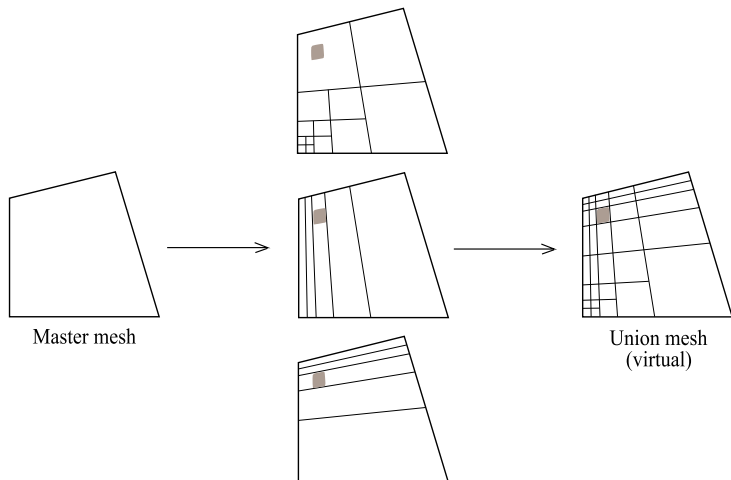
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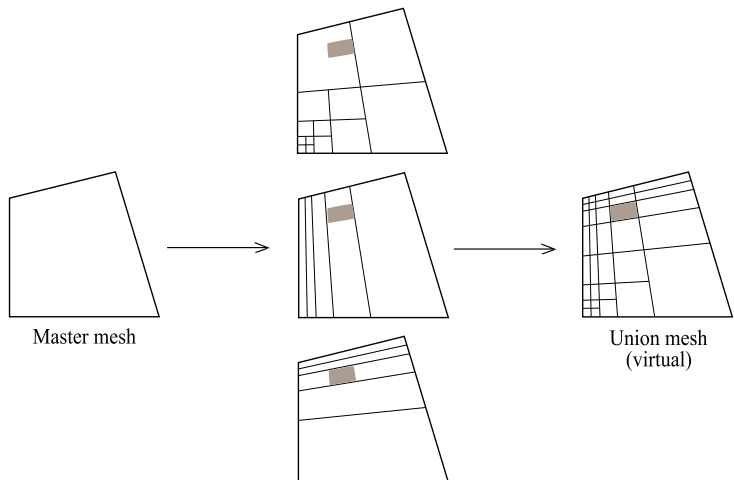
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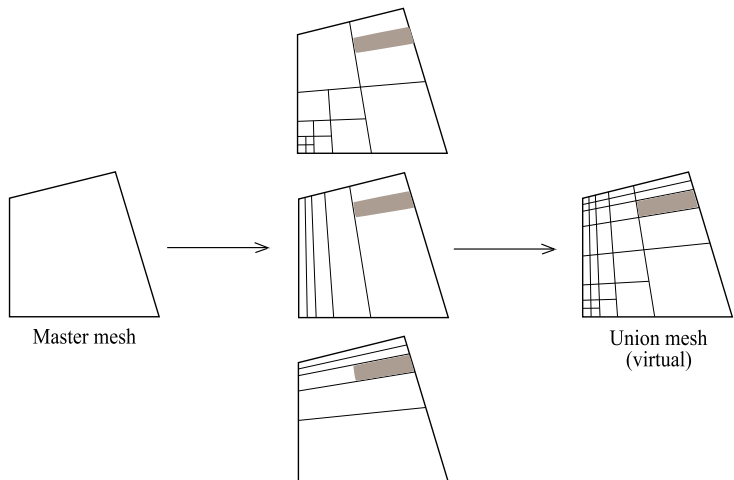
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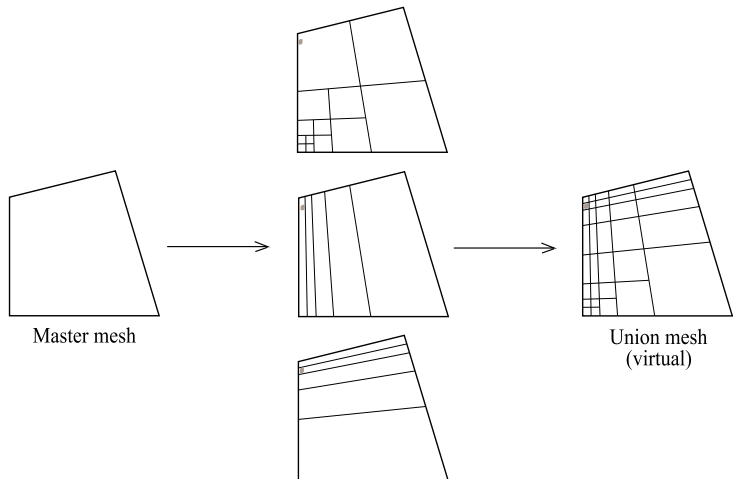
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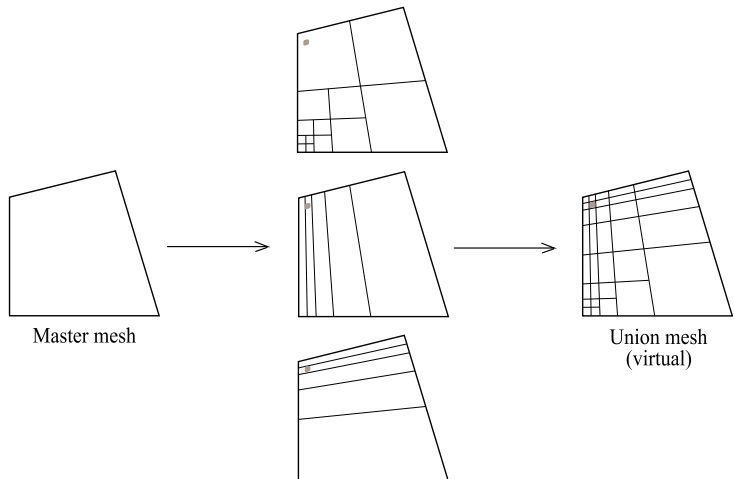
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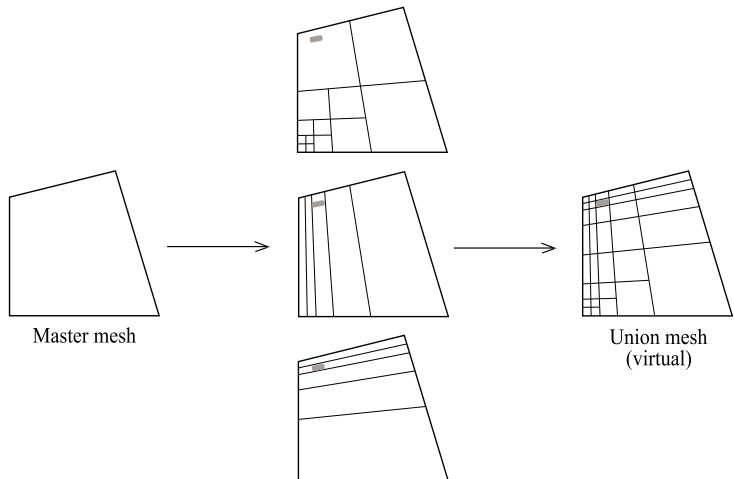
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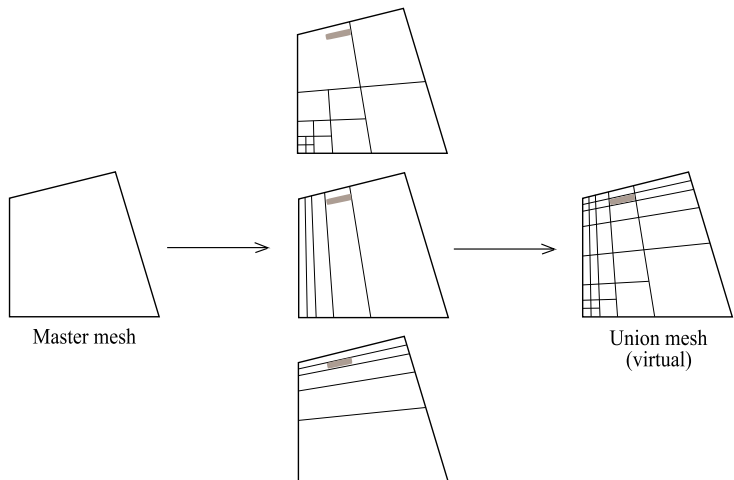
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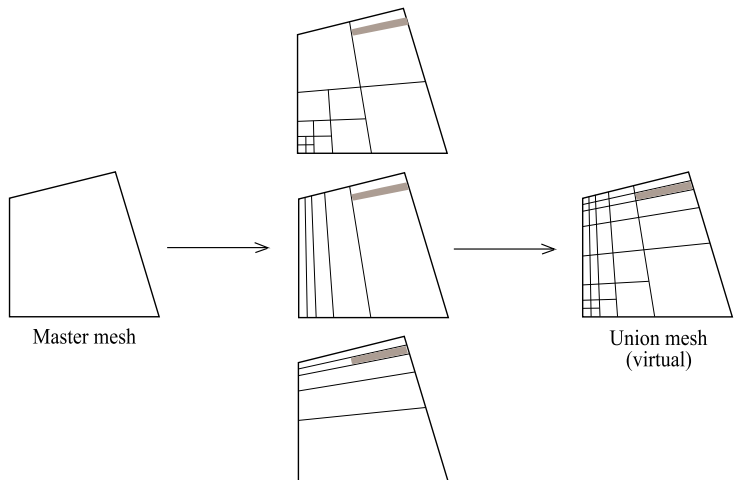
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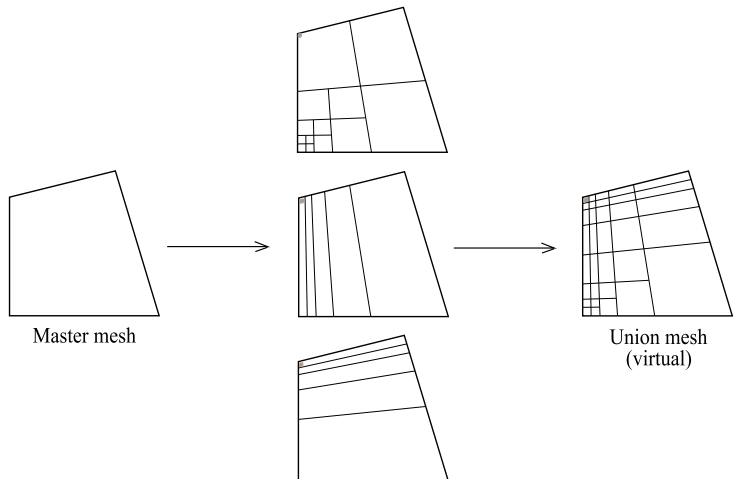
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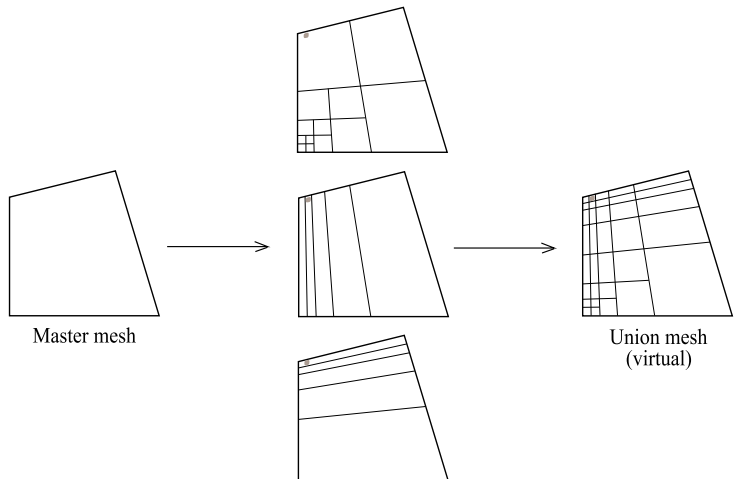
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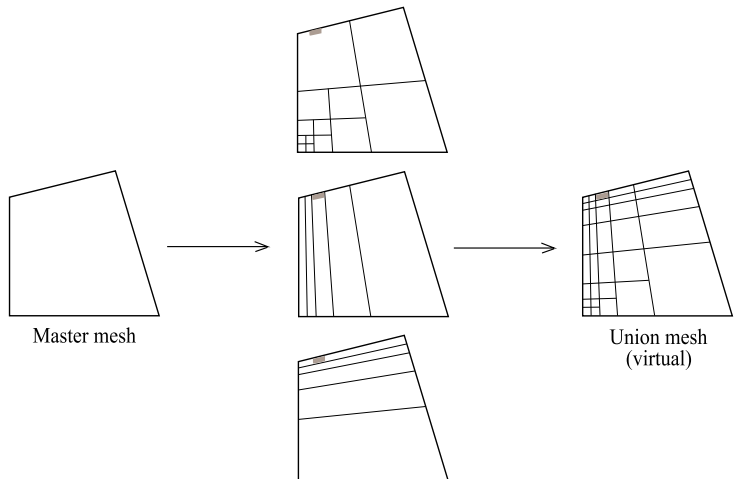
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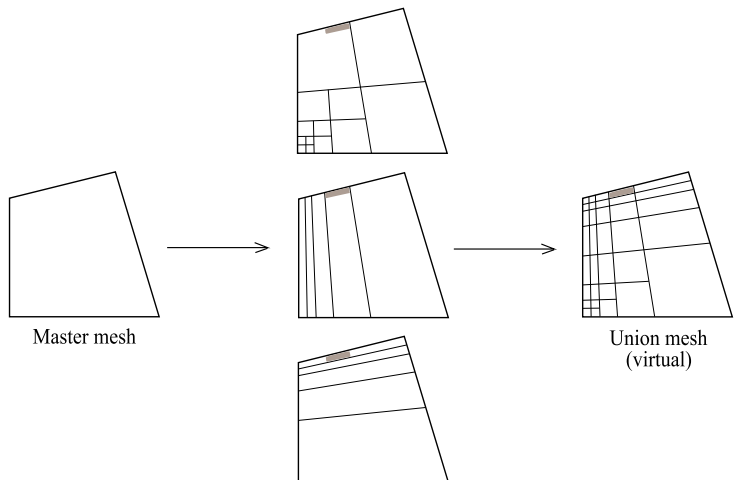
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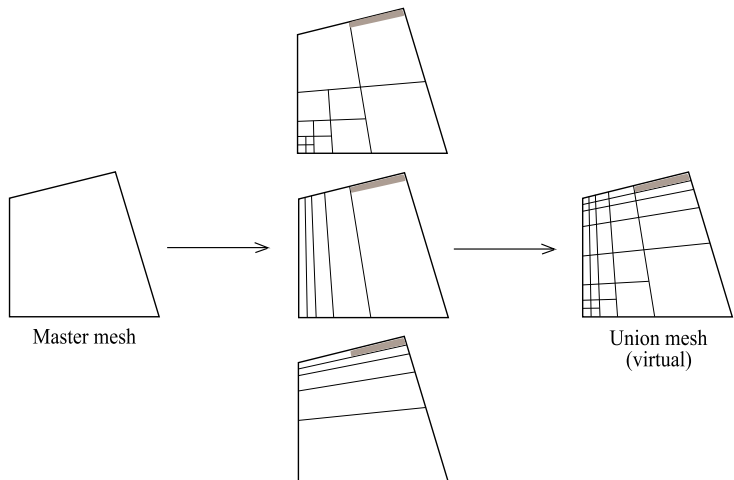
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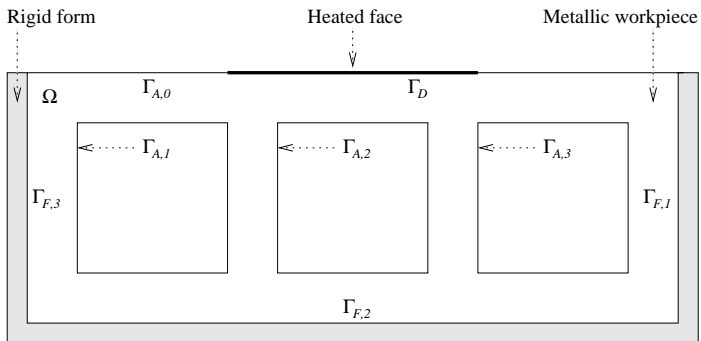
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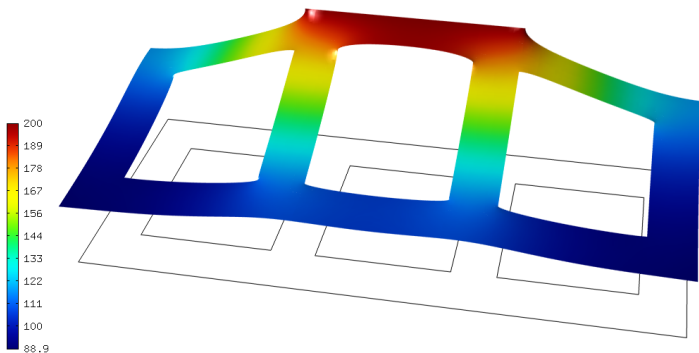
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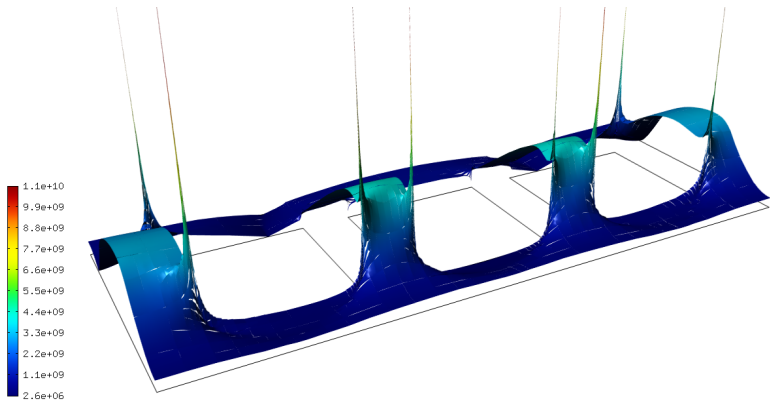
Thermoelasticity



- Solution: temperature



- Solution: stress



Thermoelasticity (step 1)

2	2	2	2	2	2	2	2	2	2
2			2			2			2
2			2			2			2
2	2	2	2	2	2	2	2	2	2

2	2	2	2	2	2	2	2	2	2
2			2			2			2
2			2			2			2
2	2	2	2	2	2	2	2	2	2

Thermoelasticity (step 2)

2	1	1	2	2	2	2	2	2	1	1	2					
	2	1							1	2						
1	1 2	[Empty]	2	[Empty]	2	[Empty]	2	[Empty]	2	1	1					
	1 1								1	1						
1	1								1	1						
1	1								1 2 1 2	1 2 1 2	1	1				
1	1 1								2 3 1 2	1 2 2 3	1 1	1				
	1 2								2	1						
2	2	1	1	1 2	2	2	1	1	2	2	2	1	1	1	2	2
	1	1	1	1		1	1	1	1		1	1	1	1	1	

2	2	2	2	2	2	2	2	2	2
2	[Empty]	[Empty]	2	[Empty]	[Empty]	2	[Empty]	[Empty]	2
2	[Empty]	[Empty]	2	[Empty]	[Empty]	2	[Empty]	[Empty]	2
2	2	2	2	2	2	2	2	2	2

Thermoelasticity (step 3)

2	1	1	2	2	2	2	2	2	2	1	1	2
	2	1								1	2	
1	1			2			2				1	1
	1	1									1	1
	1	1		1 2 1 2			1 2 1 2				1	1
	1	1		1 1			1 1				1	1
	1	1		1 1			1 1				1	1
	1	1		1 1			1 1				1	1
2	2	1	1	1	1	2	2	1	1	1	1	2
	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1

2	2	2	2	2	2	2	2	2	2
2			2			2			2
2			2			2			2
2	2	2	2	2	2	2	2	2	2

Thermoelasticity (step 4)

1	1	1	1		2	2	2	2	2	2	1	1	1	1	
1	2	1	1	2							2	1	1	2	1
1	1													1	1
1	1					2				2				1	1
1	1													1	1
1	1	2				1	2	1	2		1	2	1	1	1
1	1	1				1	1			1	1			1	1
1	1	1				1	1			1	1			1	1
1	1	1				1	1			1	1			1	1
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	2	1	1	1	1	2	1	1	2	1	1

2	2	2	2	2	2	2	2	2	2	2
2			2			2			2	2
2			2			2			2	2
2	2	2	2	2	2	2	2	2	2	2

Thermoelasticity (step 5)

1	1	1	1	2	2	2	2	2	2	1	1	1	1
1	1 1	1 1	1 1	2	2	2	2	2	2	2	1 1	1 1	1 1
1 3	1											1 1	1 3
1	1 2				2				2			1 2	1
1	1 2				1 2 1 2				1 2 1 2			1 2	1
1 3	1 1				1 1				1 1			1 1	1 3
2 1 3 2	1 1	1 1	1 2	1 1	3	3	1 1	1 2	1 2	1 1	3	3	1 2
3 2 3 2	2	1	1 1	2	2	2	2	1	1	2	2	2	2

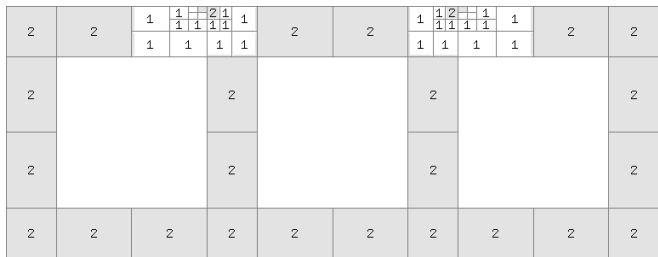
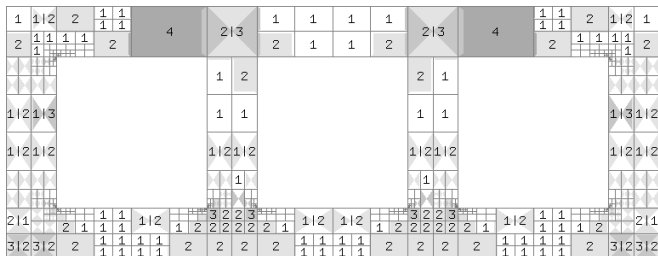
2	2	1	2	2	2	2	2	2	1	2	2
		1	1						1	1	
2				2				2			2
2				2				2			2
2	2	2	2	2	2	2	2	2	2	2	2

Thermoelasticity (step 6)

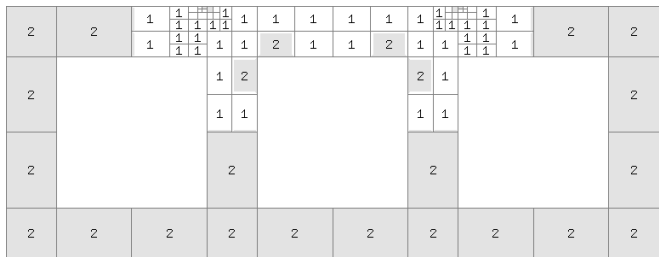
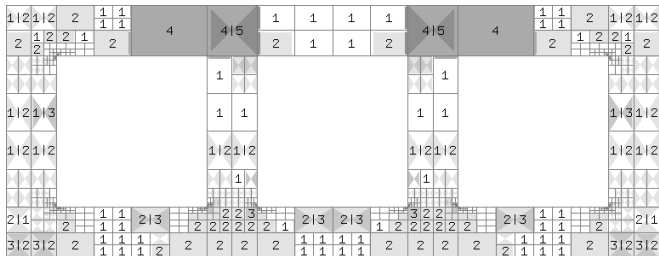
1	1 2	2	$\frac{1}{1}$	$\frac{1}{1}$		2	2	2	2	2	2	$\frac{1}{1}$	$\frac{1}{1}$	2	1 2	1
1	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	2							2	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$
1 3							1	2				2	1			1 3
	1 1														1 1	
1 2	1 3						1	1				1	1		1 3	1 2
1 2	1 2						1 2	1 2				1 2	1 2		1 2	1 2
								1 1				1 1				
2 1			$\frac{1}{1}$	$\frac{1}{1}$	1 2		3	3		$\frac{1}{2}$	1 2	1 2		1 2	$\frac{1}{1}$	$\frac{1}{1}$
		2	1	$\frac{1}{1}$	$\frac{1}{1}$	1 2				1 2				1 2	$\frac{1}{1}$	$\frac{1}{1}$
3 2	3 2	2	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	2	2	2	2	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$

2	2	1	$\frac{1}{1}$	$\frac{2}{1}$	2	1	2	2	1	2	$\frac{2}{1}$	$\frac{1}{1}$	1	2	2
		1	1	1	1						1	1	1		
2						2					2				2
2						2					2				2
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

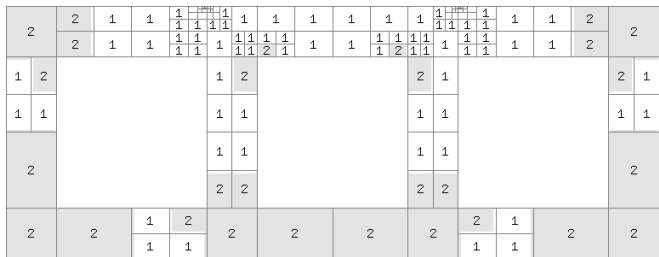
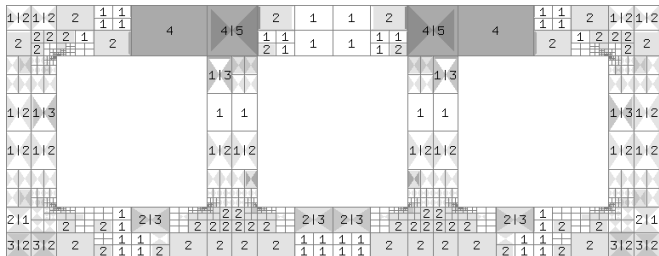
Thermoelasticity (step 7)



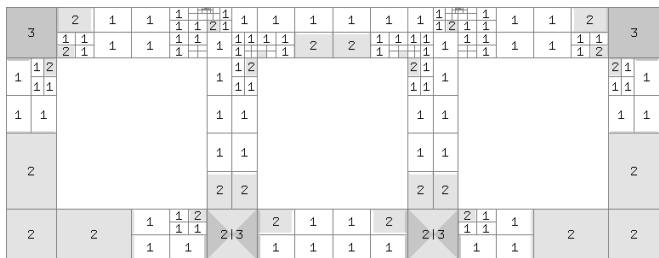
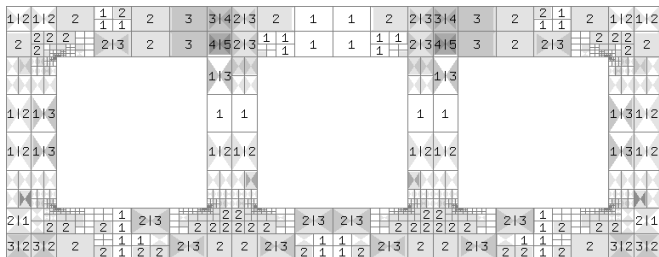
Thermoelasticity (step 8)



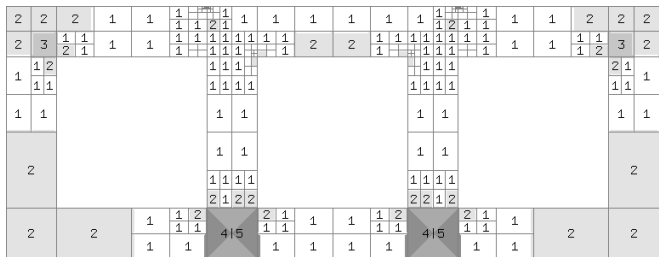
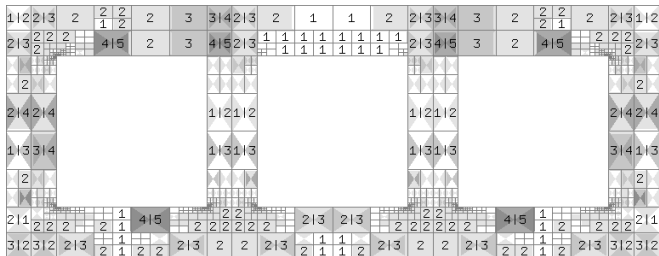
Thermoelasticity (step 9)



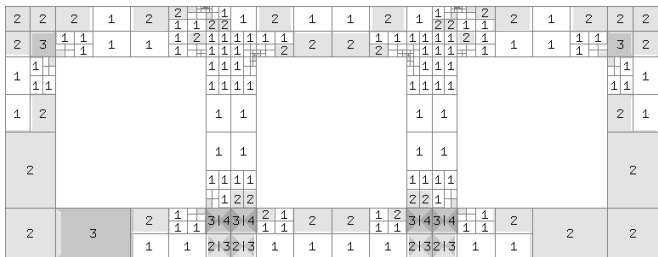
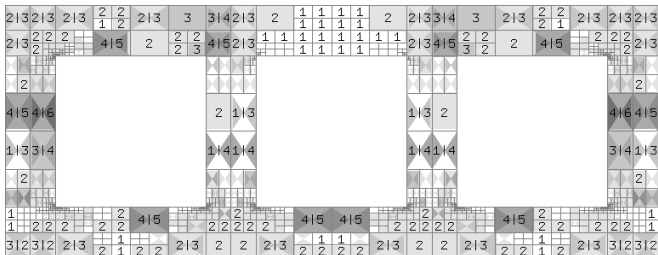
Thermoelasticity (step 10)



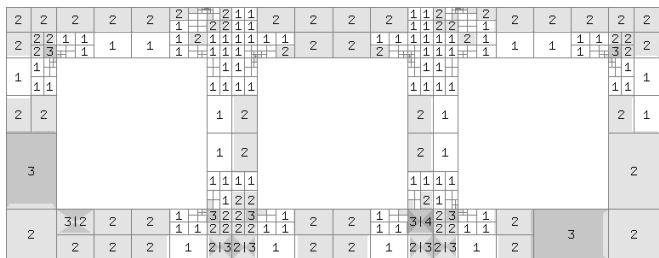
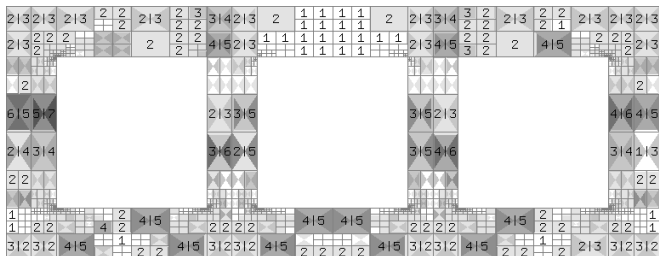
Thermoelasticity (step 11)



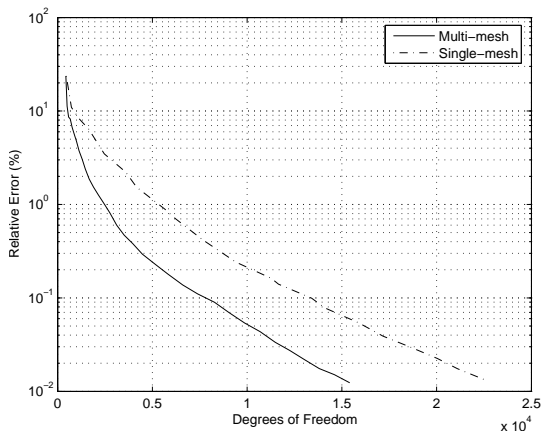
Thermoelasticity (step 12)



Thermoelasticity (step 13)



Convergence: multi-mesh vs. single-mesh



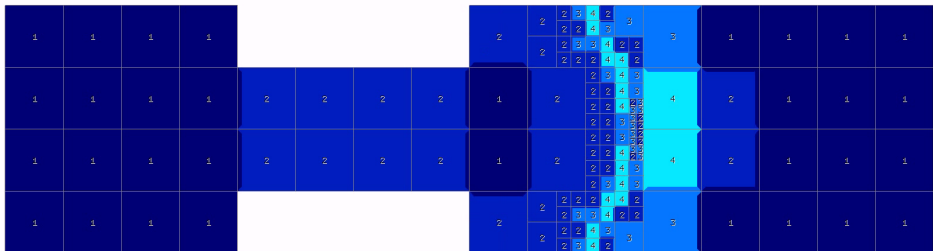
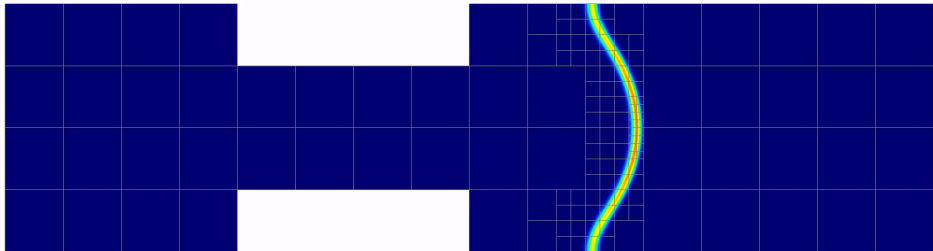
- Approximate a time-dependent PDE with time-independent one(s).
- Illustration: $\frac{\partial u}{\partial t} - \Delta u = f$

$$\frac{\partial u}{\partial t} \approx \frac{u^{n+1} - u^n}{\Delta t} \Rightarrow -\Delta t \Delta u^{n+1} + u^{n+1} = u^n + \Delta t f^{n+1}$$

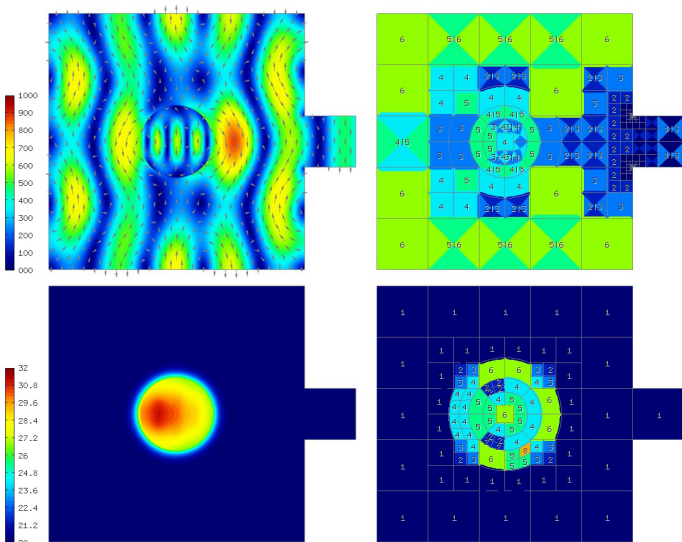
$$\frac{\partial u}{\partial t} \approx \frac{3u^{n+2} - 4u^{n+1} + u^n}{2\Delta t} \Rightarrow -2\Delta t \Delta u^{n+2} + 3u^{n+2} = 4u^{n+1} - u^n + 2\Delta t f^{n+2}$$

- Time-independent PDEs solved using *spatial adaptivity*
- “Simultaneous mesh refinement and coarsening”
- Adaptive control of time step as in embedded ODE methods

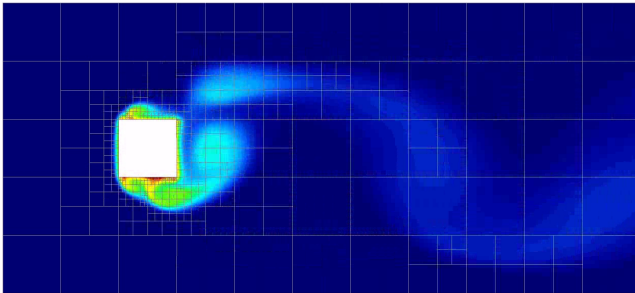
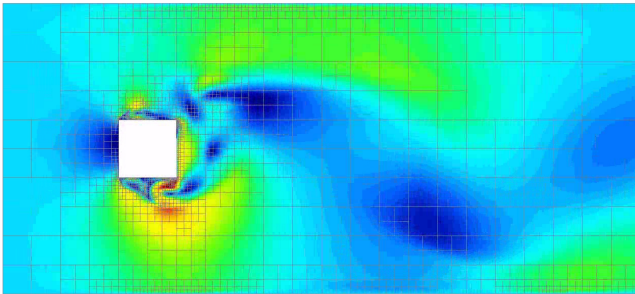
Flame propagation



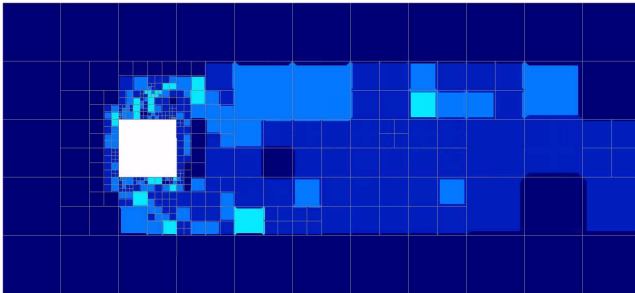
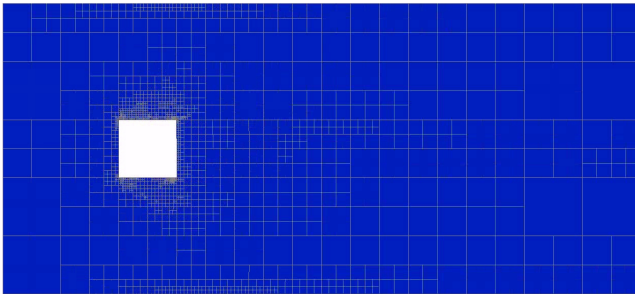
Microwave heating



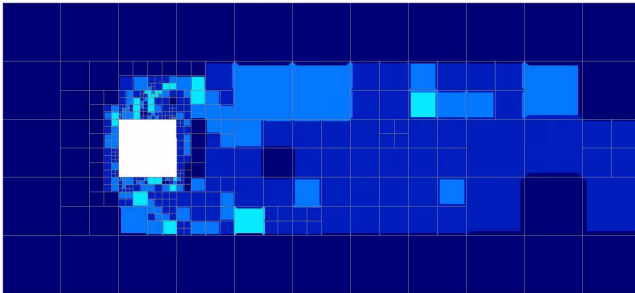
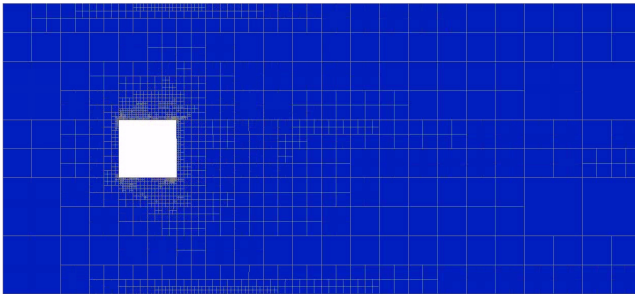
Thermally-conductive flow



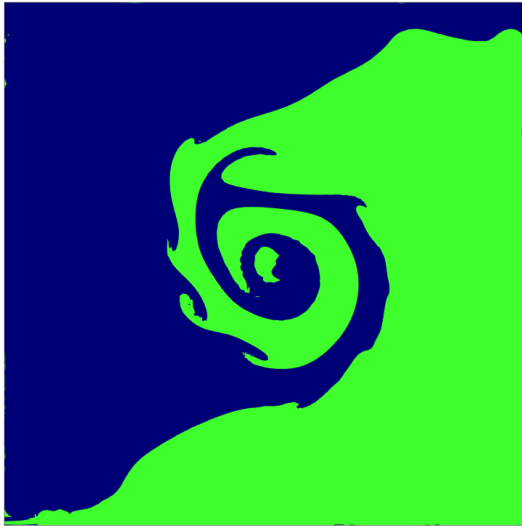
Thermally-conductive flow



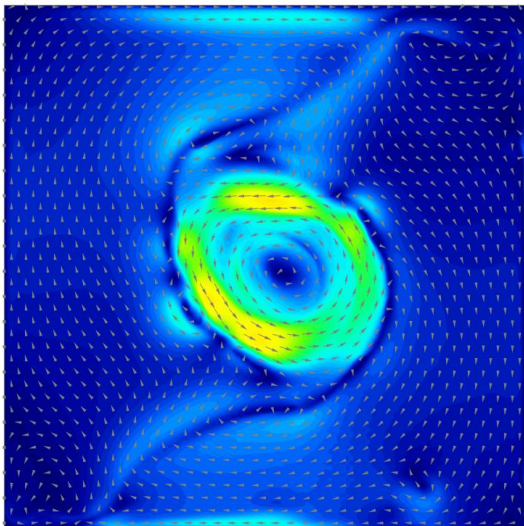
Thermally-conductive flow



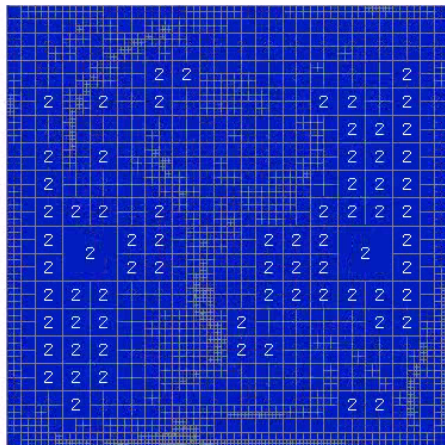
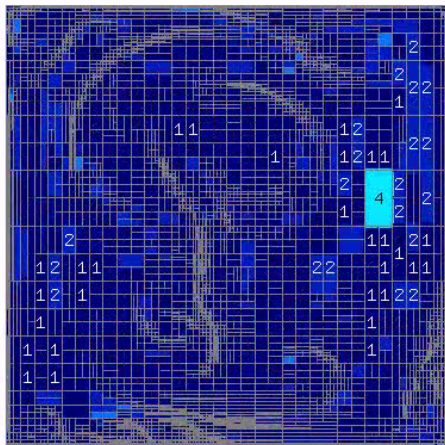
Two-phase flow



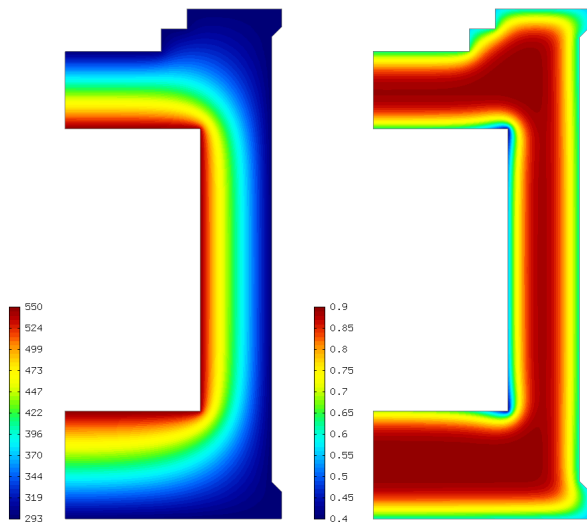
Two-phase flow



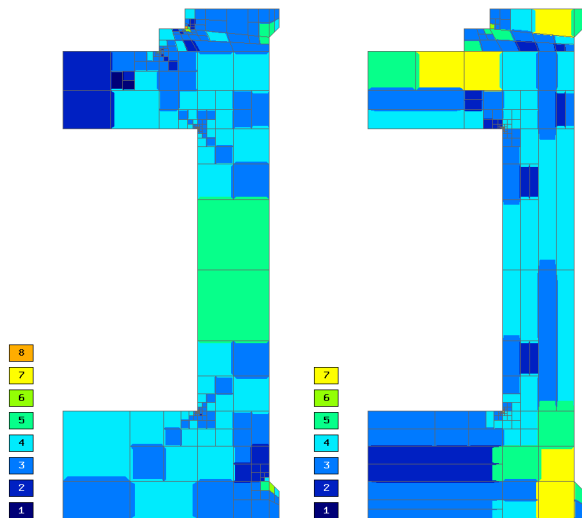
Two-phase flow



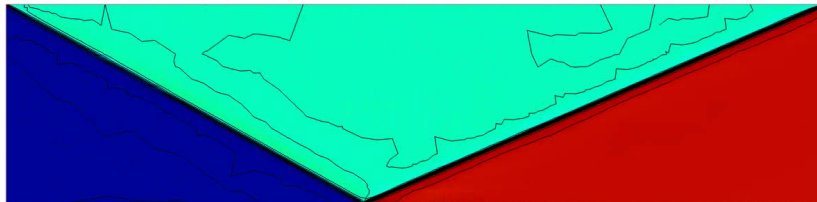
Heat and moisture transfer in concrete



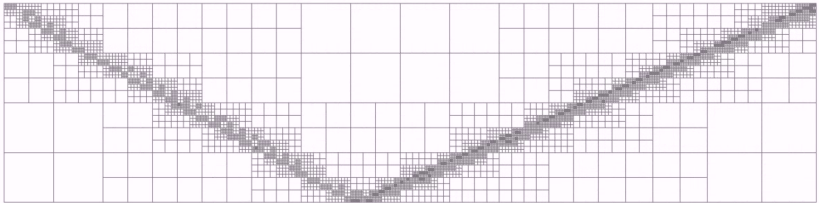
Heat and moisture transfer in concrete



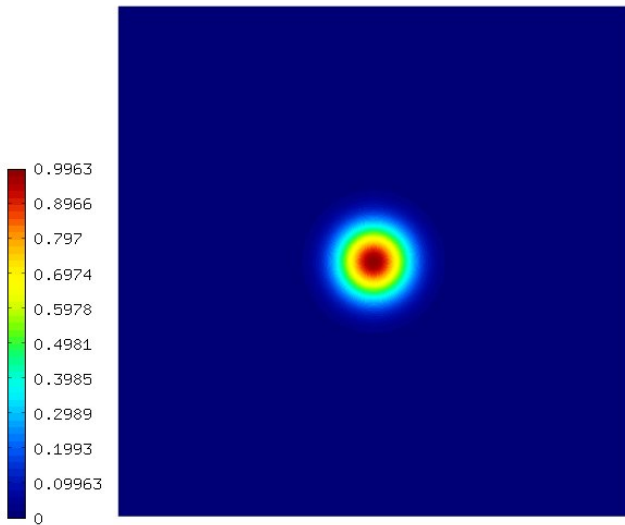
Transonic flow



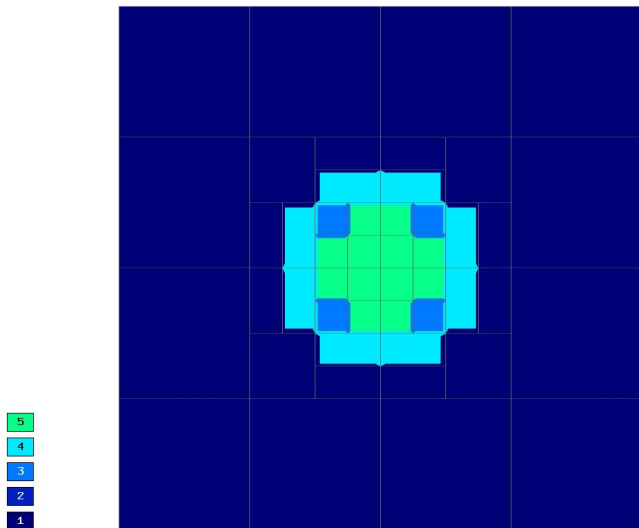
Transonic flow



Gross-Pitaevski equation



Gross-Pitaevski equation



`http://hpfem.org,`
`http://femhub.org`

Thank you for your attention!