A Very Short Introduction to Interval Analysis

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Interval analysis is a branch of numerical analysis, devoted to dealing with the accuracy issue of computer-based calculations. It all started in the late sixties with the seminal book by Moore [4]. Further well-known reference books on this topic are [1, 5, 2, 3]. Materials (course, software, benchmarks) are available in [6].

The goal of interval analysis is to design methods that cope with all kind of imprecision that hinders classical numerical techniques from providing reliable results. Such imprecision can model rounding errors as well as data uncertainties inherent to real-life problems.

The basic idea of interval analysis is to embed intervals that include the range of all possible error made, in any low-level computation. Therefore, computations are performed with the so-called *interval arithmetics*, that takes interval operands instead of real operands, e.g.,

$$[1, 2] + [2, 3] = [3, 5].$$

Indeed, if $x \in [1, 2]$ and $y \in [2, 3]$, then one can see that $x + y$ can only lie in $[3, 5]$. Interval computations are also possible with elementary functions such as *exp*, *sqr*, or *sin*, based on our knowledge of their monotonicity properties. A more complex function f can also be evaluated recursively on an interval vector $\mathbf{x}$ (also called **a box**), as long as the expression of f is a chain of elementary functions. However, this only results in a (sometimes rather crude) outer approximation of the range of f on $\mathbf{x}$. Will shall write

$$f(\mathbf{x}) \supset range(f, \mathbf{x}). \tag{1}$$

So far, interval analysis has been mostly used to solve systems of equations. For that purpose, the method combines interval arithmetics with a combinatorial search to find all the solutions in a given initial domain. Let f be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, and $\mathbf{x}_0$ be an interval vector. We can enforce the following procedure to find all the solutions of $f(x) = 0$ in $\mathbf{x}_0$:

push $\mathbf{x}_0$ on a stack
while the stack is not empty **do**
  pop a box $\mathbf{x}$ from the stack
  if width($\mathbf{x}$) $< \epsilon$ **then**
    store $\mathbf{x}$ as a potential solution
  else if $0 \in f(\mathbf{x})$ **then**
    split $\mathbf{x}$ into two interval vectors $\mathbf{x}_1$ and $\mathbf{x}_2$
    push $\mathbf{x}_1$ and $\mathbf{x}_2$ on the stack
  end if
end while

The soundness of this algorithm relies on (1). Indeed, if $0 \notin f(\mathbf{x})$, then we know that no solution exist in $\mathbf{x}$, so that this box can be safely discarded. Unless they are removed by the latter test, boxes are split until their width get smaller than a user-defined precision ϵ. The sharper the evaluation of f, the more likely a box can be removed, and clearly, this has a direct consequence on both the overall efficiency and accuracy. This is why devising sharp evaluations has been a crucial matter in interval analysis.

Existence and Uniqueness of Solutions

Each box $\mathbf{x}$ returned by the algorithm could not be proven to be infeasible, but, so far, nothing proves that it does contain a solution. Some techniques exist to guarantee the existence of a solution in $\mathbf{x}$. For instance, we can avail ourselves of Brouwer's theorem. This theorem states that any continuous function of a compact set to itself has a fixed point. Assume now that (by some linearization) we can rewrite $f(x)$ in an equivalent form $g(x) - x$. Then finding a solution of $f(x) = 0$ amounts to finding a fixed point of g. Now, if the interval evaluation of g satisfies $g(\mathbf{x}) \subseteq \mathbf{x}$, by (1) we know that for all x in $\mathbf{x}$, $g(x) \in \mathbf{x}$, and Brouwer's theorem can be applied. More sophisticated theorems can also certify that a solution is unique inside the box.

Parameterized Systems

As coefficients of equations often represent physical measurements, they are only known to lie within some intervals of confidence. So it is more significant to consider a parameterized system $f(p, x)$, where p denotes the set of parameters. The nice thing about interval theorems is that they can be extended to the parameterized case almost straightforwardly. This is as simple as plugging intervals in place of reals in the formulae. If $\mathbf{p}$ represents the domain of the parameters, then a "safe" box $\mathbf{x}$ ensures that

$$\big(\forall p \in \mathbf{p}\big)\big(\exists x \in \mathbf{x}\big) \mid f(p, x) = 0.$$

Some researchers focus today on situations where more freedom in the quantifiers is required. Given parameters p and q, one may rather look for a box $\mathbf{x}$ such that $\big(\forall x \in \mathbf{x}\big)\big(\forall p \in \mathbf{p}\big)\big(\exists q \in \mathbf{q}\big) \mid f(p, q, x) = 0$. Methods under development resort to a more complex algebraic structure called *generalized intervals*, where bounds are not constrained to be ordered (e.g., $[1, -1]$ is a valid interval).

References

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