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Fractional chromatic number of triangle free graphs with given maximal average degree

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Outlines

- Definitions
- Some known results
- Our main result

(n,k)-coloring

A k-tuple coloring, is a generalization of the usual graph coloring. Instead of just one color, we assign to each vertex a subset with k distinct colors and require that adjacent vertices have disjoint color sets.

Briefly, we write (n,k)-coloring instead.

An (n,1)-coloring is an ordinary proper n-coloring.

The fractional chromatic number of G, denoted by

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In 1995, Leader proved that the fractional chromatic number is not always attained. It depends on whether $\chi_f(G)$ is rational or not.

Homomorphism

A homomorphism from G into H is a map $\varphi : V(G) \to V(H)$ such that adjacent vertices in G are mapped into adjacent vertices in H.

Kneser graph

The Kneser graph $K_{n:k}$ is a graph with a vertex set consisting all k-element subsets of $\{1,2,\cdots,n\}$ and two vertices are adjacent if and only if the corresponding subsets are disjoint.

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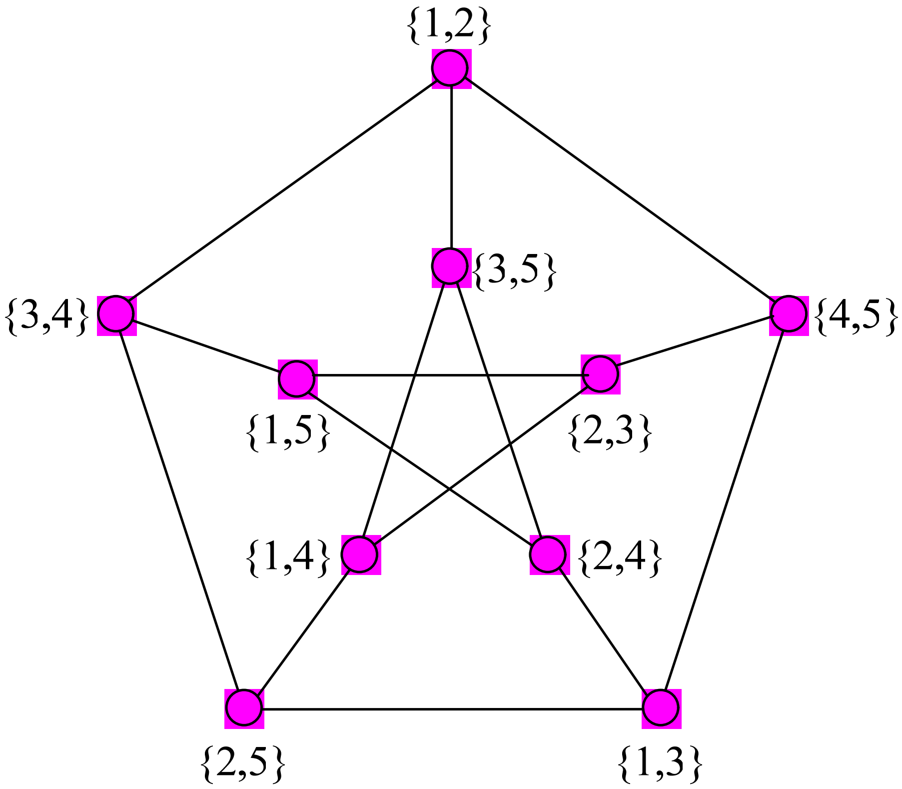
An (n,k)-coloring of G is also a homomorphism of G into $K_{n:k}$.

$K_{5:2} = P_{10}$



(k,d)-circular coloring

For positive integers k ≥ 2d, a (k,d)-circular coloring of graph G is a map $\varphi : V(G) \to \{0,\cdots,k-1\}$ such that $d \leq |\varphi(x)-\varphi(y)| \leq k-d$ for each edge $xy \in E(G)$.

A graph having such a coloring is (k,d)-circular colorable.

The circular chromatic number of G, denoted by

$$\chi_c(G) = \min\ \{\frac{k}{d} : \text{G has a (k,d)-coloring}\}.$$

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Two inequalities

The following well-known inequalities hold for every graph G:

$$\chi(G) - 1 < \chi_c(G) \le \chi(G).$$

and

$$\chi_f(G) \le \chi_c(G) \le \chi(G).$$

Conjecture

Conjecture – [Jaeger ’84]

For any integer $k \geq 1$, every 4k-edge-connected graph admits a (2k+1,k)-flow.

For planar graphs, the flow problem can be dualized to a circular coloring problem.

Conjecture – [Jaeger's conjecture restricted to planar graphs]

Every planar graph G of girth at least 4k has circular chromatic number at most $2+\frac{1}{k}$.

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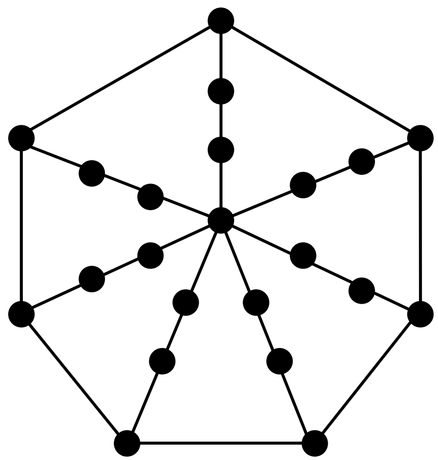
The threshold 4k is sharp by DeVos in 2000.



A counterexample of the case k=2.

Case of k=1

Theorem - [Grötzsch '59]
Every planar graph with girth at least 4 is 3-colorable.

Case of k=1

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Every planar graph with girth at least 4 is 3-colorable.

Every planar graph with girth at least 4 has circular chromatic number at most 3.

Theorem - [Nešetřil and Zhu '96]

Each planar graph G with girth at least 10k-4 suffices to

$$\chi_c(G) \leq 2 + \frac{1}{k}.$$

The same result was proved by Galuccio, Goddyn and Hell in 2001.

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Theorem - [Zhu '01]

Each planar graph G with odd girth at least 8k-3 suffices to

$$\chi_c(G) \le 2 + \frac{1}{k}.$$

Theorem - [Borodin * et al. '04]

Planar graph G with girth at least $\frac{20k-2}{3}$ has a circular chromatic number at most $2+\frac{1}{k}$.

* Borodin, Kim, Kostochka and West.

Theorem - [Klostermeyer and Zhang '02]

Planar graph G with girth at least 10k-7, has a fractional chromatic number at most $2+\frac{1}{k}$.

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Theorem - [Pirnazar and Ullman '02]

Planar graph G with odd girth at least 8k-4, has a fractional chromatic number at most $2+\frac{1}{k}$.

Theorem - [Dvořák, Škrekovski and Valla '08]

If G is a planar graph with odd-girth at least 9, then $G \to P_{10}$.

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If G is a planar graph with odd-girth at least 9, then $G \rightarrow P_{10}$.

Planar graph G of girth at least 8 suffices to $\chi_f(G) \leq 2 + \frac{1}{2}$.

Maximal average degree

Definition

$$Mad(G) = \max\{\frac{2\cdot|E(H)|}{|V(H)|}, H \subset G\}.$$

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If G is a planar graph with girth g, then Mad(G)< $\frac{2\cdot g}{g\text{-}2}$.

Theorem - [Borodin * et al. ’07]

Every triangle-free graph G with $\mathrm{Mad}(G) < 12/5$ suffices to $\chi_c(\mathrm{G}) \le \frac{5}{2}$.

*Borodin, Hartke, Ivanova, Kostochka and West.

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Corollary

Planar graph G with girth at least 12 suffices to $\chi_c(G) \leq \frac{5}{2}$.

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Our main result

Theorem – [Chen and Raspaud '08]

Let G be a simple triangle-free graph with maximal average degree.

- If Mad(G)<5/2, then $\chi_f(G) \leq 5/2$;
- If Mad(G)<9/4, then $\chi_f(G) \leq 7/3$;
- If Mad(G)<24/11, then $\chi_f(G) \leq 9/4$.

Corollary

Let G be a planar graph with girth g.

- If $g \geq 10$, then $\chi_f(G) \leq 5/2$;
- If $g \geq 18$, then $\chi_f(G) \leq 7/3$;
- If $g \geq 24$, then $\chi_f(G) \leq 9/4$.

Theorem’s proof

Suppose G is a triangle-free simple graph with Mad(G)<5/2, then $\chi_f(G) \leq 5/2$.

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- Choose a counterexample G with least vertices;
- Investigate the structural properties of G;

Structural properties

j-thread

A thread in G is a path whose internal vertices are 2-vertices. We use j-thread to denote a thread with exactly j internal 2-vertices.

Observation

Assume $P=v_0 \cdots v_4$ is a path and v_0 is precolored with a color $a \in V(K_{5:2})$. Then, $|F(v_0:v_1)|=7$, $|F(v_0:v_2)|=3$, $|F(v_0:v_3)|=1$ and $|F(v_0:v_4)|=0$.

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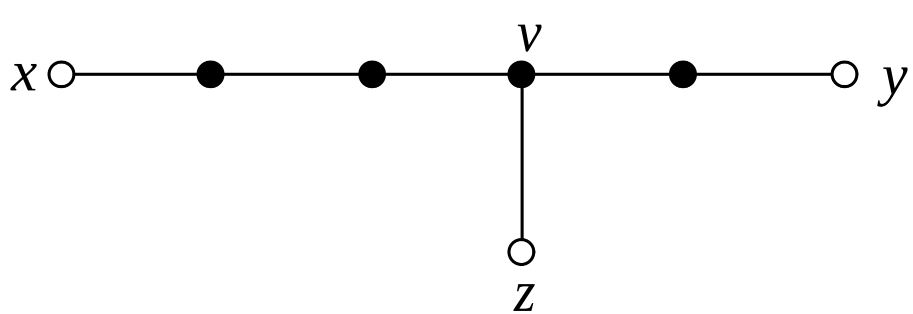
Lemma

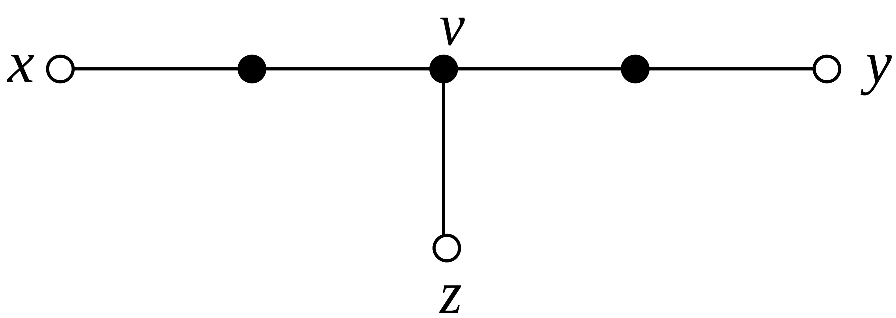
Assuming that v is a (2,1,0)-vertex. If the color of y is different to that of z, then v can be colored properly.



v is a (2,1,0)-vertex.

Lemma

Assuming v is a (1,1,0)-vertex. If the color of z is distinct to the colors of x and y, then v can be colored properly.



v is a (1,1,0)-vertex.

Theorem's proof

Suppose G is a triangle-free simple graph with Mad(G)<5/2, then $\chi_f(G) \leq 5/2$.

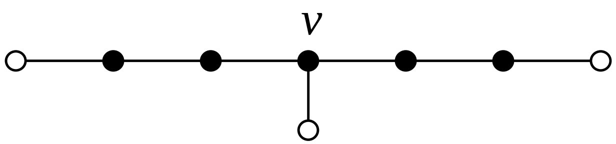
- Choose a counterexample G with least vertices;
- Investigate the structural properties of G;
- Show some reducible configurations of G;

Reducible configurations

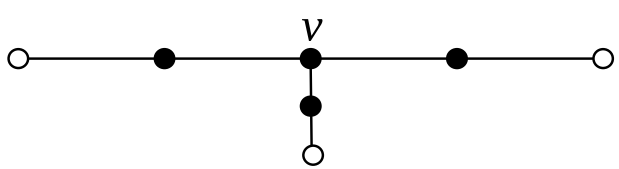
- Reducible vertices and threads of G;

✓ There is no 3^+-thread in G.

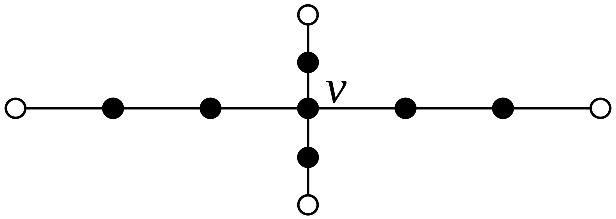
A 3^+-thread P



A $(2,2,0^+)$-vertex v



A $(1^+,1^+,1^+)$-vertex v



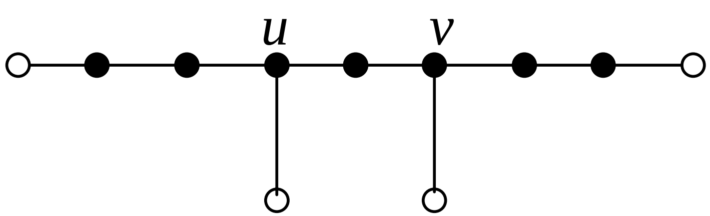
A $(1^+,1^+,2^+,2^+)$-vertex v

✓ No $(1^+,1^+,1^+)$-vertex, $(2,2,0^+)$-vertex, and $(1^+,1^+,2,2)$-vertex.

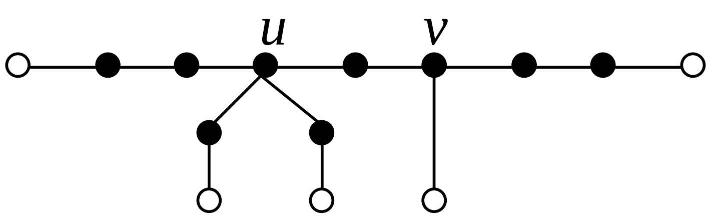
Reducible configurations

- Reducible vertices and threads of G;
- Reducible united thread structures of G;

✓ No $P_2(2,1,0)^*P_1(2,1,0)^*P_2$ and $P_2(2,1,1,1)^*P_1(2,1,0)^*P_2$.

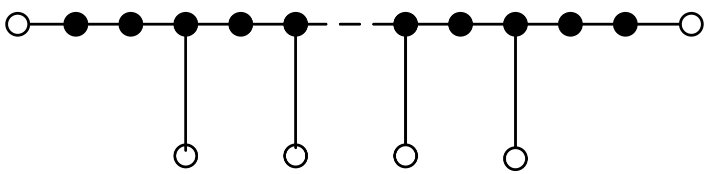


$P_2(2,1,0)^* P_1(2,1,0)^* P_2$

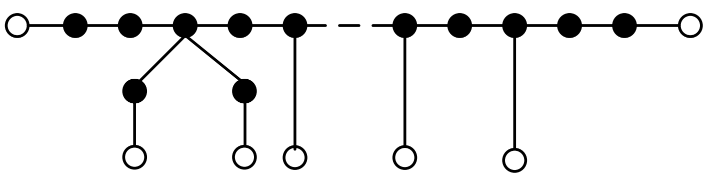


$P_2(2,1,1,1)^* P_1(2,1,0)^* P_2$

✓ No $P_2(2,1,0)^*P_1^i(2,1,0)^*P_2$ and $P_2(2,1,1,1)^*P_1^i(2,1,0)^*P_2$.

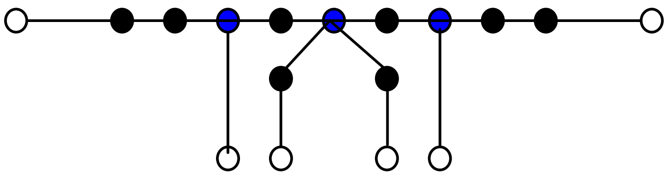


$P_2(2,1,0)^* P_1^i(2,1,0)^* P_2$

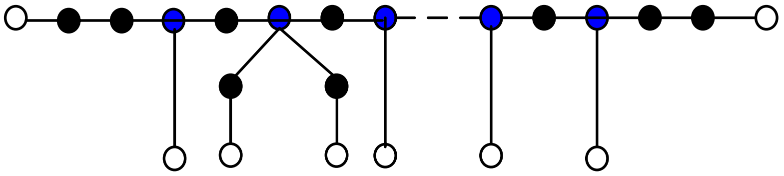


$P_2(2,1,1,1)^* P_1^i(2,1,0)^* P_2$

✓ No $P_2(2,1,0)^*P_1(1,1,1,1)^*P_1(2,1,0)^*P_2$, $P_2(2,1,0)^*P_1(1,1,1,1)^*P_1^i(2,1,0)^*P_2$ and $P_2(2,1,0)^*P_1^i(1,1,1,1)^*P_1^j(2,1,0)^*P_2$.



$P_2(2,1,0)^*P_1(1,1,1,1)^*P_1(2,1,0)^*P_2$



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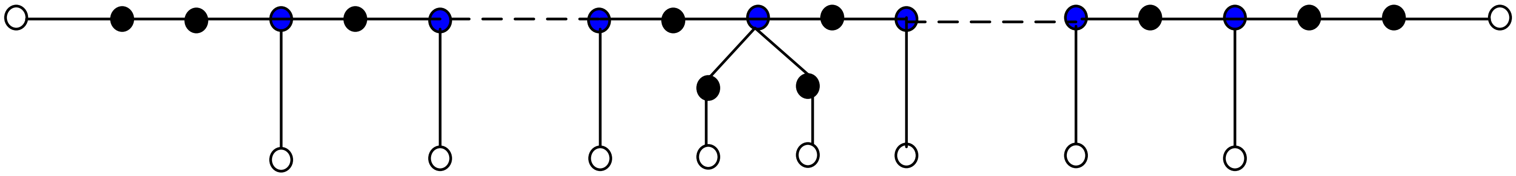


$P_2(2,1,0)^*P_1^i(1,1,1,1)^*P_1^j(2,1,0)^*P_2$

Reducible configurations

- Reducible vertices and threads of G;
- Reducible united thread structures of G;
- Reducible united thread-cycle structures of G.

✓ G dose not contain the following united thread-cycle structures:

(1) $\overline{Q_{(2,1,0)}} = P_2 (2,1,0)^* P_1$;

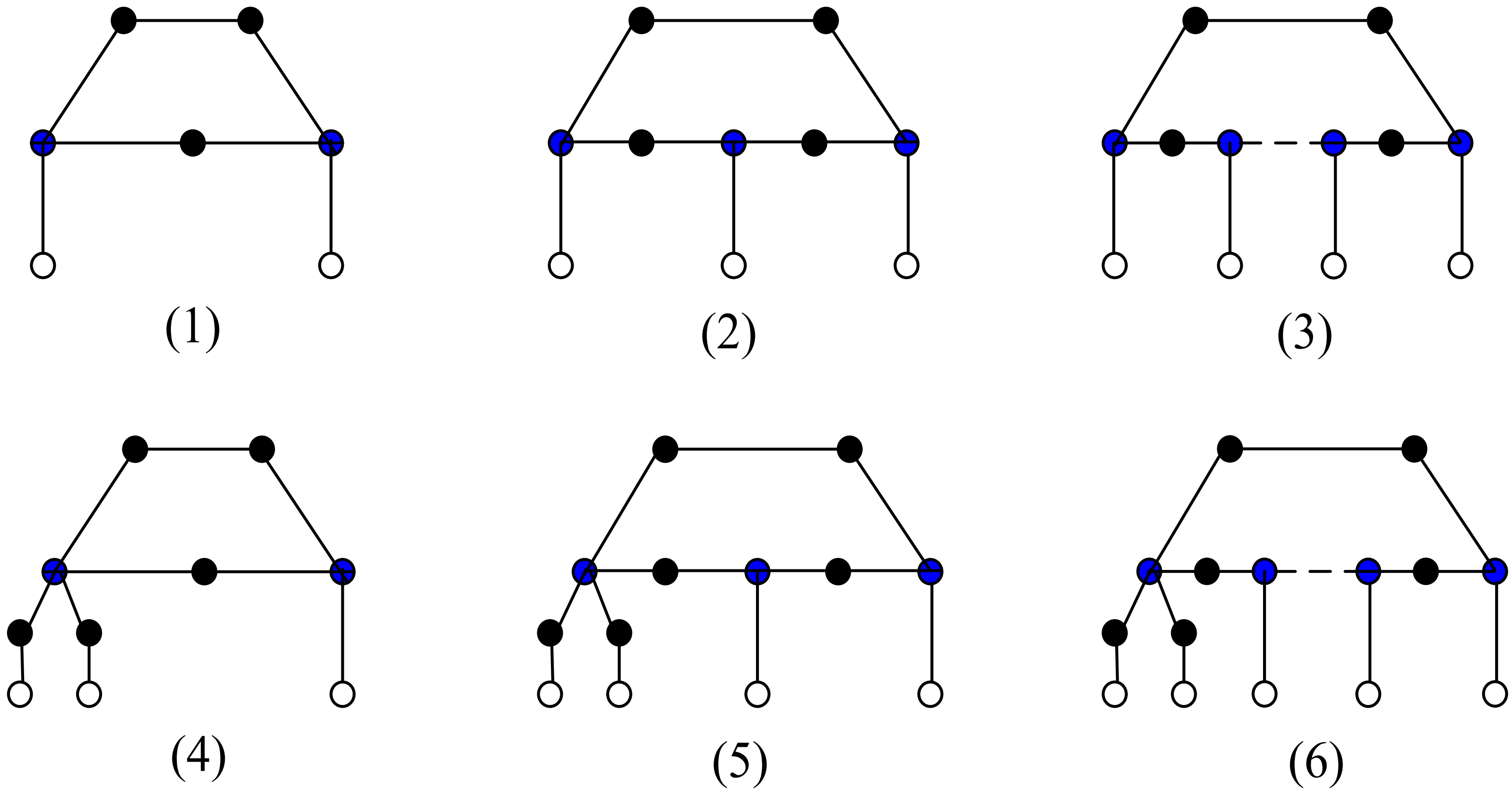
(2) $\overline{Q_{(2,1,0)}} = P_2 (2,1,0)^* P_1 (1,1,0)^* P_1$;

(3) $\overline{Q_{(2,1,0)}} = P_2 (2,1,0)^* P_1^i$;

(4) $\overline{Q_{(2,1,0)}} = P_2 (2,1,1,1)^* P_1$;

(5) $\overline{Q_{(2,1,0)}} = P_2 (2,1,1,1)^* P_1 (1,1,0)^* P_1$;

(6) $\overline{Q_{(2,1,0)}} = P_2 (2,1,1,1)^* P_1^j$.



Reducible united thread-cycle structures

Theorem's proof

Suppose G is a triangle-free simple graph with Mad(G)<5/2, then $\chi_f(G) \le 5/2$.

- Choose a counterexample G with least vertices;
- Investigate the structural properties of G;
- Show some reducible configurations of G;
- Use discharging argument to obtain a contradiction;

Compensatory path

A compensatory path for a (2,1,0)-vertex v is chosen as any shortest path F formed by concatenating threads in the following way:

First, F starts along the unique 1-thread at v. After F traversed some number of thread, let v^* be the last vertex reached.

If v^* is a (1,1,0)-vertex, then F continues along one of the other 1-thread incident to v^*, otherwise, F ends at v^*. We say v^* a sponsor of v and v a boss of v^*.

This concept was first proposed by Borodin * et al. '07.

*Borodin, Hartke, Ivanova, Kostochka and West.

- We define a weight function:

$$w(v)=2d(v)-5, \text{ for each vertex } v \in V(G).$$

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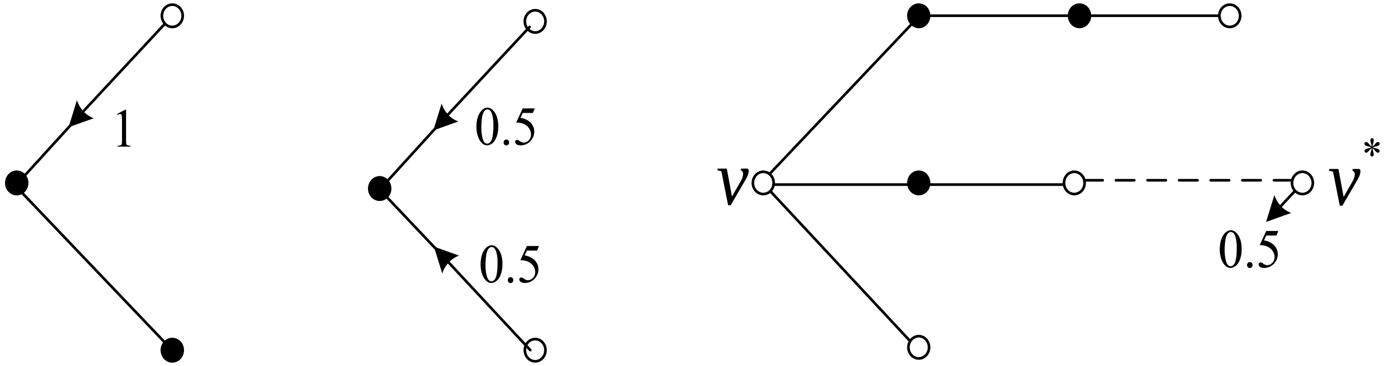
- Discharging rules:

  - Each 2-vertex in a 2-thread pulls charge 1 from its neighbor of 3^+-vertex;
  - Each 2-vertex in a 1-thread pulls charge 0.5 from each neighbor;
  - Each (2,1,0)-vertex pulls 0.5 from its sponsor.



- Using the relation:

$$\sum_{v \in V(G)} d(v) = 2|E(G)|.$$

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- Applying discharging rules, we obtain that:

$$w'(v) \geq 0, \text{ for each vertex } v \in V(G).$$

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- Applying discharging rules, we obtain that:

$$\text{w}'(\text{v}) \geq 0, \text{ for each vertex v} \in V(G).$$

- We derive the following obvious contradiction:

$$0 \leq \frac{\sum_{v\in V(G)} w'(v)}{V(G)} = \frac{\sum_{v\in V(G)} w(v)}{V(G)} = \frac{\sum_{v\in V(G)} 2d(v)-5}{V(G)} \leq 2Mad(G) - 5 < 2\cdot\frac{5}{2} - 5 = 0.$$

Theorem's proof

Suppose G is a triangle-free simple graph with Mad(G)<5/2, then $\chi_f(G) \le 5/2$.

- Choose a counterexample G with least vertices;
- Investigate the structural properties of G;
- Show some reducible configurations of G;
- Use discharging argument to obtain a contradiction;
- Hence, no counterexample can exist.

Thanks for your attention!