Nonequilibrium dynamics on complex topologies: models for epidemics and opinions

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Outline

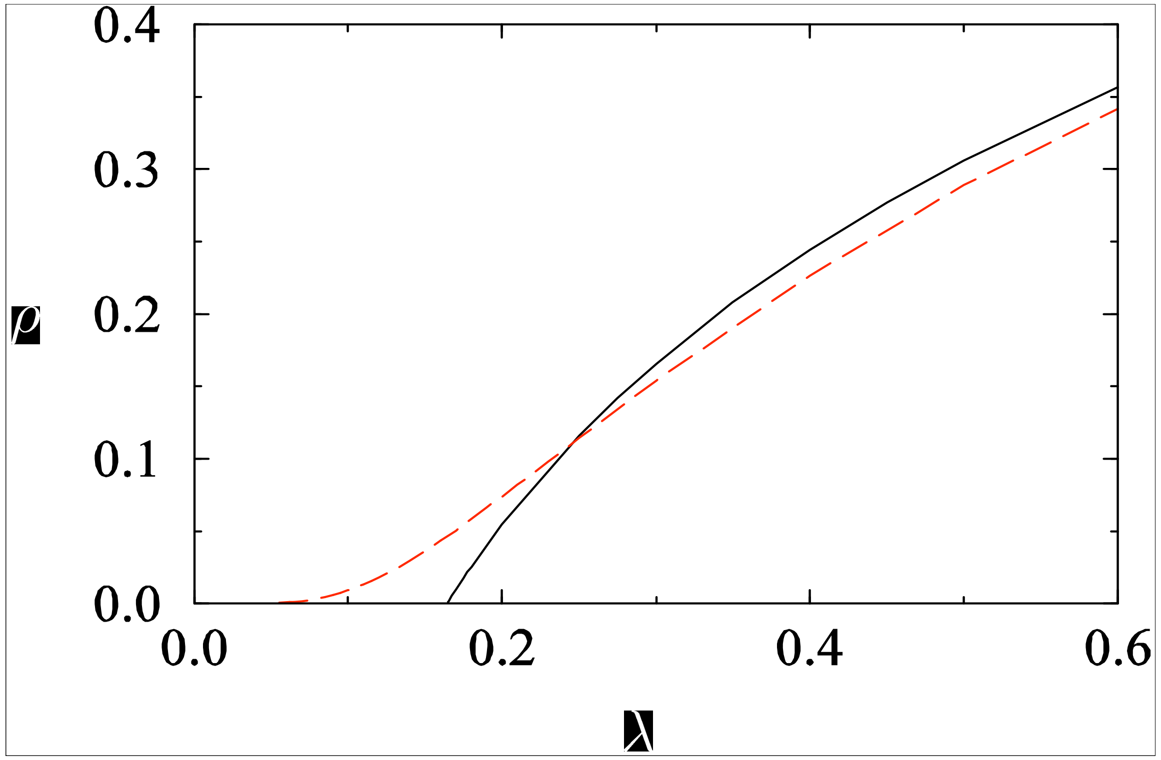
- Introduction
- Models for epidemics: SIS and SIR
- Disordered contact process: rare region effects
- Opinion dynamics: voter model on networks
- Conclusions

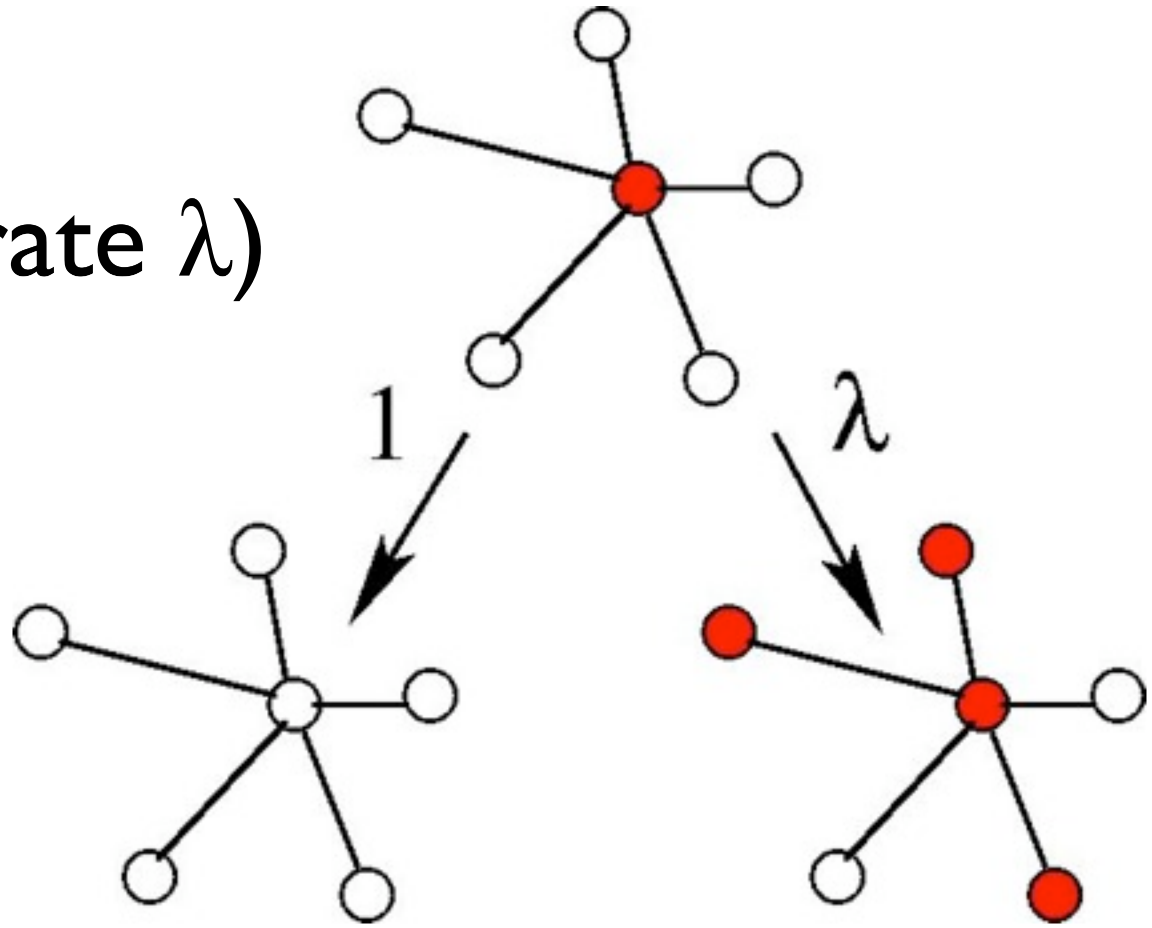
Introduction

- Statistical physics approach to interdisciplinary research
- Complex topologies are the natural substrate
- Highly nontrivial interplay between structure and dynamics

Susceptible-Infected-Susceptible (SIS) model

- Two possible states: susceptible and infected
- Two possible events for infected nodes:
  - Recovery (rate 1)
  - Infection to neighbors (rate λ)





Heterogeneous Mean-Field theory for SIS

<u>*Pastor-Satorras and Vespignani (2001)*</u>

- Degree distribution $P(k) \sim k^{-\gamma}$
- ρ_k = density of infected nodes of degree k

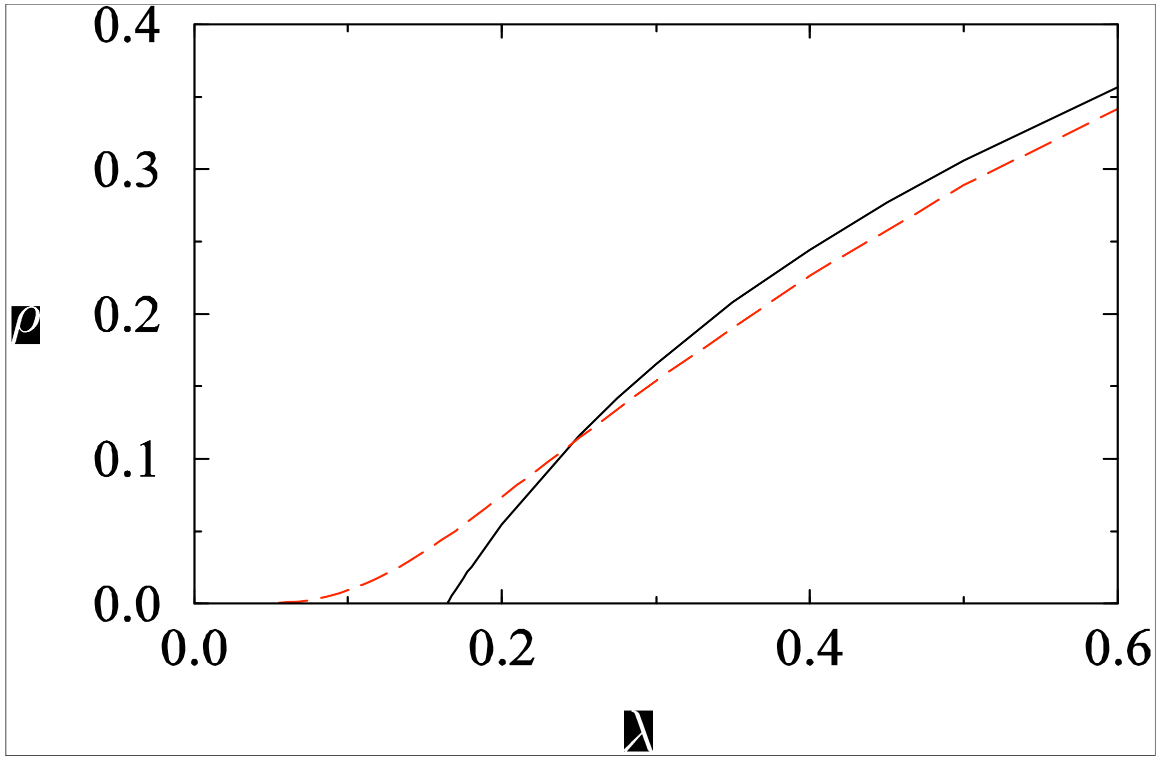
$$\dot{\rho_k} = -\rho_k + \lambda k[1-\rho_k] \sum_{k'} P(k'|k)\rho_{k'}$$

$$\lambda_c = \frac{1}{\frac{\langle k^2 \rangle}{\langle k \rangle}}$$

Scale-free networks

$\gamma < 3$

<u>zero epidemic threshold</u>



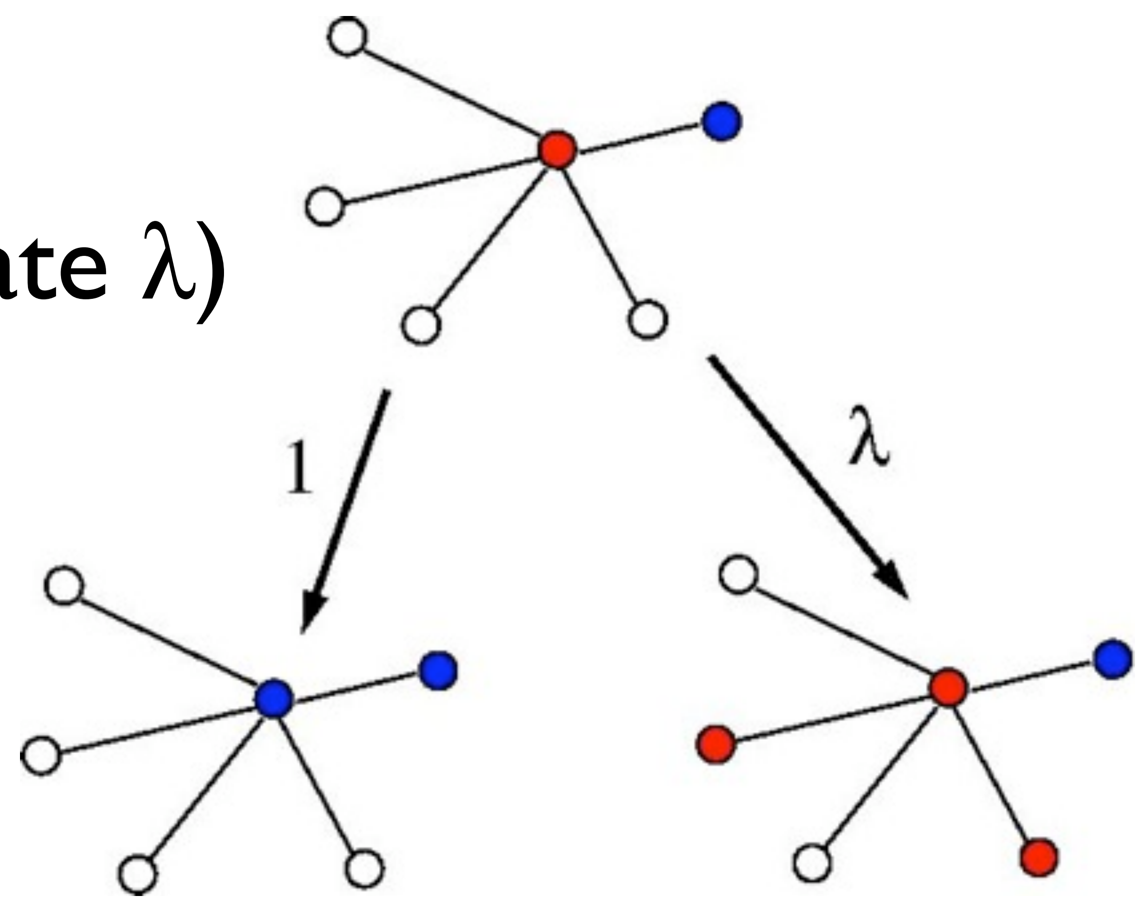
Susceptible-Infected-Removed (SIR) model

- Three possible states: susceptible, infected and removed.
- Two possible events for infected nodes:

▶ Death/recovery (rate 1)

▶ Infection to neighbors (rate λ)

▶ Transition between healthy and infected



HMF for SIR

- HMF theory

$$\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}$$

- Zero epidemic threshold for scale-free networks
- Finite epidemic threshold for scale-rich networks

Beyond HMF for SIS

- Wang et al., 2003
  Λ_N = largest eigenvalue of adjacency matrix

$$\lambda_c = \frac{1}{\Lambda_N}$$

- Chung et al., 2005

$$\Lambda_N = \begin{cases} c_1 \sqrt{k_c} & \sqrt{k_c} > \frac{\langle k^2 \rangle}{\langle k \rangle} \ln^2(N) \\ c_2 \frac{\langle k^2 \rangle}{\langle k \rangle} & \frac{\langle k^2 \rangle}{\langle k \rangle} > \sqrt{k_c} \ln(N) \end{cases}$$

k_c = largest degree in the network

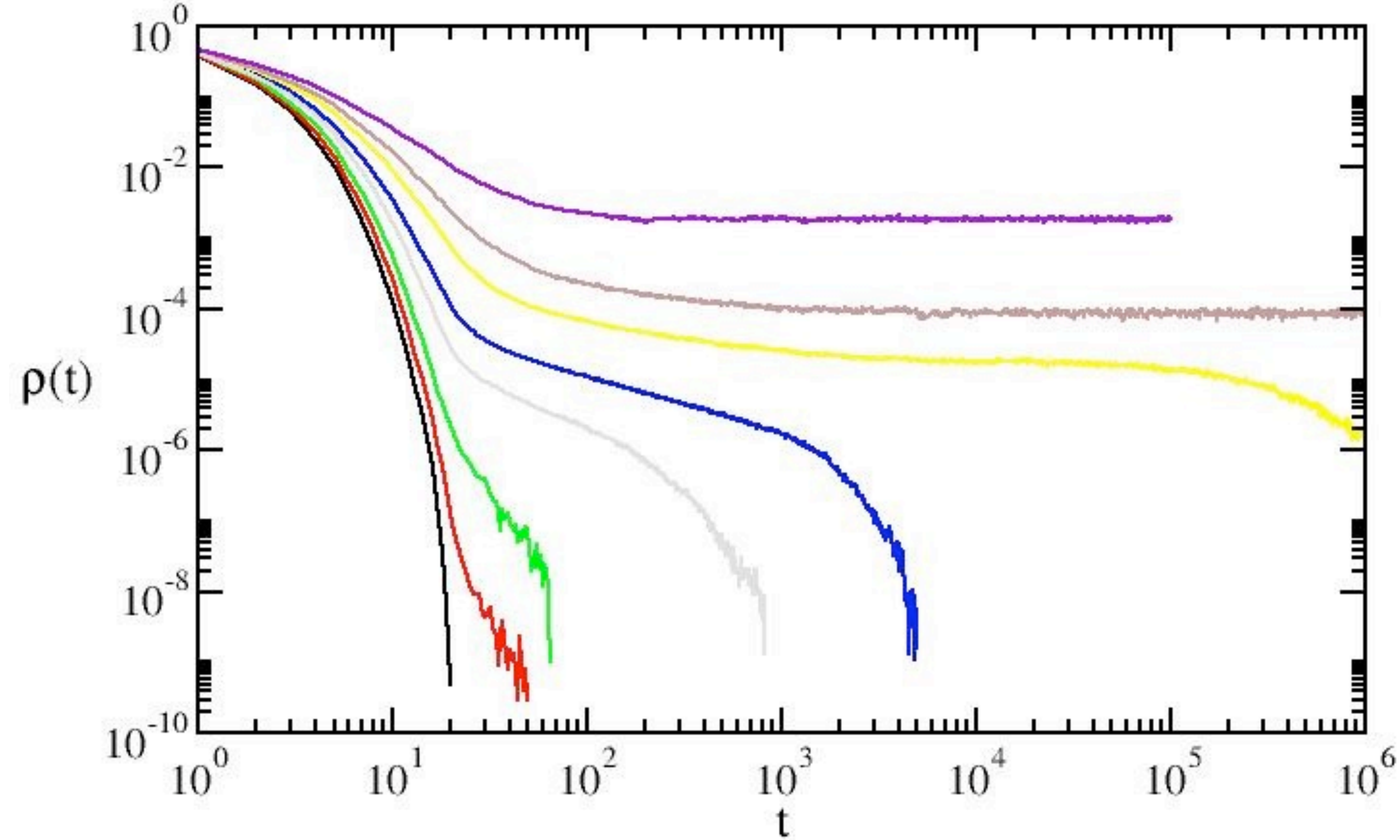
Beyond HMF for SIS

- Summing up

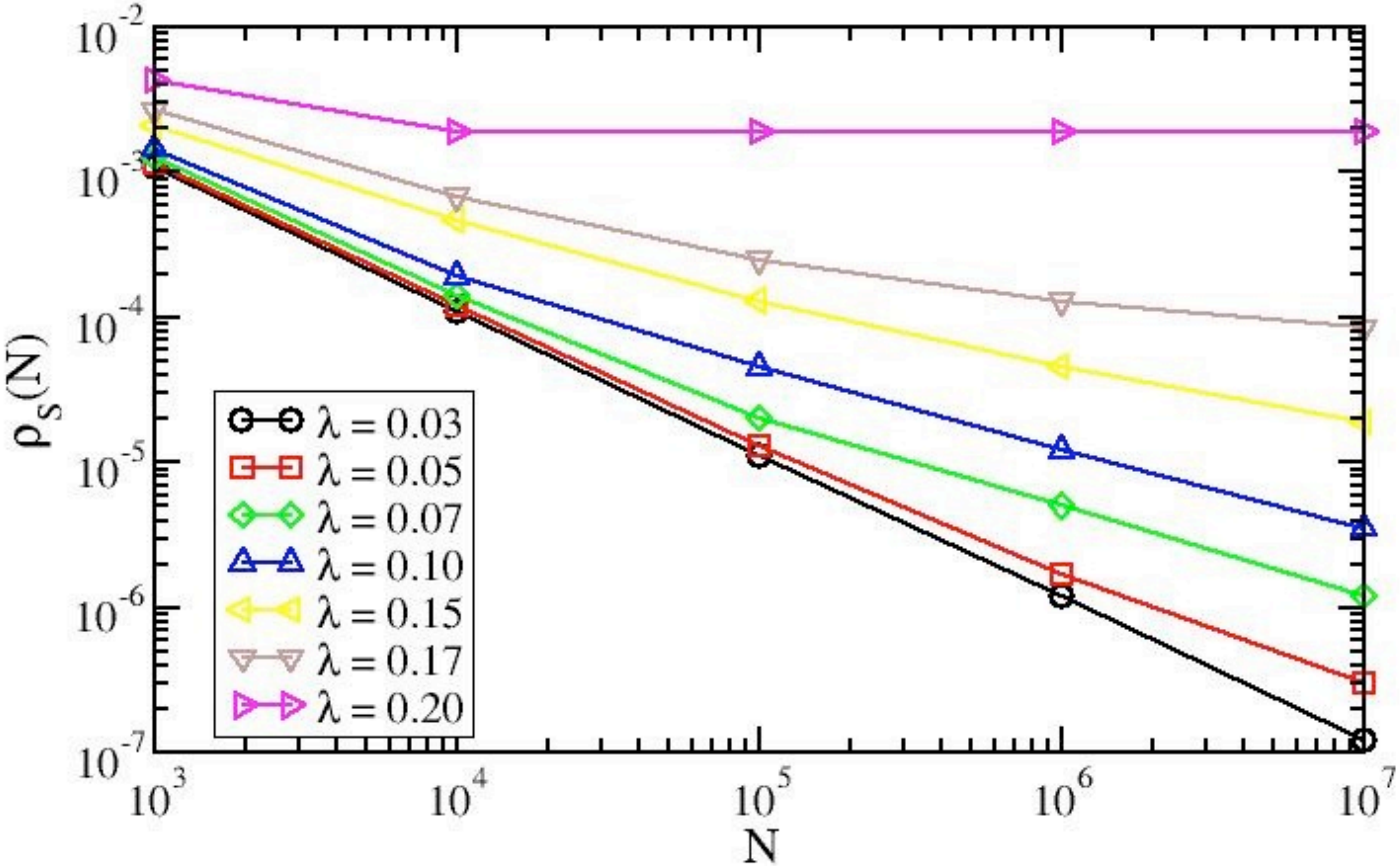
$$\lambda_c \simeq \begin{cases} 1/\sqrt{k_c} & \gamma > 5/2 \\ \frac{\langle k \rangle}{\langle k^2 \rangle} & 2 < \gamma < 5/2 \end{cases}$$

- In any uncorrelated quenched random network with power-law distributed connectivities, the epidemic threshold goes to zero as the system size goes to infinity.
- This has nothing to do with the scale-free nature of the degree distribution.

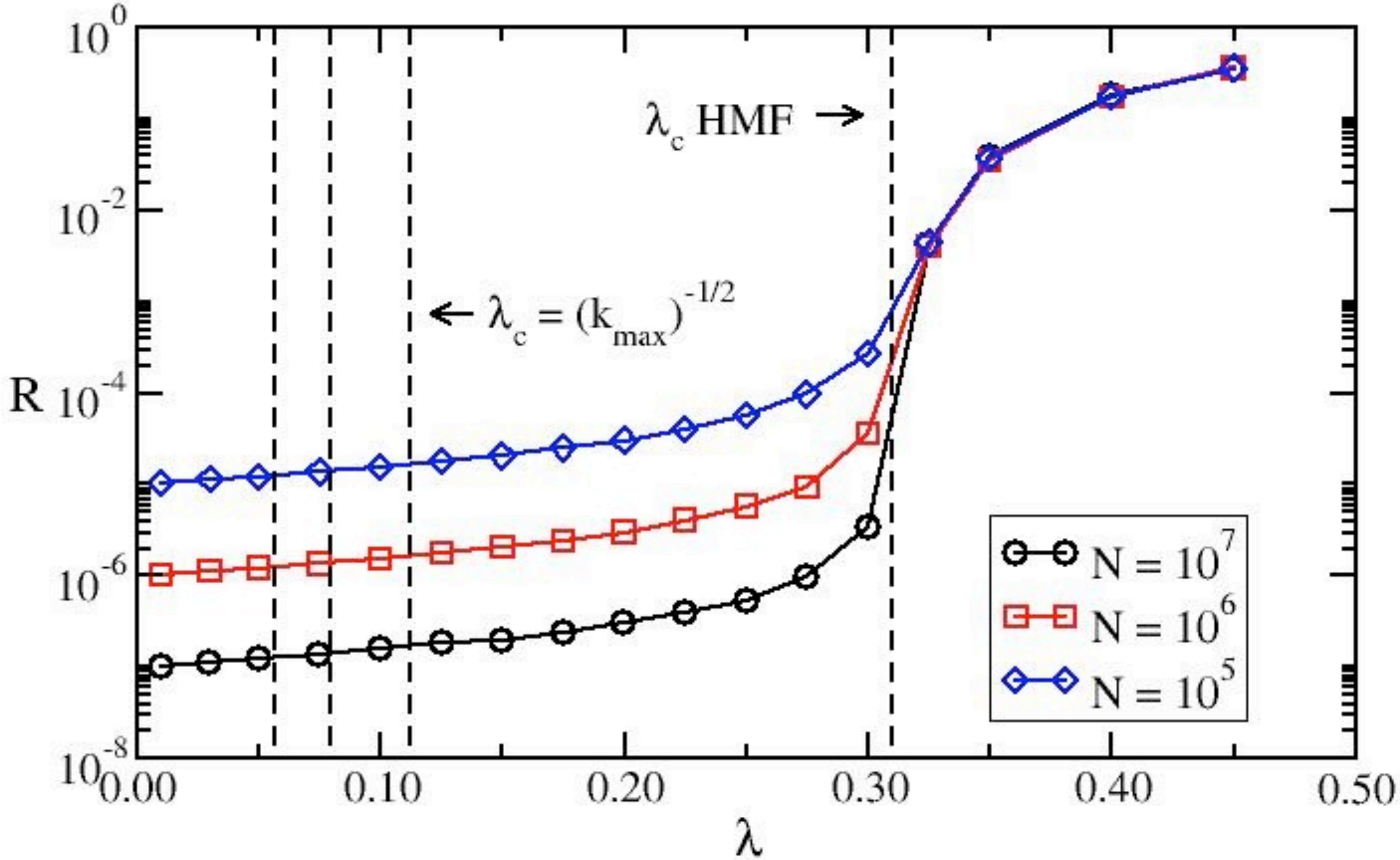
SIS $\gamma = 4.5$



Finite Size Scaling
SIS $\gamma = 4.5$



SIR $\gamma = 4.5$



Mathematical origin of HMF failure for SIS

- HMF is equivalent to using annealed networks with adjacency matrix

$$a_{ij} = \frac{k_i k_j}{\langle k \rangle N}$$

- This matrix has a unique nonzero eigenvalue

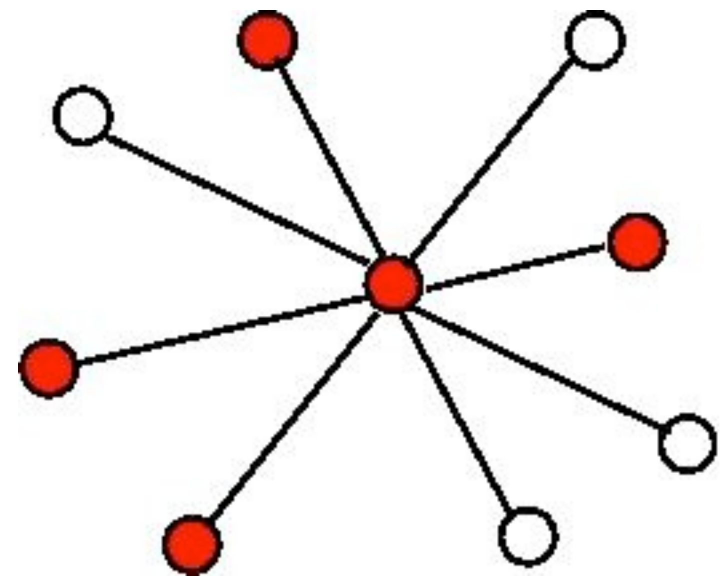
$$\Lambda_N = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

Physical origin of HMF failure for SIS

- Star graph with nodes

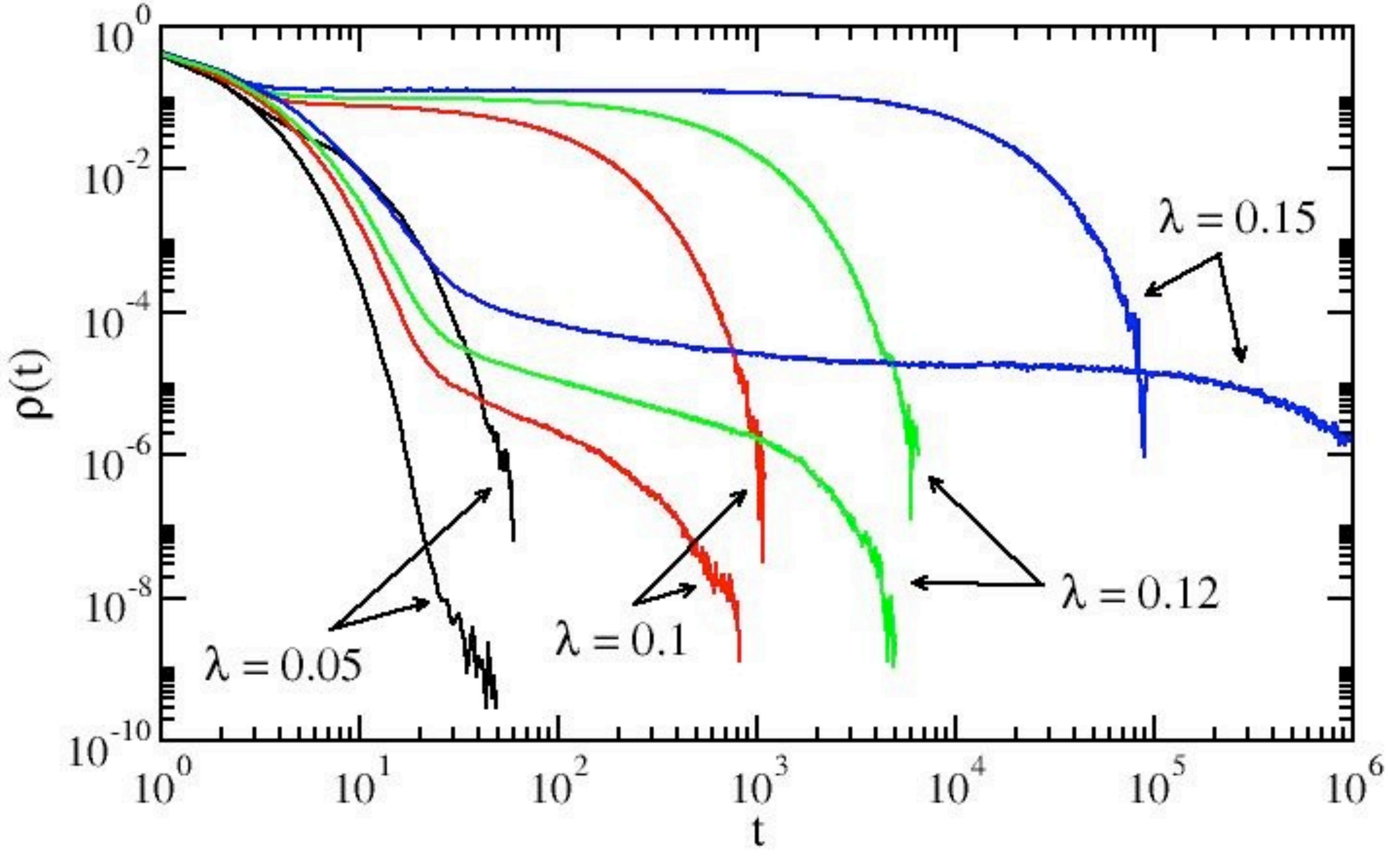
$$\rho_{max} \propto (\lambda^2 k_{max} - 1)$$

$$\rho_1 \propto (\lambda^2 k_{max} - 1)$$

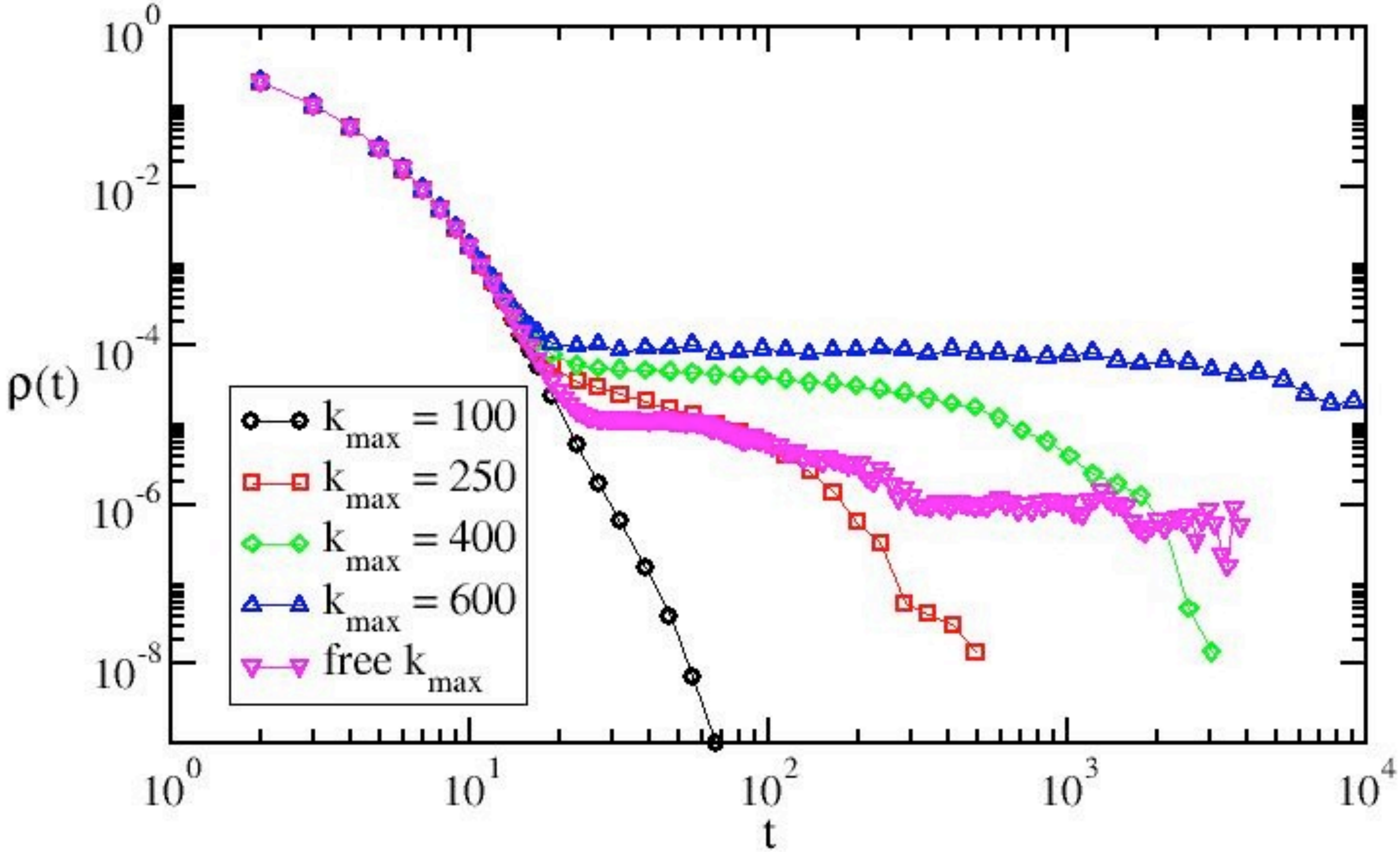


- For $\lambda > 1/\sqrt{k_{max}}$ the hub and its neighbors are a self-sustained core of infected nodes, which spread the activity to the rest of the system.

Star vs full graph same k_{max}



Fluctuations of k_{max}

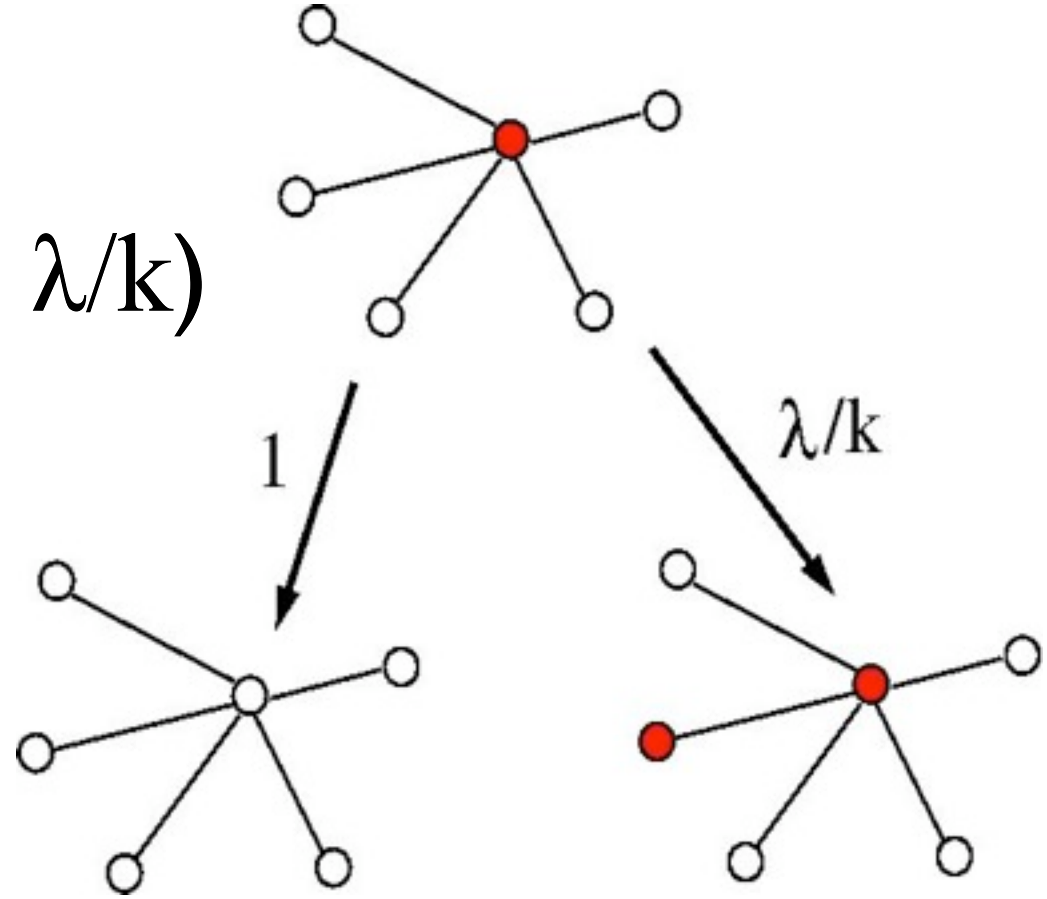


Summary on epidemics

- Zero epidemic threshold for SIS on scale-rich networks.
- Finite epidemic threshold for SIR on scale-rich networks.
- Conjecture: zero threshold for all models with steady state.
- Caveat: annealed networks are important for real epidemics.

Contact Process (CP)

- Two possible states: susceptible and infected
- Two possible events for infected nodes:
  - Recovery (rate 1)
  - Infection to neighbors (rate λ/k)



- Phase-transition with finite threshold

Quenched (disordered) contact process

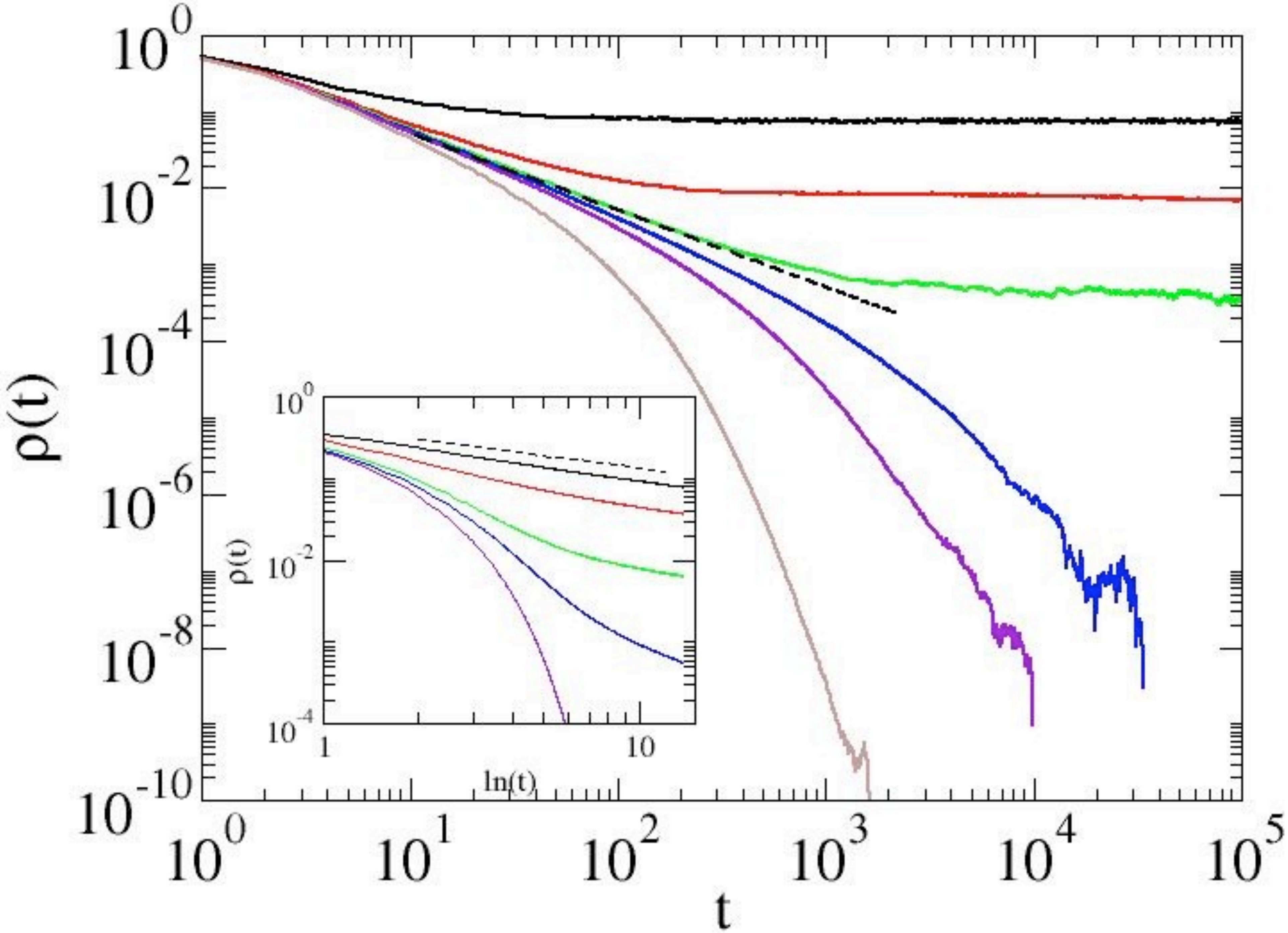
- A fraction q of nodes has reduced infection rate: $\lambda r \quad 0 \leq r \leq 1$
- A fraction 1-q of nodes has normal infection rate: λ

$$\lambda_c(q,r) = \frac{\langle k \rangle}{\langle k \rangle - 1} \, \frac{1}{1-q(1-r)}.$$

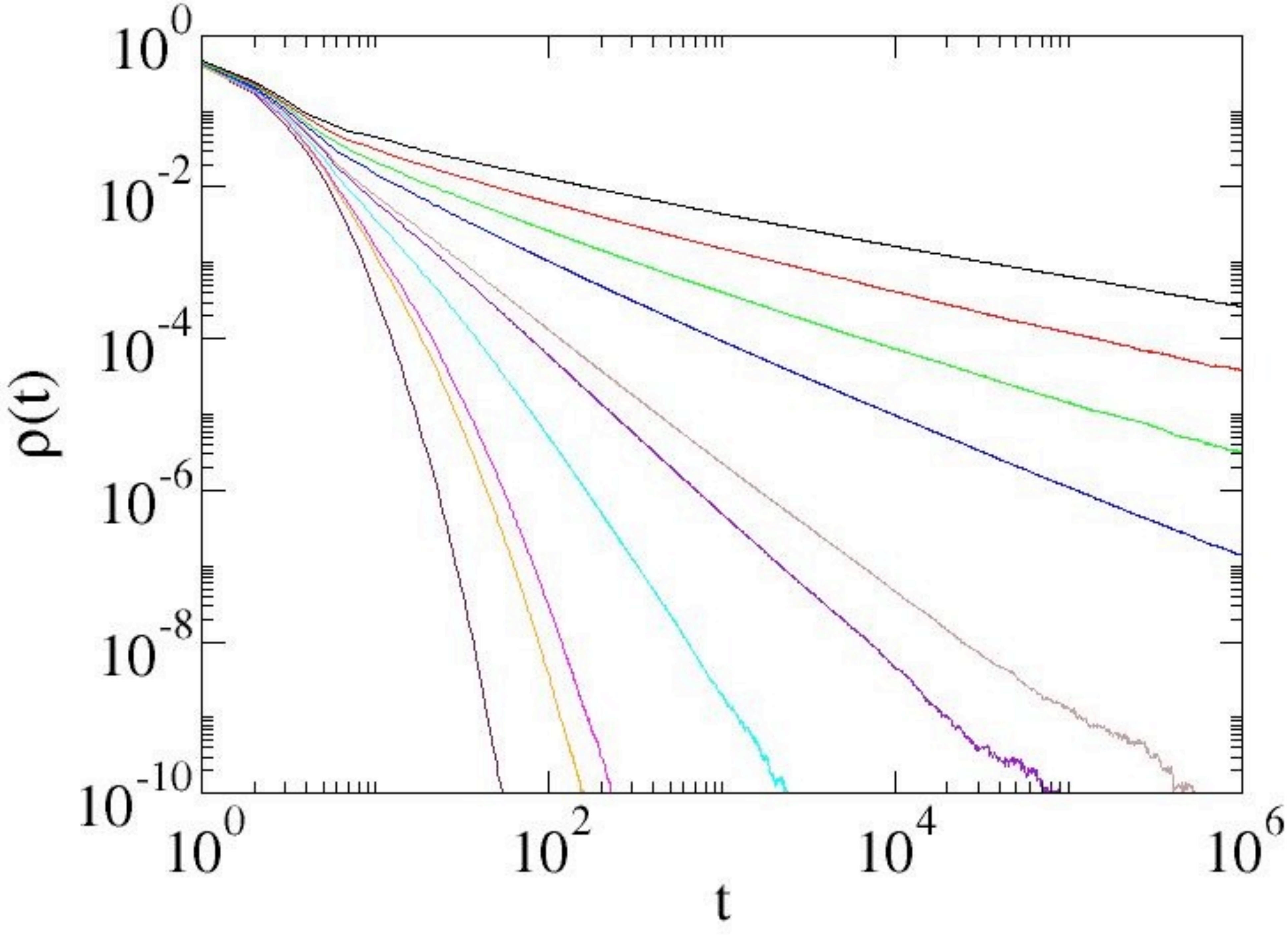
- For q > q_{perc} nodes with normal infection rate form only small clusters

$$q_{perc} = 1 - 1/\langle k \rangle$$

$q < q_{perc}$



$q > q_{perc}$



Rare regions effect

- For the "dirty" system, the threshold is larger than the threshold for the pure system.

$$\lambda_c^{dirty} > \lambda_c^{pure}$$

- For $\lambda_c^{pure} < \lambda < \lambda_c^{dirty}$, there are rare local clusters of "pure" nodes, which are above the threshold, i.e. in the active phase.

- Activity in pure clusters lives until a coherent fluctuation destroys it. This occurs in a time

$$\tau(s) \simeq t_0 \exp[A(\lambda)s]$$

Rare regions effect

- Size distribution of "pure" clusters

$$P(s) \sim \frac{1}{\sqrt{2\pi}p} s^{-3/2} e^{-s(p-1-\ln(p))} \qquad p = \langle k \rangle (1-q)$$

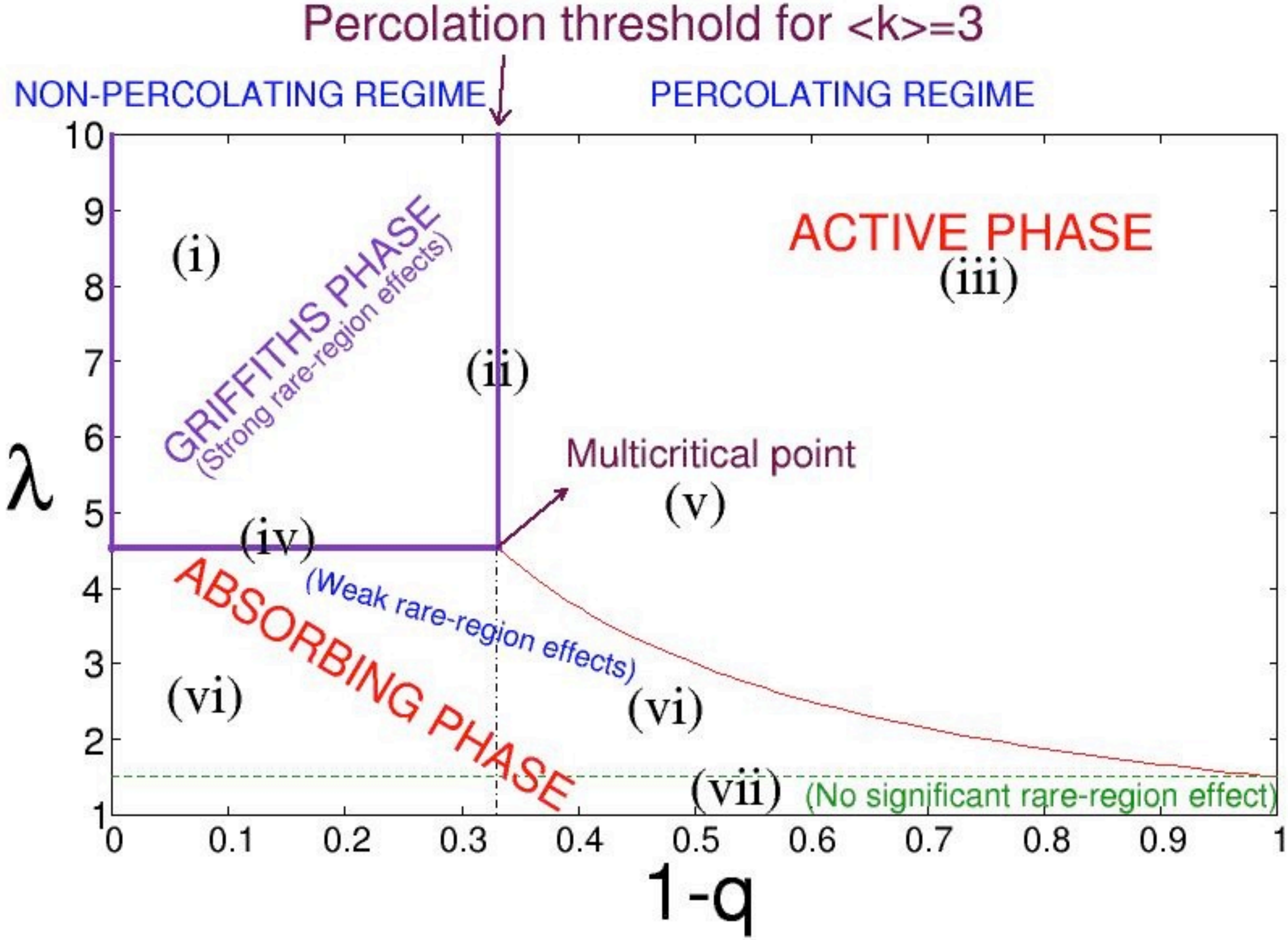
- Overall activity decay

$$\rho(t) \sim \int ds\ s\ P(s) \exp\left[-t/(t_0 e^{A(\lambda)s})\right] \sim t^{-\gamma(p,\lambda)}$$

$$\gamma(p,\lambda) = -(p-1-\ln(p))/A(\lambda)$$

Generic power-law decay
with continuously varying exponents

Phase diagram



Conclusions

- Models for epidemics have zero threshold also on scale-rich networks if they have a steady state. HMF may fail.
- Quenched disorder may yield generic power law decays.
- Voter dynamics is strongly affected by scale-free nature. Heterogeneous pair approximation works.

- C. Castellano and R. Pastor-Satorras, "Thresholds for epidemic spreading in networks" (soon in arXiv)
- M. A. Munoz, R. Juhasz, C. Castellano and G. Odor, "Griffiths phases in networks" (soon in arXiv)
- E. Pugliese and C. Castellano, "Heterogeneous pair approximation for voter models on networks", EPL, 88, 58004 (2009)