Polynomial kernels for 3-leaf power graph modification problems

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Plan

Edition - Motivation

- Experimental errors.
- Graph modification problems.
- NP-hard (ex : CLUSTER EDITION).

Graph modification problems

$\mathcal{F}$-*edition* : transform a graph G with at most k edges modification into a graph that belongs to $\mathcal{F}$.

Different techniques

- Approximation algorithms
- Exact exponential algorithms
- Probabilistic algorithms
- Parameterized algorithms

Plan

Parameterized algorithm

A problem parameterized by $k \in \mathbb{N}$ is said to be fixed-parameter tractable (in *FPT*) if it can be solved in time $f(k).n^{O(1)}$.

Remarks

The function f considered can be anything and depends only on the parameter k. Thus, the function $f(k) = 2^{2^{2^k}}$ is good.

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Kernelization

Given be a parameterized problem Π and $(I, k) \in \Pi$, a kernelization is defined as follows :

$$(I, k) \xrightarrow{\text{reduction rules}} (I', k')$$

$$(I \in sol(\Pi) \Leftrightarrow I' \in sol(\Pi))$$

$$|I'| \leq h(k) \quad \text{et} \quad k' \leq k$$

Theorem

$\Pi \in FPT \Leftrightarrow \Pi$ has a kernel (size : exponential).

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Consequences

- *Pre-processing*
- $\Rightarrow$ Reducing the size of a given input
- $\Rightarrow$ Resolution on kernels
- $\Rightarrow$ Additive complexity ($O(g(k) + poly(n))$).

CLUSTER EDITION : known results

- An $O(k^2)$ kernel can be built in $O(n + m)$ (Protti et al., 2007).
- An $O(k)$ kernel can be built in $O(n.m^2)$ (Guo, 2007).

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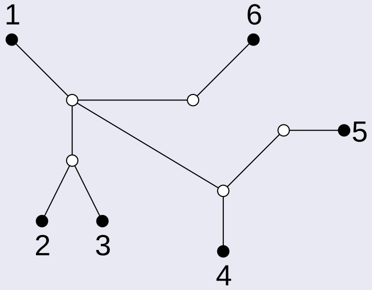
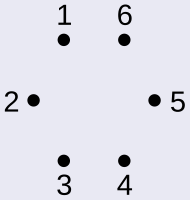
Example : 3-*leaf power*





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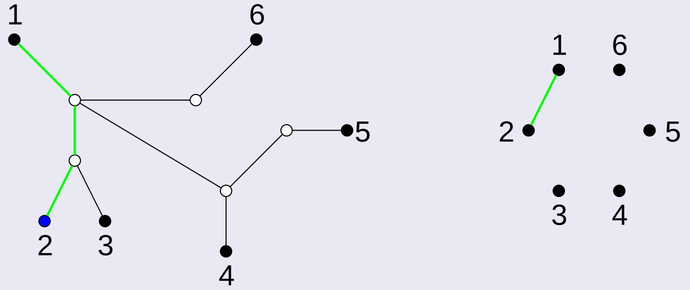

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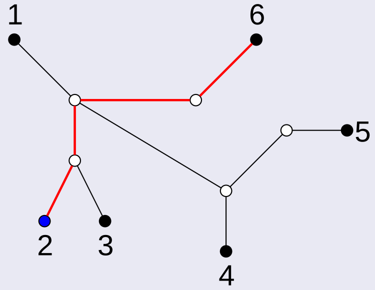

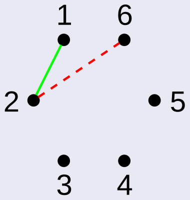

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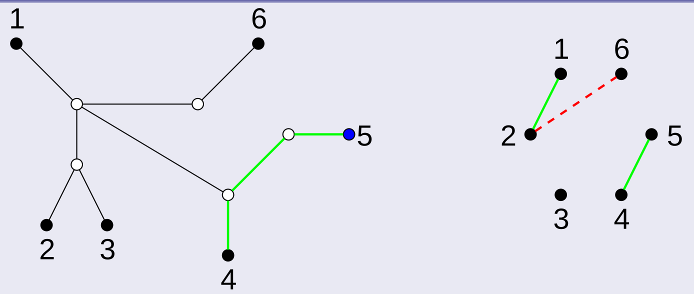

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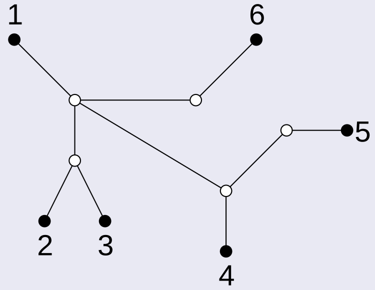

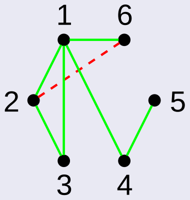

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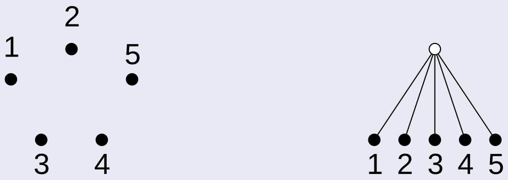
1-LEAF POWER

1-*leaf powers* are exactly independent sets.



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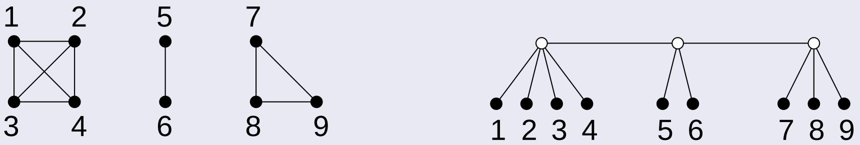
2-LEAF POWER (CLUSTER GRAPH)

2-*leaf powers* are disjoint union of cliques.



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Properties

- Every *leaf power* is chordal.
- The p-LEAF POWER class of graphs is closed under induced subgraph and true twin addition.

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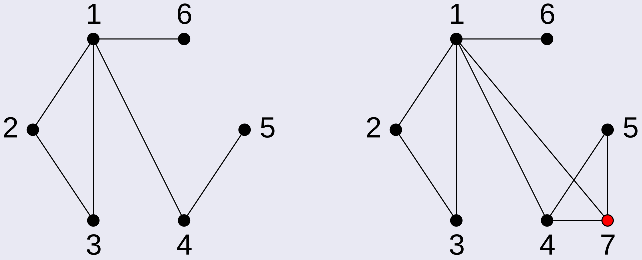
True twin



Recognition

- Polynomial for $p \leq 5$ (Brandstädt (05,06), Chang et Ko 07).
- Open for $p > 5$.

Edition

	$p = 2$	$p \geq 3$
Edition	NP-hard (KM, 86)	NP-hard (Dom et al., 05)
Deletion	NP-hard (NSS, 99)	NP-hard (Dom et al., 05)
Completion	P	NP-hard (Dom et al., 05)

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FPT results

- 3-LEAF POWER EDITION $\in$ *FPT* (Dom et al., 05)
- 4-LEAF POWER EDITION $\in$ *FPT* (Dom et al., 08)

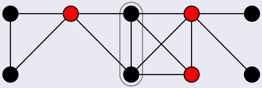
Plan

Critical clique (Lin et al, 00)

A critical clique is a maximal clique module.

Example

G



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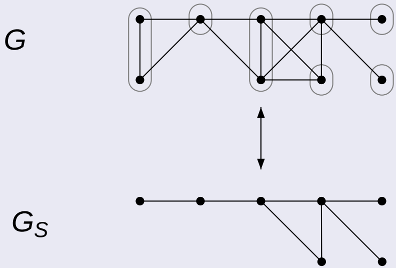
G_S : Example



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Consequence

A graph is a 3-*leaf power* if and only if its critical clique graph is a tree.

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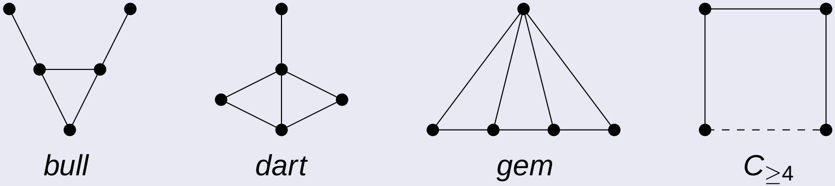
Forbidden induced subgraphs



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A graph is a 3-*leaf power* if and only if its critical clique graph is a tree.

Lemma (Dom et al., 05)

One can always find an optimal 3-*leaf power edition* that does not break any critical clique.

Lemma

Given $\mathcal{F}$ an hereditary graph family closed under true twin addition, there always exists an optimal edition that does not break any critical clique.

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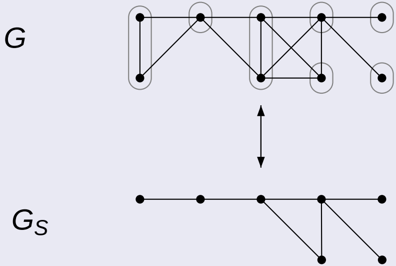
G_S : Example



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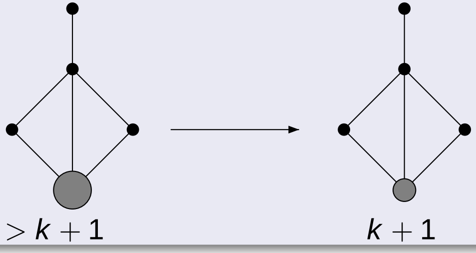
Problem

Question : is there a polynomial kernel for the 3-LEAF POWER EDITION problem ? (Dom, Guo, Hüffner et Niedermeier., 05).

Plan

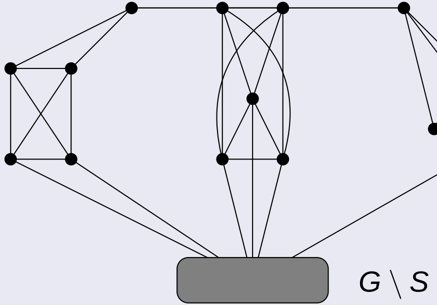
Connected components and critical cliques

- Remove from G every connected component C such that $G[C]$ is 3-leaf power.
- If G has a critical clique K of size $|K| > k+1$, then remove $|K| - k - 1$ vertices of K from $V(G)$.



Branch

An induced subgraph $G[S]$, $S \subseteq V$, is a branch if S is the disjoint union of critical cliques K_i such that the subgraph of G_S induced by $\cup_i K_i$ is a tree.



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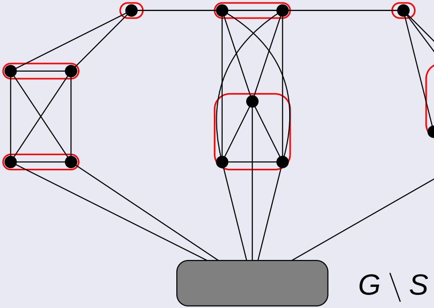

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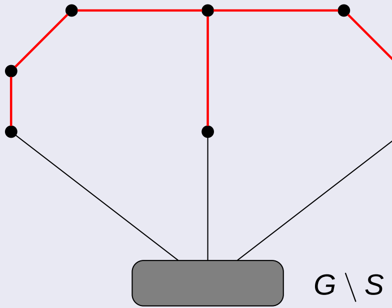
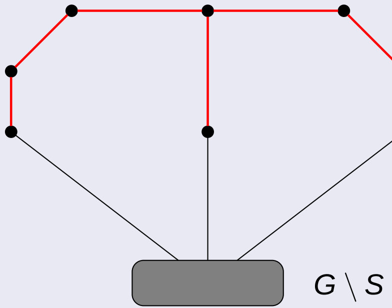


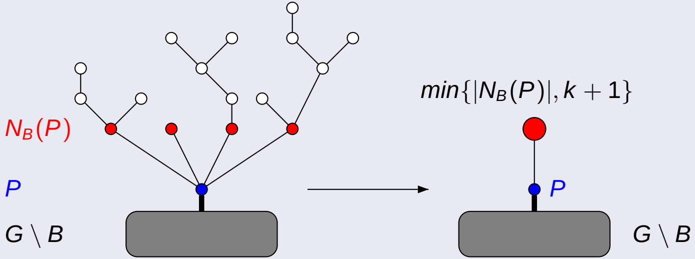
Notation

- An attachment point is a critical clique which has neighbors in $G \setminus S$.
- An i-branch is a branch B with i attachment points.



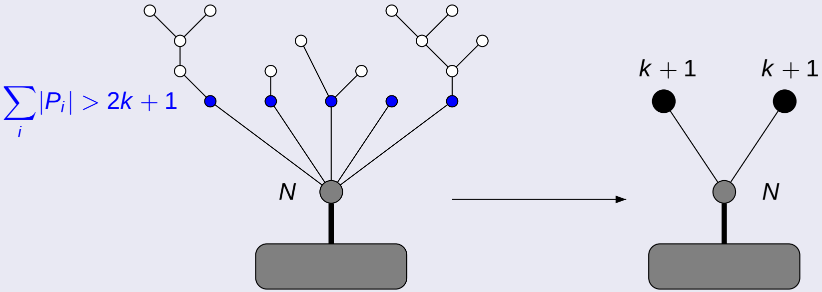
Rule : 1-branch

If G contains a 1-branch B with attachment point P, then remove from G the vertices of $B \setminus P$ and add a new critical clique of size $\min\{|N_B(P)|, k+1\}$ adjacent to P.



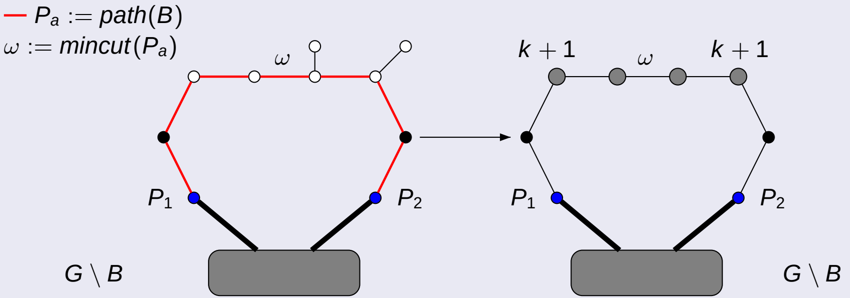
Rule : several 1-branches

Idea. If too many 1-branches are attached to the same neighborhood N in G, then transform N into a critical clique.



Rule : 2-branch

Let B be a 2-branch whose $|path(B)|$ contains at least 8 critical cliques. It is safe to do the following :



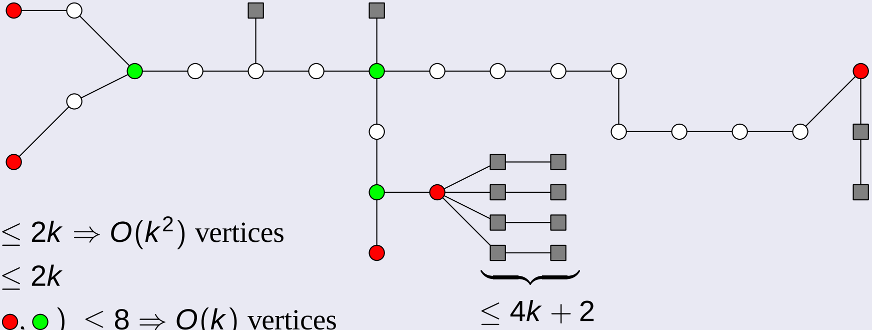
Plan

Size : strategy

Let G be a reduced graph and $|F| \leq k$ s.t. $H = G + F$ is a 3-*leaf power*.

- Count the number of vertices of H_S.

Size : strategy



|●| $\leq 2k \Rightarrow O(k^2)$ vertices

|●| $\leq 2k$

d_T(●, ●) $\leq 8 \Rightarrow O(k)$ vertices

Size : strategy

- Count the number of vertices of H_S.
- Deduce the number of vertices of G_S ($|G_S| \leq |H_S| + 4k$ *(Protti et. al, 07)*).
- Conclude by the rule which bound the number of vertices of every critical clique of G.

Theorem

An $O(k^3)$ kernel can be built in linear time for both 3-LEAF POWER EDITION and 3-LEAF POWER DELETION.

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Remark

For the 3-LEAF POWER COMPLETION problem, the 2-branch reduction rule is no longer safe. Thus, in order to build a cubic kernel for this problem, another reduction rule is needed.

Theorem

An $O(k^3)$ kernel can be built in linear time for both 3-LEAF POWER EDITION and 3-LEAF POWER DELETION.

Rule : long cycle

If G has a clean 2-branch such that $|path(B)| \geq k + 4$ then there is no 3-leaf power completion of size at most k for G.

Plan

Results

- Cubic kernels for 3-LEAF POWER EDITION and 3-LEAF POWER DELETION.
- Cubic kernel for 3-LEAF POWER COMPLETION.

Perspectives and open problems

- Polynomial kernel for 4-LEAF POWER EDITION
- Recognition of *p-leaf powers*, $p \geq 6$
- Tractability of p-LEAF POWER EDITION, $p \geq 5$

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