On the Unique Games Conjecture

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The Unique Games Conjecture (UGC)

- Proposed by Subhash Khot in 2002 [Kho02]
- It states that a problem called Unique Games (UG) is hard to approximate
- Gap-preserving reductions from UG $\rightarrow$ inapproximability results for several other problems
- The conjecture motivated work in the analysis of boolean functions, geometry ...

Outline

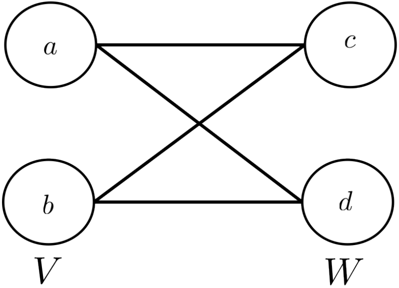
1. Game, what game?
   - Label cover
   - Why Label cover?
2. The conjecture
3. Implications of UGC
   - Analysis of boolean functions
   - Metric embeddings
   - Inapproximability
     - MaxCut
     - UGC and SDP
4. UGC: True or False?

Plan

Label Cover Problem (LC)

Input:

- A bipartite graph $((V, W), E)$



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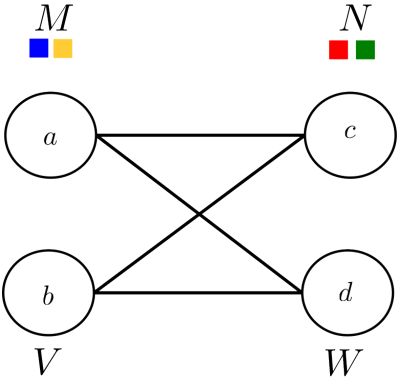

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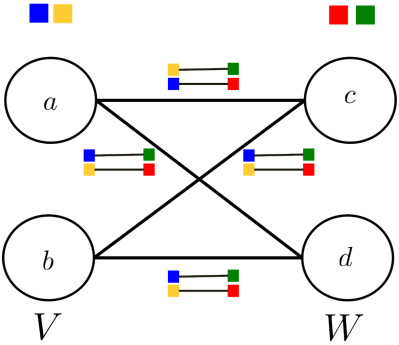

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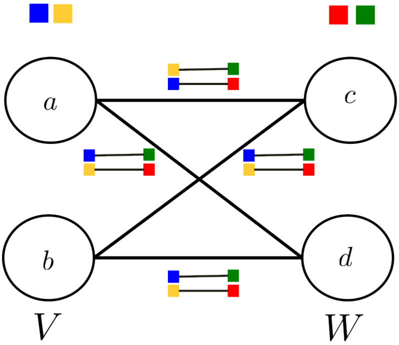
A labeling of the vertices of G:
$l : V \to N$ and $l : W \to M$.
An edge (v, w) is satisified if
$\pi_{v,w}(l(v)) = l(w)$



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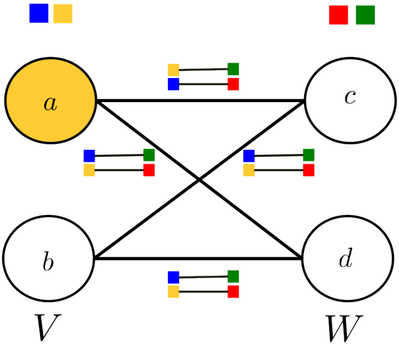

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For an instance $\mathcal{U}$ of LC, $OPT(\mathcal{U})$ = fraction of edges satisfied by an optimal labeling of $\mathcal{U}$.

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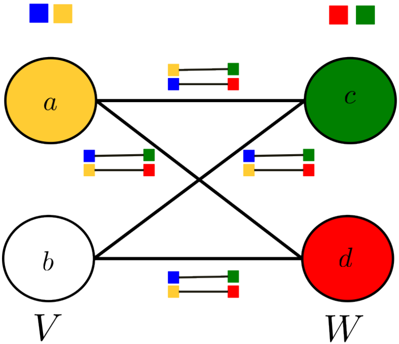

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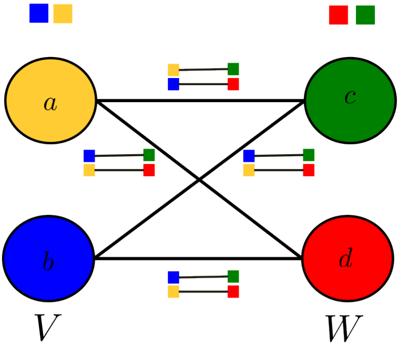

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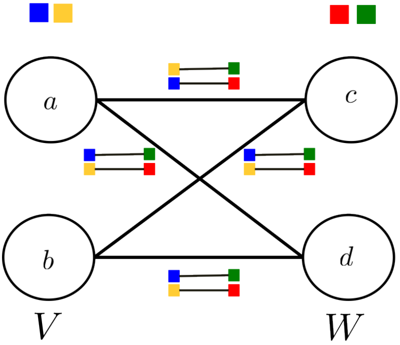
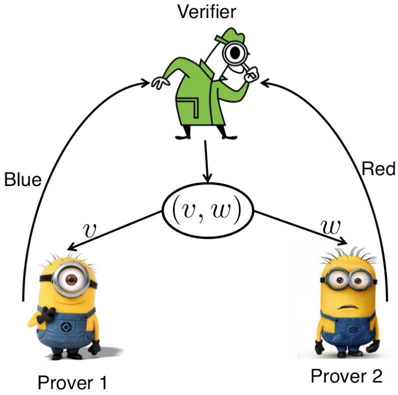
Where is the game?



Label Cover



2-Prover 1-Round Game

Unique Label Cover (ULC)

The label cover problem $\mathcal{L} = (G, M, N, \pi_{vw})$ is called unique if:

- $M = N$
- $\forall (v, w) \in E$, $\pi_{v,w}$ is a bijection (permutation)

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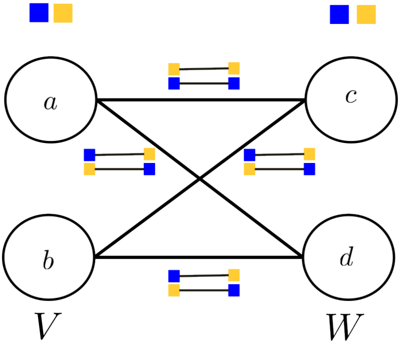

How is Label Cover (LC) useful?

It is all due to the following theorem:

Theorem

For $\epsilon > 0$, it is NP-hard to decide whether a Label Cover problem:

- *satisfies all edges ($OPT = 1$)*
- *satisfies at most a fraction ϵ of the edges ($OPT \leq \epsilon$)*

Proved with PCP theorem [AS98] + Raz's Parallel Repetition Lemma [Raz98]

- Reductions from $LC(1, \epsilon)$ have allowed to prove inapproximability results for many problems.
- Where does this problem fall short? $\rightarrow$ 2-CSPs
- Mostly because of the "many-to-one"ess of the constraints.
- How about having a stronger result? the same inapproximability theorem for Unique Label Cover?

Plan

The Unique Games Conjecture[Kho02]

Conjecture

For $\epsilon, \delta > 0$, it is NP-hard to decide whether a Unique Label Cover problem:

- *satisfies at least $1 - \epsilon$ fraction of the edges (OPT $\geq 1 - \epsilon$)*
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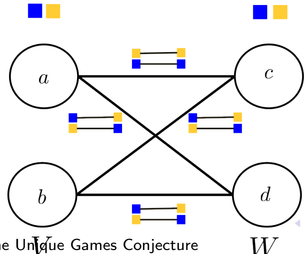
Why $1 - \epsilon$ and not 1?
$\rightarrow$ Deciding if all edges can be satisfied is easy.



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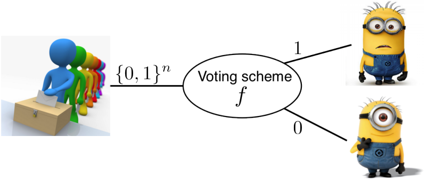
To the Unique Games Conjecture are associated:

- Non-conditional results
  - Analysis of boolean functions
  - Metric embeddings
- Conditional results: Inapproximability

Majority is Stablest [MOO05]



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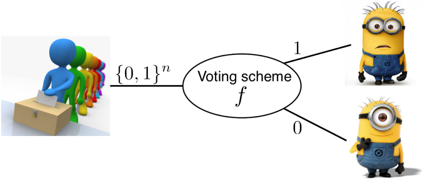

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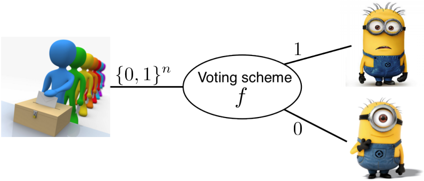

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- Influence of voter i in a scheme f:

$$Pr_{x \in \{1,-1\}^n}(f(x_1, \ldots, x_i, \ldots, x_n) \neq f(x_1, \ldots, -x_i, \ldots, x_n))$$

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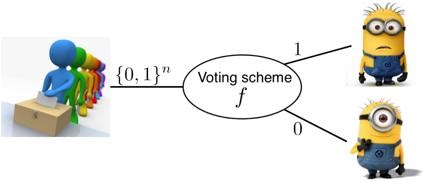

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- Noise stability$_\rho$ of a scheme f: Probability that the result does not change if a random fraction ρ of voters flip their votes.

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- Schemes that are far from being dictatorships

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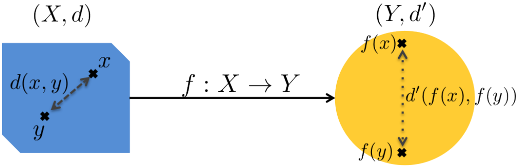
The answer: Majority is Stablest

The "Majority is Stablest" (MIS) theorem [MOO05] states that the Majority function maximizes noise stability among balanced boolean functions on the discrete cube with "small" influences.

Metric embedding[KV05]



$$L \times d(x, y) \leq d'(f(x), f(y)) \leq C \times L \times d(x, y)$$

$$\implies f \text{ has distortion } C$$

Metric Embedding: Goemans-Linial Conjecture

- **Goemans Linial Conjecture:** Every negative type metric embeds into l_1 with constant distortion.
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(*d* is a negative type metric if $\sqrt{d}$ is isometrically embeddable in l_2.)
- Insights from UGC have helped constructing a negative metric that embeds in l_1 with distortion at least $log(log(n))$ [KV05]
$\rightarrow$ The conjecture is false

Metric Embedding: Goemans-Linial Conjecture

Goemans-Linial Conjecture is true $\rightarrow$ $O(1)$-approximation for a graph partitioning problem $\mathcal{P}$ (sparsest cut) with some SDP relaxation $\mathcal{S}$.

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- Results on approximation ULC with some SDP relaxation

$\Rightarrow$ The ratio of the approximation of $\mathcal{P}$ with $\mathcal{S}$ is not constant
$\Rightarrow$ The Goemans-Linial Conjecture is false

Some UGC inapproximability results [Kho10]

Problem	Best Approx. Known	Best In-approx known	Best Inap-prox. known under UGC
Vertex Cover	2	1.36	$2 - \epsilon$
MaxCut	0.878	$\frac{16}{17} + \epsilon$	$0.878 + \epsilon$
Max Acyclic Sub-graph	0.5	$\frac{65}{66} + \epsilon$	$0.5 + \epsilon$
Any CSP $\mathcal{C}$ with integrality gap $\alpha_{\mathcal{C}}$	$\alpha_{\mathcal{C}}$		$\alpha_{\mathcal{C}} + \epsilon$

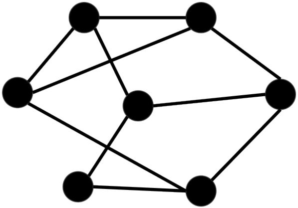
Max-Cut: definition

Input:

- A graph $G = (V, E)$

Output:

-



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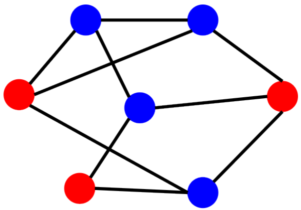

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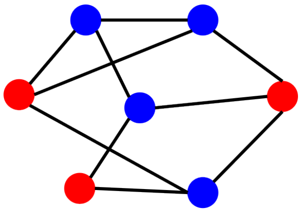

MaxCut is NP-hard and hard to approximate within $\frac{16}{17}$ [Hås01].

MaxCut: Goemans Williamson algorithm [GW95]

Quadratic Program

$$\text{max:} \quad \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

$$\text{s.t.:} \quad x_i^2 = 1 \quad \forall i \in V$$

SDP relaxation

$$\text{max:} \quad \sum_{(i,j)\in E} \frac{1 - v_i . v_j}{2}$$

$$\text{s.t:} \quad v_i . v_i = 1 \quad \forall i \in V$$

$$v_i \in \mathbb{R}^n \quad \forall i \in V$$

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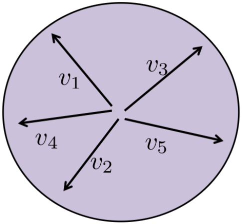
The algorithm

- Solve the relaxed SDP
-
-



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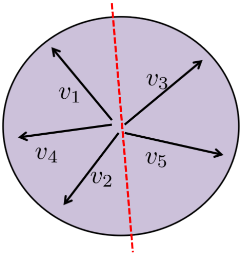

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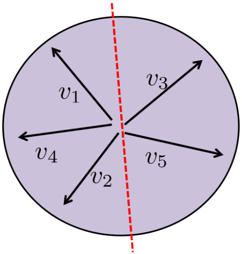

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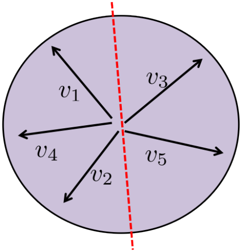

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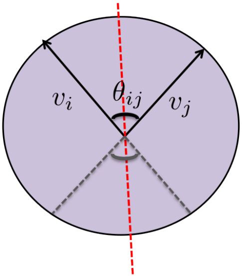
What is the ratio achieved by the algorithm?

MaxCut: ratio of the Goemans Williamson algorithm[GW95]

- OPT:= value of the MaxCut
- OPT_{SDP}:= value obtained by the SDP relaxation
- $\mathbb{E}(C)$:= Expectation of the value of the cut obtained by the algorithm



MaxCut: ratio of the Goemans Williamson algorithm[GW95]

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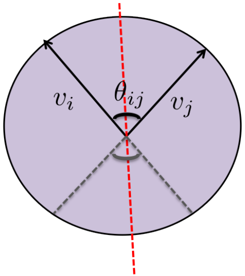
-



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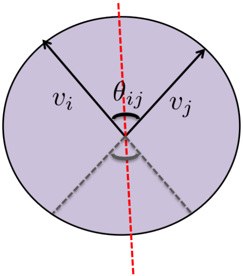

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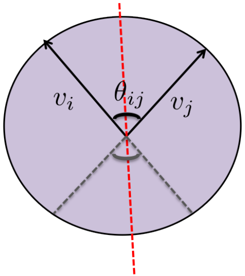

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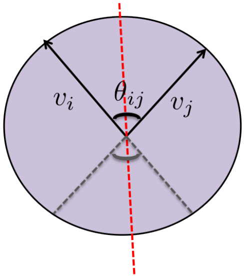

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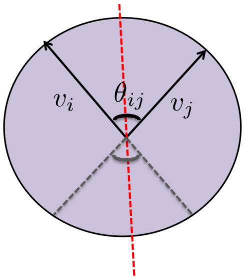

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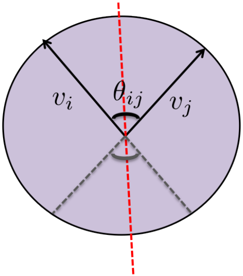
Let $\alpha_{GW} = \min_{0\leq\theta\leq\pi} \frac{2}{\pi} \frac{\theta_{ij}}{1 - \cos(\theta_{ij})} \simeq 0.878$



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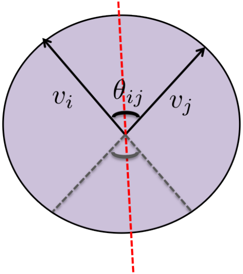

Let $\alpha_{GW} = \min_{0 \leq \theta \leq \pi} \frac{2}{\pi} \frac{\theta_{ij}}{1-\cos(\theta_{ij})} \simeq 0.878$ then

$$\mathbb{E}(C) \geq \alpha_{GW} OPT_{SDP} \geq \alpha_{GW} OPT$$

From Unique Label Cover to Max-Cut [KKMO04]

$ULC(\delta) \rightarrow$ distinguishing between the cases:

- $OPT \geq 1 - \delta$
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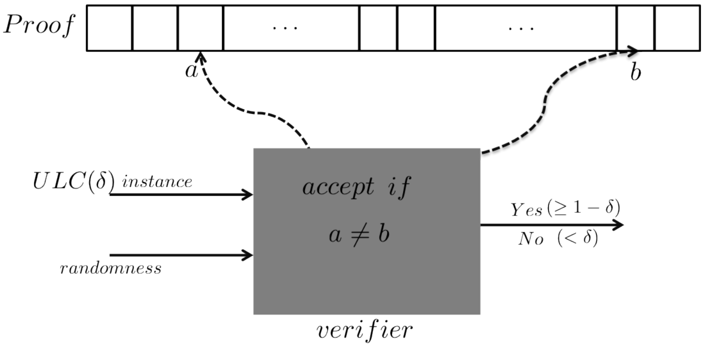
Theorem

For every $\epsilon > 0$ there exists δ such that there is a PCP for $ULC(\delta)$ in which the verifier reads two bits from the proof and accepts iff they are unequal, and which has completeness c and soundness s such that $\frac{s}{c} = \alpha_{GW} + \epsilon$

2-bit PCP for ULC(δ)



- Completeness: If $OPT(ULC) \geq 1-\delta$, then there is a proof that the verifier accepts with probability $\geq c$.
- Soundness: If $OPT(ULC) \leq \delta$, then all proofs are accepted with probability $\leq s$.

Corollary

Assuming UGC, MaxCut is hard to approximate within $\alpha_{GW} + \epsilon$.

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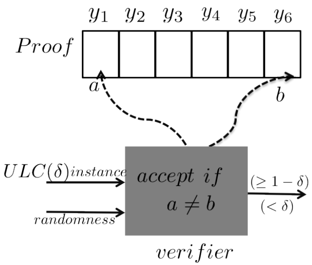

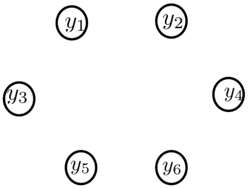

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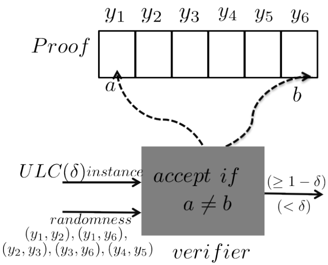

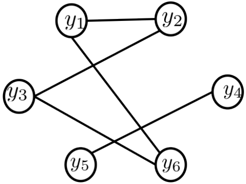

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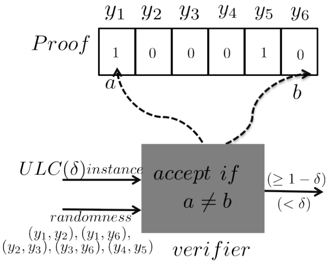

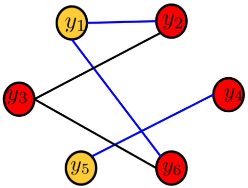

Probability of passing the test of a proof = fraction of edges of the corresponding cut

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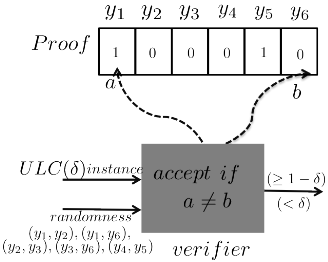

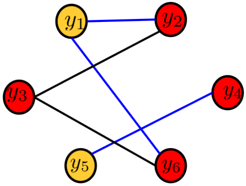

There is a proof that the verifier accepts with probability $\geq c \rightarrow$ there is a cut with value $\geq c|E|$

All proofs are accepted with probability $\leq s \rightarrow$ all cuts have value $\leq s|E|$

Corollary

Assuming UGC, MaxCut is hard to approximate within $\alpha_{GW} + \epsilon$.

- $OPT(ULC) \geq 1 - \delta \Rightarrow MaxCut \geq c|E|$
- $OPT(ULC) \leq \delta \Rightarrow MaxCut \leq s|E|$

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$$\Rightarrow \text{MaxCut is hard to approximate within } \frac{s}{c} = \alpha_{GW} + \epsilon$$

How to build a 2-bit PCP for ULC?

Proof	label v_1	label v_2	$\cdots$	label v_n

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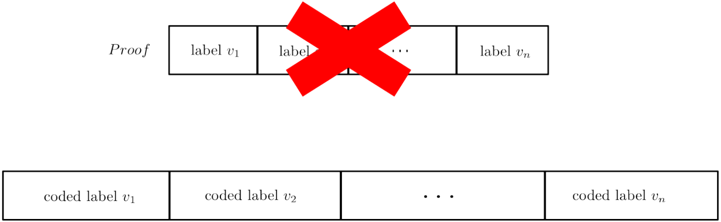

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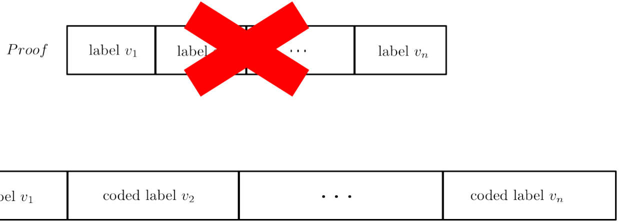

- The verifier expects codewords of the labels
- The verifier has to have completeness c ans soundness s

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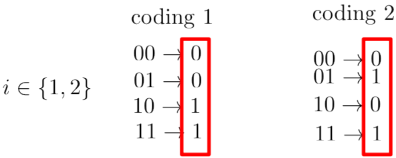

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$\rightarrow$ Distinguish dictatorships from functions far from being dictatorships

- A dictatorship depending on coordinate i can be decoded into label i.
- Functions far from dictatorships cannot be decoded

How to build a 2-bit PCP for ULC?

The PCP with only two bits can be designed thanks to:

- Unique games (permutations)
- Majority is stablest

UGC and SDP

- Under UGC, the SDP-based algorithm provides the best approximation for MaxCut.
-

UGC and SDP

- Under UGC, the SDP-based algorithm provides the best approximation for MaxCut.
- A stronger result [Rag08]: UGC $\rightarrow$ for every MAX-CSP, the simplest SDP relaxation is the best possible poly-time approximation.

UGC and SDP [Rag08]

- For every MAX-CSP there is a semi-definite programming relaxation $\mathcal{S}$
- Assuming UGC, no other polynomial time algorithm can provide a better approximation than $\mathcal{S}$

Plan

UGC: True or False?

True?

- Validates the quality of SDP relaxations

False?

UGC: True or False?

True?

- Validates the quality of SDP relaxations
- It provides very "neat" inapproximability results

False?

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UGC: True or False?

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False?

- The results can still hold even if the conjecture is false
- A sub-exponential time algorithm has been designed [ABS10]

UGC: True or False?

True?

- Validates the quality of SDP relaxations
- It provides very "neat" inapproximability results
- There is no algorithm to refute it
- $GapULC_{C(\delta)\delta,\delta}$ is NP-hard [FR04]

False?

- The results can still hold even if the conjecture is false
- A sub-exponential time algorithm has been designed [ABS10]
- $C(\delta)\delta \to 0$ as $\delta \to 0$

Thank you

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