



Résolution des équations de Maxwell-Vlasov sur des maillages quelconques par une méthode de volumes finis

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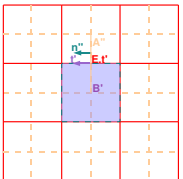
Système de Maxwell (mode T.M)

$$\begin{aligned}
 \frac{\partial B}{\partial t} + \nabla \times \mathbf{E} &= 0 && \text{dans } [0, T] \times \Omega \\
 \frac{\partial \mathbf{E}}{\partial t} - c^2 \nabla \times B &= -\frac{1}{\epsilon_0} \mathbf{J} && \text{dans } [0, T] \times \Omega \\
 \alpha \mathbf{E} \cdot \mathbf{t} - \beta B &= 0 && \text{dans } [0, T] \times \partial\Omega \\
 \mathbf{E}(0, \cdot) &= \mathbf{E}^0 && \text{dans } \Omega \\
 B(0, \cdot) &= B^0 && \text{dans } \Omega \\
 \text{avec } \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} && \text{dans } [0, T] \times \Omega
 \end{aligned}$$

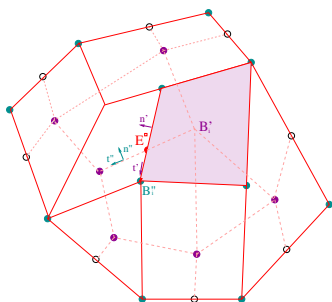
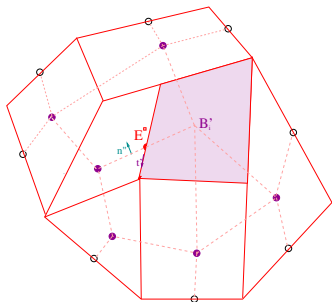


Schéma de Yee

$$\begin{aligned}
 1/ \quad & \int_{C'_i} \left(\frac{\partial B}{\partial t} + \nabla \times \mathbf{E} = 0 \right) \\
 & \Rightarrow \int_{C'_i} \frac{\partial B}{\partial t} + \sum_{j \in AC'_i} \int_{A'_j} \mathbf{E} \cdot \mathbf{t}'_j = 0 \\
 & \rightsquigarrow |C'_i| \frac{\partial B'_i}{\partial t} + \sum_{j \in AC'_i} |A'_j| \mathbf{E}_j \cdot \mathbf{t}'_j = 0
 \end{aligned}$$



$$\begin{aligned}
 2/ \quad & \int_{A''_j} \left(\frac{\partial \mathbf{E}}{\partial t} - c^2 \nabla \times B = -\frac{1}{\epsilon_0} \mathbf{J} \right) \cdot \mathbf{n}''_j \\
 & \Rightarrow \int_{A''_j} \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{n}''_j - c^2 \int_{A''_j} \nabla B \cdot \mathbf{t}''_j = -\frac{1}{\epsilon_0} \int_{A''_j} \mathbf{J} \cdot \mathbf{n}''_j \\
 & \rightsquigarrow |A''_j| \frac{\partial \mathbf{E}_j}{\partial t} \cdot \mathbf{t}'_j - c^2 (B'_2 - B'_1) = -\frac{1}{\epsilon_0} |A''_j| \mathbf{J}_j \cdot \mathbf{n}''_j
 \end{aligned}$$



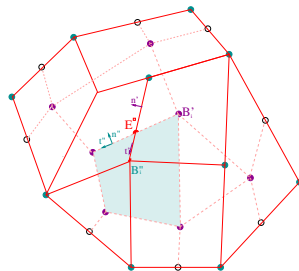
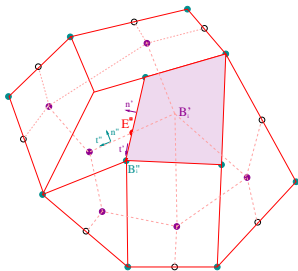
$$1/ \quad |C'_i| \frac{\partial B'_i}{\partial t} + \sum_{j \in AC'_i} |A'_j| \mathbf{E}_j \cdot \mathbf{t}'_j = 0$$

$$2/ \quad |A''_j| \frac{\partial \mathbf{E}_j}{\partial t} \cdot \mathbf{n}''_j - c^2 (B'_2 - B'_1) = -\frac{1}{\epsilon_0} |A''_j| \mathbf{J}_j \cdot \mathbf{n}''_j$$

problème : $\mathbf{n}''_j \neq \mathbf{t}'_j$!



Discretisation de la loi de Faraday sur le maillage primal et dual



$$\int_{C'_i} \left(\frac{\partial B}{\partial t} + \nabla \times \mathbf{E}^\diamond = 0 \right)$$

$$\Leftrightarrow |C'_i| \frac{\partial B'_i}{\partial t} + \sum_{j \in AC'_i} |A'_j| \mathbf{E}_j^\diamond \cdot \mathbf{t}'_j = 0$$

$$\Leftrightarrow \frac{\partial B'_i}{\partial t} + (\nabla' \times \mathbf{E}^\diamond)_i = 0$$

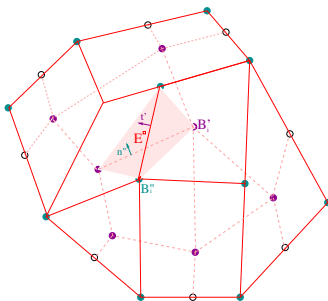
$$\int_{C''_i} \left(\frac{\partial B}{\partial t} + \nabla \times \mathbf{E}^\diamond = 0 \right)$$

$$\Leftrightarrow |C''_i| \frac{\partial B''_i}{\partial t} + \sum_{j \in AC''_i} |A''_j| \mathbf{E}_j^\diamond \cdot \mathbf{t}''_j = 0$$

$$\Leftrightarrow \frac{\partial B''_i}{\partial t} + (\nabla'' \times \mathbf{E}^\diamond)_i = 0$$



Discretisation de la loi d'Ampere-Maxwell sur le maillage diamant



$$\int_{C_j^\diamond} \left(\frac{\partial \mathbf{E}}{\partial t} - c^2 \nabla \times \mathbf{B} = -\frac{1}{\epsilon_0} \mathbf{J} \right)$$

$$\rightsquigarrow |C_j^\diamond| \frac{\partial \mathbf{E}_j^\diamond}{\partial t} + c^2 ((B'_2 - B'_1) |A'_j| \mathbf{t}'_j + (B''_2 - B''_1) |A''_j| \mathbf{t}''_j) = -\frac{1}{\epsilon_0} |C_j^\diamond| \mathbf{J}_j^\diamond$$

$$\rightsquigarrow \frac{\partial \mathbf{E}_j^\diamond}{\partial t} - c^2 (\nabla^\diamond \times \mathbf{B}''^n)_j = -\frac{1}{\epsilon_0} \mathbf{J}_j^\diamond$$



$$\begin{aligned}
 B'^{n+1} &= B'^n - \Delta t \nabla' \times \mathbf{E}^{\diamond n+1/2} \\
 B''^{n+1} &= B''^n - \Delta t \nabla'' \times \mathbf{E}^{\diamond n+1/2} \\
 \mathbf{E}^{\diamond n+1/2} &= \mathbf{E}^{\diamond n-1/2} + c^2 \Delta t \nabla^{\diamond} \times B'^n - \frac{\Delta t}{\epsilon_0} \mathbf{J}^{\diamond n}
 \end{aligned}$$

Pour un maillage cartésien \Downarrow

$$B'_i{}^{n+1} = B'_i{}^n - \frac{\Delta t}{|C'_i|} \sum_{j \in \partial AC'_i} |A'_j| \mathbf{E}_j^{\diamond n+1/2} \cdot \mathbf{t}'_{ij}$$

$$B''_i{}^{n+1} = B''_i{}^n - \frac{\Delta t}{|C''_i|} \sum_{j \in AC''_i} |A''_j| \mathbf{E}_j^{\diamond n+1/2} \cdot \mathbf{t}''_{ij}$$

$$\mathbf{E}_j^{\diamond n+1/2} \cdot \mathbf{t}'_{ij} = \mathbf{E}_j^{\diamond n-1/2} \cdot \mathbf{t}'_{ij} - \frac{c^2 \Delta t}{|A''_j|} (B'_{ij} - B'_i) - \frac{\Delta t}{\epsilon_0} \mathbf{J}_j^{\diamond n} \cdot \mathbf{t}'_{ij}$$

$$\mathbf{E}_j^{\diamond n+1/2} \cdot \mathbf{t}''_{ij} = \mathbf{E}_j^{\diamond n-1/2} \cdot \mathbf{t}''_{ij} - \frac{c^2 \Delta t}{|A'_j|} (B''_{ij} - B''_i) - \frac{\Delta t}{\epsilon_0} \mathbf{J}_j^{\diamond n} \cdot \mathbf{t}''_{ij}$$



Discretisation de la divergence sur le maillage primal et dual

$$\begin{aligned}
 \frac{1}{|C'_i|} \int_{C'_i} \nabla \cdot \mathbf{E} &= \frac{1}{|C'_i|} \sum_{j \in AC'_i} \int_{A'_j} \mathbf{E} \cdot \mathbf{n}'_j \\
 &\approx \frac{1}{|C'_i|} \sum_{j \in AC'_i} |A'_j| \mathbf{E}_j^\diamond \cdot \mathbf{n}'_{ij} \\
 &:= (\nabla' \cdot \mathbf{E}^\diamond)_i \\
 \frac{1}{|C''_i|} \int_{C''_i} \nabla \cdot \mathbf{E} &= \frac{1}{|C''_i|} \sum_{j \in AC''_i} \int_{A''_j} \mathbf{E} \cdot \mathbf{n}''_j \\
 &\approx \frac{1}{|C''_i|} \sum_{j \in AC''_i} |A''_j| \mathbf{E}_j^\diamond \cdot \mathbf{n}''_{ij} \\
 &:= (\nabla'' \cdot \mathbf{E}^\diamond)_i
 \end{aligned}$$



Propriétés des opérateurs discrets :

- $\nabla' \cdot (\nabla^\diamond \times B''') = 0$
- $\nabla'' \cdot (\nabla^\diamond \times B''') = 0$
- $\langle \nabla''' \times \mathbf{E}^\diamond, B''' \rangle_{'''} = \langle \mathbf{E}^\diamond, \nabla^\diamond \times B''' \rangle_\diamond + \langle \mathbf{E}^\diamond \cdot \mathbf{t}', B''' \rangle_{\partial\Omega}$

ou,

$$\langle H''', B''' \rangle_{'''} := \frac{1}{2} \left(\sum_{i=1}^{N'} |C'_i| H'_i B'_i + \sum_{i=1}^{N''} |C''_i| H''_i B''_i \right)$$

$$\langle \mathbf{E}^\diamond, \mathbf{H}^\diamond \rangle_\diamond := \sum_{i=1}^{N^\diamond} |C_i^\diamond| \mathbf{E}_i^\diamond \cdot \mathbf{H}^\diamond$$

$$\langle \mathbf{H}^\diamond, B''' \rangle_{\partial\Omega} := \frac{1}{4} \sum_{A'_j \subset \partial\Omega} |A'_j| H_j^\diamond (B''_{j2} + B''_{j1} + 2B'_j)$$



- Discrétisation adéquate de ρ et \mathbf{J}
- Condition initiales satisfaisant la loi de Gauss discrète \Rightarrow Loi de Gauss discrète satisfaite à chaque instant

$$\nabla' \cdot \mathbf{E}^{\diamond n+1/2} = \frac{\rho'^{n+1/2}}{\epsilon_0}$$

$$\nabla'' \cdot \mathbf{E}^{\diamond n+1/2} = \frac{\rho''^{n+1/2}}{\epsilon_0}$$

Discrétisation adéquate \leftrightarrow Conservation de l'équation de charge discrète

$$\frac{\rho'^{n+1/2} - \rho'^{n-1/2}}{\Delta t} + \nabla' \cdot \mathbf{J}^{\diamond n} = 0$$

$$\frac{\rho''^{n+1/2} - \rho''^{n-1/2}}{\Delta t} + \nabla'' \cdot \mathbf{J}^{\diamond n} = 0$$



Exemple d'une discrétisation adéquate de ρ et \mathbf{J} :

$$\rho_i^{n+1/2} = \frac{1}{|C'_i|} \int_{C'_i} \rho(\mathbf{X}, t^{n+1/2}) d\mathbf{X}$$

$$\rho''_i^{n+1/2} = \frac{1}{|C''_i|} \int_{C''_i} \rho(\mathbf{X}, t^{n+1/2}) d\mathbf{X}$$

$$\mathbf{J}_i^{\diamond n} \cdot \mathbf{n}'_i = \frac{1}{\Delta t} \int_{t^{n-1/2}}^{t^{n+1/2}} \frac{1}{|A'_i|} \int_{A'_i} \mathbf{J} \cdot \mathbf{n}'_i d\mathbf{X} dt$$

$$\mathbf{J}_i^{\diamond n} \cdot \mathbf{n}''_i = \frac{1}{\Delta t} \int_{t^{n-1/2}}^{t^{n+1/2}} \frac{1}{|A''_i|} \int_{A''_i} \mathbf{J} \cdot \mathbf{n}''_i d\mathbf{X} dt$$



Conservation d'une énergie électromagnétique discrète :

$$\begin{aligned} \mathbb{E}^n &:= \frac{\epsilon}{2} \left(\left| \mathbf{E}^{\diamond n+1/2} \right|_{\diamond}^2 + c^2 \langle B'^{m n}, B'^{m n+1} \rangle_{',m} \right) \\ &= \mathbb{E}^0 \end{aligned}$$



Stabilité :

- ▶ Schéma stable sous une condition CFL

$$c\Delta t < \min_i \min_{A_j^i \subset \partial C_i \setminus \partial\Omega} \sqrt{\frac{2|A_j^i| |C_i| \sin \theta_j}{(1 + |\cos \theta_j|) |\partial C_i|}}$$

- ▶ condition CFL pour un maillage cartésien

$$\frac{c\Delta t}{h} \leq \frac{1}{\sqrt{2}}$$

⇒ CFL du schéma de Yee



Pour des fonctions régulières

$$\|er^n\| := \sqrt{|er^{\diamond n+1/2}|_{\diamond}^2 + \frac{1}{2} (|er'^n|^2 + |er''^n|^2)} \leq K(h + \Delta t^2)$$

Ou,

$$\begin{aligned} er^n &:= (er'^n, er''^n, er^{\diamond n+1/2}) \\ er^{\diamond n+1/2} &:= \mathbf{E}^{\diamond n+1/2} - \Pi^{\diamond n+1/2} \mathbf{E} \\ er'^n &:= \mathbf{B}'^n - \Pi'^n \mathbf{B} \\ er''^n &:= \mathbf{B}''^n - \Pi''^n \mathbf{B} \end{aligned}$$

$\Pi^{\diamond n+1/2} \mathbf{E}$: Projection du champs électrique exact sur le maillage diamant

$\Pi'^n \mathbf{B}$: Projection du champs magnétique exact sur le maillage primal

$\Pi''^n \mathbf{B}$: Projection du champs magnétique exact sur le maillage dual



Convergence

$$\frac{\Pi'^{n+1}B - \Pi'^n B}{\Delta t} + \nabla' \times \Pi^{\diamond n+1/2} \mathbf{E} = r'^n$$

$$\frac{\Pi''^{n+1}B - \Pi''^n B}{\Delta t} + \nabla'' \times \Pi^{\diamond n+1/2} \mathbf{E} = r''^n$$

$$\frac{\Pi^{\diamond n+1/2} \mathbf{E} - \Pi^{\diamond n-1/2} \mathbf{E}}{\Delta t} - c^2 \nabla^{\diamond} \times \Pi'^n B + \frac{1}{\epsilon_0} \mathbf{J}^{\diamond n} = r^{\diamond n}$$



$$\frac{er'^{n+1} - er'^n}{\Delta t} + \nabla' \times er^{\diamond n+1/2} = r'^n$$

$$\frac{er''^{n+1} - er''^n}{\Delta t} + \nabla'' \times er^{\diamond n+1/2} = r''^n$$

$$\frac{er^{\diamond n+1/2} - er^{\diamond n-1/2}}{\Delta t} - c^2 \nabla^{\diamond} \times er'^n = r^{\diamond n}$$



$$er^{n+1} = \mathcal{M}_h er^n + \Delta t r^n = \mathcal{M}_h^{n+1} er^0 + \Delta t \sum_{k=0}^n \mathcal{M}_h^{n-k} r^k$$



Convergence

$$\|er^n\| \leq \|\mathcal{M}_h^{n+1} er^0\| + \Delta t \sum_{k=0}^n \|\mathcal{M}_h^{n-k} r^k\| \leq K \left(\|er^0\| + \Delta t \sum_{k=0}^n \|r^k\| \right)$$

Projection de la solution exacte :

$$(\Pi^{\diamond n+1/2} \mathbf{E})_j \cdot \mathbf{t}'_j := \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \frac{1}{|A'_j|} \int_{A'_j} \mathbf{E} \cdot \mathbf{t}'_j$$

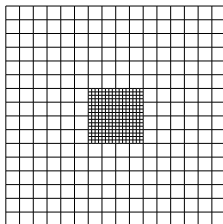
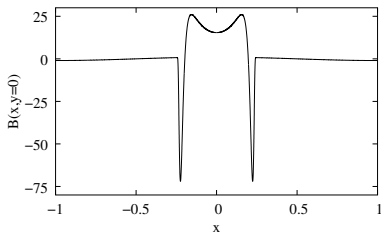
$$(\Pi^{\diamond n+1/2} \mathbf{E})_j \cdot \mathbf{t}''_j := \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \frac{1}{|A''_j|} \int_{A''_j} \mathbf{E} \cdot \mathbf{t}''_j \Rightarrow \begin{array}{l} \|r^n\| \leq K(h + \Delta t^2) \\ \|er^0\| \leq K(h + \Delta t^2) \end{array}$$

$$(\Pi'^n B)_i := \frac{1}{\Delta t} \int_{t^{n-1/2}}^{t^{n+1/2}} B(C'_i, \cdot)$$

$$(\Pi''^n B)_i := \frac{1}{\Delta t} \int_{t^{n-1/2}}^{t^{n+1/2}} B(C''_i, \cdot)$$

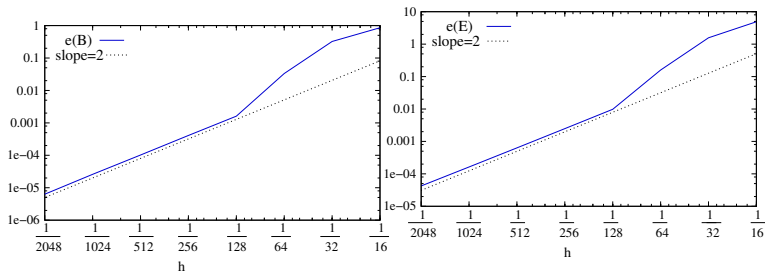


Etude de convergence sur des maillages non conformes





Convergence



$$e(B) = \sqrt{\frac{\int_{9T}^{10T} |B - \Pi B|_{L^2}^2}{\int_{9T}^{10T} |\Pi B|_{L^2}^2}}$$

B : Champs magnétique numérique

ΠB : Projection du champs magnétique

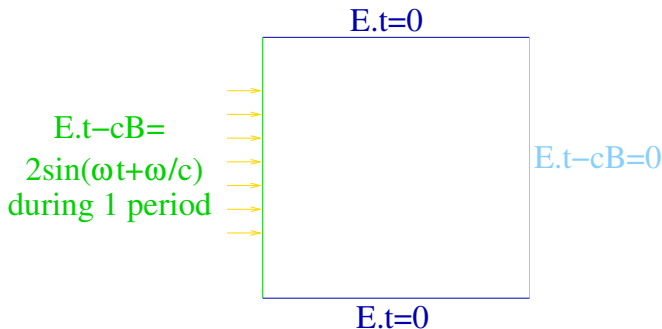
$$e(E) = \sqrt{\frac{\int_{9T}^{10T} |E - \Pi E|_{L^2}^2}{\int_{9T}^{10T} |\Pi E|_{L^2}^2}}$$

E : Champs électrique numérique

ΠE : Projection du champs électrique exact

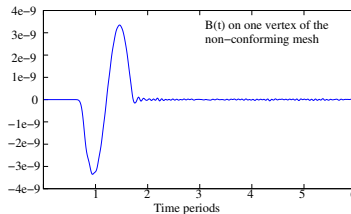
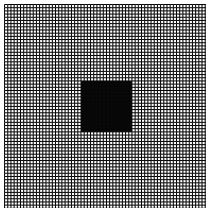
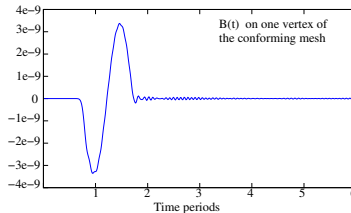
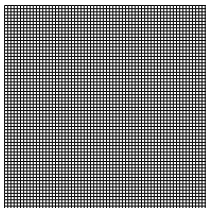


Onde rentrante avec sortie absorbante





Reflections parasites



⇒ La non conformité du maillage n'amplifie pas les réflexions parasites.



Equation de Vlasov

$$\frac{\partial f}{\partial t} + \mathbf{V} \cdot \nabla_{\mathbf{x}} f + \frac{q}{m} (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = 0$$

Couplage avec Maxwell

$$\rho(\mathbf{X}, t) = q \int_{\mathbb{R}^3} f(\mathbf{X}, \mathbf{V}, t) d\mathbf{V}$$

$$\mathbf{J}(\mathbf{X}, t) = q \int_{\mathbb{R}^3} \mathbf{V} f(\mathbf{X}, \mathbf{V}, t) d\mathbf{V}$$



Méthode PIC

Approximation de la fonction de distribution :

$$\tilde{f}(\mathbf{X}, \mathbf{V}, t) = \sum_{p=1}^{N_p} \omega_p \delta(\mathbf{X} - \mathbf{X}_p(t)) \delta(\mathbf{V} - \mathbf{V}_p(t))$$

avec,

N_p : nombre de particules numériques

ω_p : poids numérique de la particule p

$$\frac{\partial \mathbf{X}_p(t)}{\partial t} = \mathbf{V}_p(t)$$

$$\frac{\partial \mathbf{V}_p(t)}{\partial t} = \frac{q}{m} (\mathbf{E}(\mathbf{X}_p(t), t) + \mathbf{V}(\mathbf{X}_p(t), t) \times \mathbf{B}(\mathbf{X}_p(t), t))$$



→ Initialisation : distribution initiale \Rightarrow positions et vitesses initiales des particules

Technique appliquée : quiet start (quasi monte Carlo)

↪ Interpolation des champs aux positions des particules



↪ Calcul des vitesses et des positions : pousseur de Boris

$$\frac{\mathbf{V}_k^{n+1} - \mathbf{V}_k^n}{\Delta t} = \frac{q}{m} \left(\mathbf{E}_k^{n+1/2} + \frac{\mathbf{V}_k^{n+1} + \mathbf{V}_k^n}{2} \times \frac{\mathbf{B}_k^{n+1} + \mathbf{B}_k^n}{2} \right)$$

$$\frac{\mathbf{X}_k^{n+1/2} - \mathbf{X}_k^{n-1/2}}{\Delta t} = \mathbf{V}_k^n$$

↪ Calcul de la densité du charge et du courant

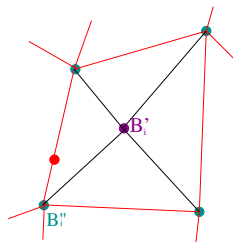
↪ Calcul des champs électromagnétiques : Maxwell





Interpolation des champs aux positions des particules

$$\begin{aligned}
 \text{Pour } \mathbf{X}_k^{n+1/2} &\in C_{i_k}^\diamond \cap C'_{i_k} \cap C''_{i_k} \\
 \mathbf{E}_k^{n+1/2} &= \mathbf{E}_{i_k}^{\diamond n+1/2} \\
 E_k^{n+1/2} &= \frac{1}{2} (E'_{i_k}{}^{n+1/2} + E''_{i_k}{}^{n+1/2}) \\
 \mathbf{B}_k^n &= \mathbf{B}_{i_k}^{\diamond n} \\
 B_k^n &= \frac{1}{2} (B'_{i_k}{}^n + B''_{i_k}{}^n)
 \end{aligned}$$





NGP (Nearest Grid Point)

$$\rho_i^{n+1/2} = \frac{q}{|C'_i|} \sum_{p=1}^{Np} \omega_p \mathbf{1}_{C'_i}(\mathbf{X}_p^{n+1/2})$$

$$\rho''_i{}^{n+1/2} = \frac{q}{|C''_i|} \sum_{p=1}^{Np} \omega_p \mathbf{1}_{C''_i}(\mathbf{X}_p^{n+1/2})$$

$$\mathbf{J}_j^{\diamond n+1/2} \cdot \mathbf{n}'_j = \frac{q}{\Delta t |A'_j|} \sum_{p=1}^{Np} \omega_p \frac{\mathbf{V}_p^n \cdot \mathbf{n}'_j}{|\mathbf{V}_p^n \cdot \mathbf{n}'_j|} \mathbf{1}_{A'_j}(\mathbf{X}_p^{n-1/2}, \mathbf{X}_p^{n+1/2})$$

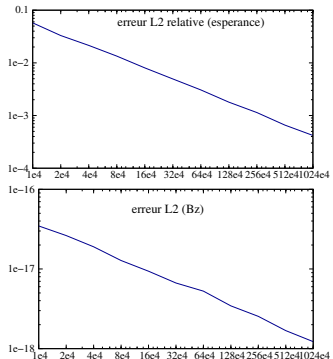
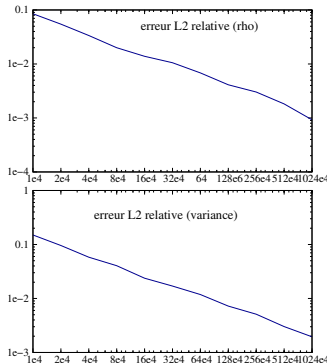
$$\mathbf{J}_j^{\diamond n+1/2} \cdot \mathbf{n}''_j = \frac{q}{\Delta t |A''_j|} \sum_{p=1}^{Np} \omega_p \frac{\mathbf{V}_p^n \cdot \mathbf{n}''_j}{|\mathbf{V}_p^n \cdot \mathbf{n}''_j|} \mathbf{1}_{A''_j}(\mathbf{X}_p^{n-1/2}, \mathbf{X}_p^{n+1/2})$$



Conservation de l'équation de charge discrète



Convergence en fonction du nombre des particules



$$f^0 = \frac{N}{\Pi\sqrt{\Pi}} \exp\left(-\frac{V_x^2 + V_y^2 + V_z^2}{V_{th}^2}\right) \quad \mathbf{E}^0 = 0 \quad \mathbf{B}^0 = 0$$



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