A local bound on the chromatic number of a line graph

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Reed conjectured that for any graph G, $\chi(G) \leq \lceil \frac{1}{2}(\Delta(G) + 1 + \omega(G) \rceil$. This was proved for line graphs [1] not too long ago, then proved for quasi-line graphs and claw-free graphs very recently. I propose a stronger conjecture.

For a vertex v let $\omega(v)$ denote the size of the largest clique containing v. I conjecture that for any graph G, $\chi(G) \leq \max_v \lceil \frac{1}{2}(d(v) + 1 + \omega(v)) \rceil$. This was recently proved for claw-free graphs with $\alpha(G) \leq 3$. It seems like it will be easy to prove for all claw-free graphs if we can prove it for line graphs of multigraphs, and that is what I would like to do.

The natural approach is by minimum counterexample or induction. So the "Local Strenthening" of Reed's conjecture represents a strengthening of the induction hypothesis. It is sometimes easier to prove than Reed's conjecture and sometimes harder. The way we proved Reed's conjecture for line graphs is this: if the induction step does not go through easily, we can deduce that $\Delta(G)$ is at least $\frac{3}{2}\Delta(H) - 1$, where H is the underlying multigraph of the line graph G. We cannot say the same thing for a minimum counterexample to the Local Strengthening. That's where things stand now. It would lend a lot of credibility to the Local Strenthening if we can prove it for line graphs.

References

[1] A. King, B. Reed, and A. Vetta. An upper bound for the chromatic number of line graphs. *European Journal of Combinatorics*, 2007. doi:10.1016/j.ejc.2007.04.014.