Direct kinematics of planar parallel manipulators

Jean-Pierre MERLET
INRIA Sophia-Antipolis
BP 93 06902 Sophia-Antipolis, France
E-mail: Jean-Pierre.Merlet@sophia.inria.fr

Abstract

We address the problem of finding all the solutions of the direct kinematics for every possible architectures of planar fully parallel manipulators. We show that for this problem all the possible kinematic chains can be reduced to a set of three basic chains and we explain how to calculate the solutions of the forward kinematics for all the combinations of these basic chains and consequently for all the possible architectures of planar parallel robots.

1 Introduction

A planar fully parallel robot is constituted of a moving platform connected to the ground by three independent kinematic chains having three independent one d.o.f. joints, one of which is actuated. The actuation scheme should provide the control of the three d.o.f. of the moving platform and the degree of mobility of the moving platform should be 0 when the actuators are locked. The joint will be denoted by R for a revolute joint and P for a prismatic joint. Consequently each of the independent kinematic chains will be denoted by a set of three letters indicating the succession of joints starting from the ground. The possible combinations are therefore : RRR, RPR, RRP, RPP, PRR, PPR, PRP, PPP, this last combination being excluded as the three joints are not independent. If we assume that the three kinematics chains are identical all the possible manipulators are presented in figure 1. In this paper the planar parallel manipulators will be defined by the sequence describing the three kinematic chains (e.g. 3-RRR will be the name of the robot with three RRR kinematic chains). The centers of the joints attached to the ground will be denoted by A_i and the centers of the joints attached to the moving platform will be denoted B_i.

At this time we have not defined the actuated joint: in fact each joint of a chain may be actuated as soon as the following rule is satisfied:

the chain obtained when locking the actuated joint is not of the PP type

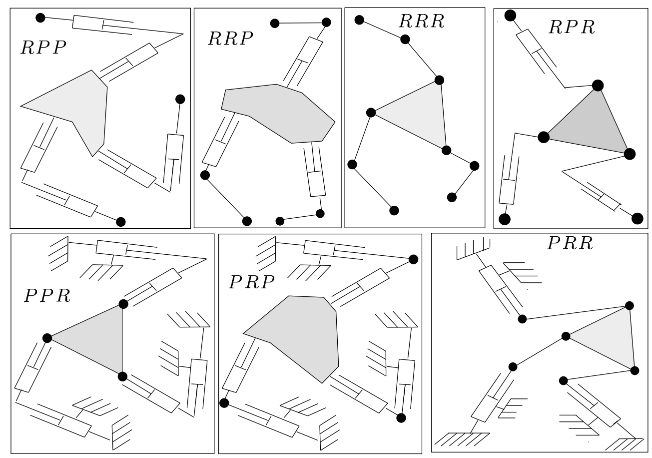

Figure 1: The possible planar fully parallel manipulators with three identical kinematic chains

Indeed in the opposite case when the actuator is locked point B_i of the chain may be located at any position in the plane. As this is true for each of the three chains the moving platform may translate in the plane and its degree of mobility is no more 0. Using this rule we may now determine all the 18 possible chains and we will underline the actuated joint (table 1).

Note that to get independence in the kinematic chain with two prismatic joints it is necessary that the joints axis should not be parallel. The kinematic chains of any planar fully parallel robots should be one of the chains listed in the table.

$\underline{R}RR$	$R\underline{R}R$	$RR\underline{R}$	$\underline{R}PR$	$R\underline{P}R$	$RP\underline{R}$
$R\underline{P}P$	$RP\underline{P}$	$\underline{P}RR$	$P\underline{R}R$	$PR\underline{R}$	$\underline{P}RP$
$PR\underline{P}$	$\underline{P}PR$	$P\underline{P}R$	$\underline{R}RP$	$R\underline{R}P$	$RR\underline{P}$

Table 1: All the possible chains for a planar parallel robot

2 Direct kinematics

2.1 Equivalent chains

The direct kinematics problem consists in finding all the possible postures of the moving platform when all the actuators of the robot are locked. Our purpose is to show that the direct kinematics of all fully parallel robots may be solved. We will first study all the possible kinematic chains and show that when their actuator is locked they are equivalent to a chain among the set of three basic chains presented in figure 2. Let us examine for example the $\underline{R}RR$ chain. When the first revolute joint is locked, clearly the extremity M of the link attached to this joint is fixed. Hence the remaining link (attached at M and at the revolute joint of the moving platform) can only turn around the revolute joint at M. Therefore this chain can be substituted by the chain of type 1. Clearly every chain whose free joints (the joints which are not actuated) are of the revolute type can be substituted by a type 1 chain.

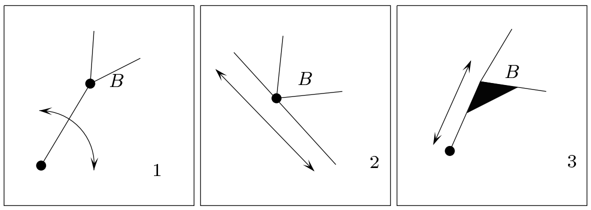

Figure 2: The three equivalent chains for all the possible kinematics chains

Consider now the chain $\underline{R}PR$: when the first revolute joint is locked point B can only translate along the line D parallel to the prismatic actuator axis while the moving platform can rotate around the joint at B (figure 3). Consequently this chain can be substituted by a type 2 kinematic chain. Clearly every chain whose free joints are prismatic and revolute (in this order, starting from the ground) can be substituted by a type 2 chain.

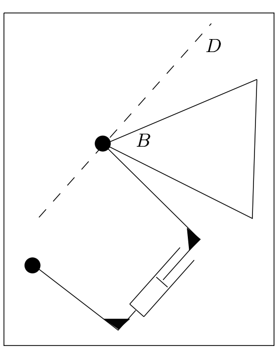

Figure 3: The equivalent chain for a $\underline{R}PR$ chain

Finally consider the chain $R\underline{P}P$: when the prismatic joint is locked point B can rotate around A and translate along the prismatic joint axis, but the angle between this axis and the platform should remain constant (figure 4). Hence this chain can be substituted by a type 3 chain. Clearly every chain whose free joints are revolute and prismatic (in this order, starting from the ground) can be substituted by a type 3 chain.

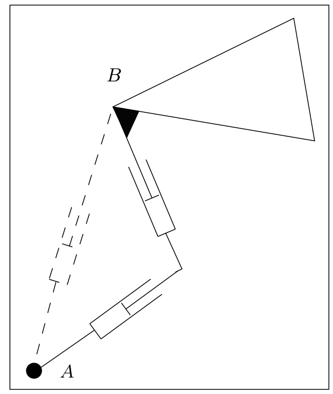

Figure 4: The equivalent chain for a $R\underline{P}P$ chain

This analysis can be performed for all the 18 possible chains and it is found that for all the chains the equivalent chain is of type 1, 2 or 3. The equivalence are summarized in table 2. If we are able to compute the direct kinematics for the robots having all the possible combinations of the three basic equivalent chains we will be able to compute the direct kinematics for all the planar parallel robots.

$\underline{R}RR$ 1	$R\underline{R}R$ 1	$RR\underline{R}$ 1	$\underline{R}PR$ 2	$R\underline{P}R$ 1	$RP\underline{R}$ 3
$R\underline{P}P$ 3	$RP\underline{P}$ 3	$\underline{P}RR$ 1	$P\underline{R}R$ 2	$PR\underline{R}$ 2	$\underline{P}RP$ 3
$PR\underline{P}$ 2	$\underline{P}PR$ 2	$P\underline{P}R$ 2	$\underline{R}RP$ 3	$R\underline{R}P$ 3	$RR\underline{P}$ 1

Table 2: The equivalent chain for all the 18 possible kinematic chains

2.2 Manipulator with identical chains

In this section we will assume that the robot has three identical kinematics chain whose equivalent type is either 1, 2 or 3.

2.2.1 3-type 1 mechanism

The direct kinematics of this mechanism has been well studied under the name of $3-\underline{R}RR$ robot (which is equivalently from the direct kinematics point of view to the $3-R\underline{P}R$ robot). This mechanism has been studied in [4], [5],[6],[7], [8],[9] but Gosselin [1],[2] was the first to complete the analysis of this mechanism by showing that it is possible to compute an univariate polynomial of order 6 enabling to determine all the postures of the platform. It was shown that the possible postures are defined by the intersection of the coupler curve of the four bar mechanism $A_1B_1B_3B_2A_2$ with the circle described by B_3 when it rotates around A_3. As the coupler curve has degree 6 and full circularity the number of real intersection points with a circle is at most 6. The polynomial of Gosselin is therefore optimal.

2.2.2 3-type 2 mechanism

This mechanism is described in figure 5: point B_1 may slide along the x axis and we denote by λ the amplitude of the translation, point B_2 slide on a line with unit vector $\mathbf{u}(u_x, u_y)$, going through point $A_2(x_b, y_b)$ while point B_3 slide on a line with unit vector $\mathbf{v}(v_x, v_y)$, going through point $A_3(x_a, y_a)$. A posture of the robot may be defined by the value of λ and the orientation angle θ. We may write two equations that indicate

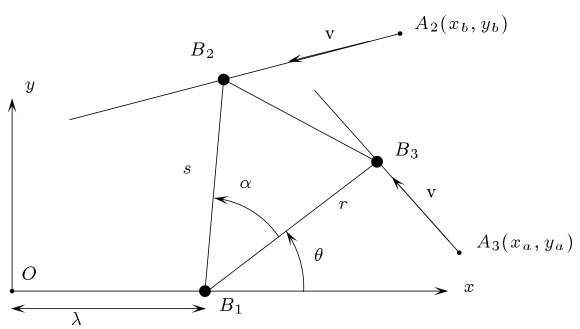

Figure 5: The 3-type 2 mechanism

that points B_2, B_3 slide along the given lines:

$$(\lambda + r\cos(\theta) - x_a)v_y - (r\sin(\theta) - y_a)v_x = 0 \quad (1)$$

$$(\lambda + s\cos(\theta+\alpha) - x_b)u_y - (s\sin(\theta+\alpha) - y_b)u_x = 0 \quad (2)$$

Any of these two equations can be used to calculate λ which is then substituted into the remaining equation. This leads to an equation of the type $A\cos(\theta)+B\sin(\theta) = C$ which admit two solutions in θ. As there is an unique value of λ for a given θ the direct kinematics of this mechanism may have two solutions. The geometrical interpretation is that as B_2, B_3 move on their lines point B_1 lie on an ellipse: the possible locations of B_1 are therefore the intersection of a line and an ellipse.

2.2.3 3-type 3 mechanism

This mechanism is described in figure 6: the posture of the robot can be computed using the length r of link A_1B_1 and θ the angle between link A_1B_1 and the x axis. The angles α_i denote constant angles between the platform and the prismatic joint axis. The coordinates

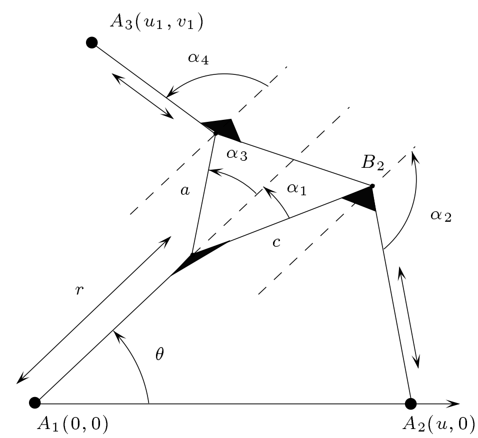

Figure 6: The 3-type 3 mechanism

of B_2 and the components of the unit vector $\mathbf{v_2}$ of the axis of the prismatic joint (link 2) are:

$$B_2 = \begin{cases} r\cos\theta + c\cos(\theta+\alpha_1) \\ r\sin\theta + c\sin(\theta+\alpha_1) \end{cases} \quad \mathbf{v_2} = \begin{cases} \cos(\theta+\alpha_2) \\ \sin(\theta+\alpha_2) \end{cases}$$

Similarly the coordinates of B_3 and the components of the unit vector $\mathbf{v_3}$ of the axis of the prismatic joint (link 3) are:

$$B_3 = \begin{cases} r\cos\theta + a\cos(\theta+\alpha_3) \\ r\sin\theta + a\sin(\theta+\alpha_3) \end{cases} \quad \mathbf{v_3} = \begin{cases} \cos(\theta+\alpha_3) \\ \sin(\theta+\alpha_3) \end{cases}$$

We may now write two equations which indicate that the vectors $(\mathbf{A_3B_3}, \mathbf{v_3})$ have the same direction as have the vectors $(\mathbf{A_2B_2}, \mathbf{v_2})$:

$$(r\cos\theta + c\cos(\theta+\alpha_1) - u)\sin(\theta+\alpha_2) - (r\sin\theta + c\sin(\theta+\alpha_1))\cos(\theta+\alpha_2) = 0 \quad (3)$$

$$(r\cos\theta + c\cos(\theta+\alpha_3) - u_1)\sin(\theta+\alpha_4) - (r\sin\theta + c\sin(\theta+\alpha_1) - v_1)\cos(\theta+\alpha_4) = 0 \quad (4)$$

Using the first equation we can compute r as a function of θ and substitute its value in the second equation. This leads to an equation of the type $A\cos(\theta) + B\sin(\theta) = C$ which admit two solutions in θ. As there is an unique value of r for a given θ the direct kinematics of this mechanism may have two solutions.

2.3 Manipulator with two identical chains

In this section we will assume that the robot has only two identical kinematics chains whose equivalent type is either 1, 2 or 3.

2.3.1 2-type 1 and 1 type 2

We assume that link 1, 2 are of type 1 and link 3 of type 2. Point B_3 lie therefore both on the coupler curve of the four bar mechanism $A_1B_1B_3B_2A_2$ and on the line of the type 2 chain. Consequently there may be up to 6 real intersection points and the direct kinematics will have at most 6 solutions. The mechanism is described in figure 7. Using the notation of the figure we may

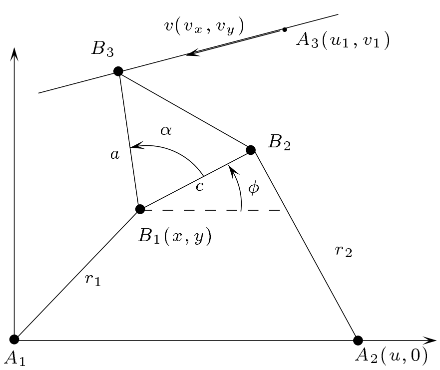

Figure 7: The 2-type 1-1-type 2 mechanism

write two equations stating that points B_1, B_2 are at distance r_1, r_2 from A_1, A_2:

$$x^2 + y^2 = r_1^2 \tag{5}$$
$$(x + c\cos\phi - u)^2 + (y + c\sin\phi)^2 = r_2^2 \tag{6}$$

Then we write that point B_3 should be on the line going through $A_3(u_1, v_1)$ with unit vector $\mathbf{v}(v_x, v_y)$:

$$(x + a\cos(\phi+\alpha) - u_1)v_y - (y + a\sin(\phi+\alpha) - v_1)v_x = 0 \tag{7}$$

Equations (6)-(5) and (7) are linear in x, y. Their values are computed and substituted in equation (5), leading to an equation in $\cos\phi, \sin\phi$. After substitution of the sine and cosine by their values as function of $T = tan(\phi/2)$ we get a 6-order polynomial in T, which enables to compute the solution of the direct kinematics.

2.3.2 2-type 1 and 1 type 3

This mechanism is presented in figure 8. In that case

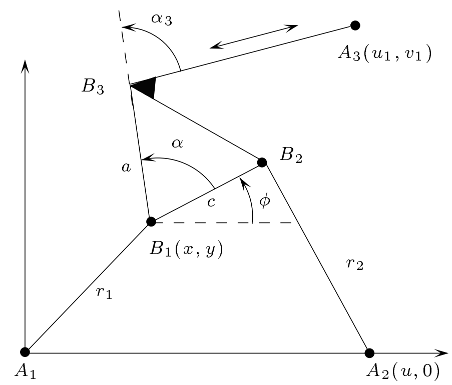

Figure 8: The 2-type 1-1-type 3 mechanism

equations (5),(6) are still valid but the third equation is obtained by writing that for given x, y, ϕ the line going through B_3, whose axis is the prismatic joint axis, meet A_3:

$$(x + a\cos(\phi+\alpha) - u_1)\sin(\phi+\alpha+\alpha_3) - (y + a\sin(\phi+\alpha) - v_1)\cos(\phi+\alpha+\alpha_3) = 0 \tag{8}$$

where α_3 is the angle between B_1B_3 and the prismatic joint axis. Equations (6)-(5) and (8) are still linear in x, y. Using the same process as in the previous section we get a 6-order polynomial in T, which enables to compute the solution of the direct kinematics. A geometric interpretation is to consider the inverted slider crank $A_1B_1B_2B_3A_3$. The coupler curve described by B_2 is a sextic with full circularity [3] meaning that the number of real intersection points of the coupler curve with the circle described by B_2 when it rotates around A_2 is at most 6. Every intersection point defines a solution for the direct kinematics of the mechanism. Therefore we have effectively computed a minimal order polynomial.

2.3.3 2-type 2 and 1 type 1

This mechanism is presented in figure 9. Here we assume that the chains 1 and 2 are of type 2 and the chain 3 of type 1. We have seen that B_3 as part of the mechanism $A_1B_1B_3B_2A_2$ describes an ellipse. The direct kinematics solution are obtained when B_3 belongs also to the circle centered in A_3. Consequently the maximum number of solution is 4.

Using the notation of figure 5 we will write first that B_2 should belong to the line going through A_2

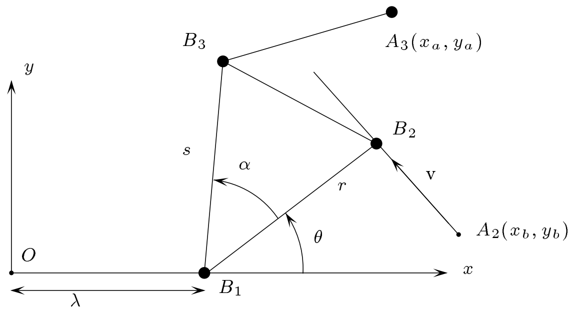

Figure 9: The 2-type 2-1-type 1 mechanism

with unit vector $\mathbf{v}(v_x, v_y)$:

$$(\lambda + r\cos(\theta) - x_a)v_y - (r\sin(\theta) - y_a)v_x = 0 \quad (9)$$

Then we write that point B_3 should be at distance r_3 from A_3:

$$(\lambda + s\cos(\theta+\alpha) - x_b)^2 + (s\sin(\theta+\alpha) - y_b)^2 = r_3^2 \quad (10)$$

Using equation (9) we compute the value of λ and substitute this value in equation (10) which become an equation in sine and cosine of θ. Using the classical half-angle transformation we get a 4-order polynomial in $T = tan(\theta/2)$.

2.3.4 2-type 2 and 1 type 3

This mechanism is presented in figure 10. Equation (9)

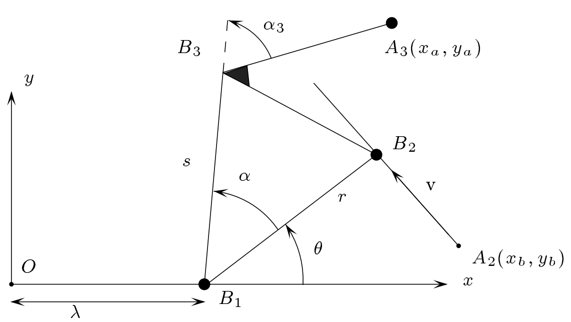

Figure 10: The 2-type 2-1-type 3 mechanism

is still valid. A second equation is obtained by writing that the line going through A_3 with direction defined by the axis of the prismatic joint of chain 3 has to meet A_3:

$$(\lambda + s\cos(\theta+\alpha) - x_b)\sin(\theta+\alpha+\alpha_3) - \\ (s\sin(\theta+\alpha) - y_b)\cos(\theta+\alpha+\alpha_3) = 0 \quad (11)$$

where α_3 is the angle between B_1B_3 and the prismatic joint axis. Using equation (9) we compute the value of λ and substitute the result in equation (11) which become an equation in sine and cosine of θ. Using the classical half-angle transformation we get a 4-order polynomial in $T = tan(\theta/2)$. A geometrical interpretation can be given if we consider the mechanism $A_1B_1B_2B_3A_3$: the coupler curve described by this mechanism is of degree 4. The solution of the direct kinematics are obtained by intersecting the coupler curve with the line described by B_2 as member of chain 2: consequently there is at most 4 intersection points.

2.3.5 2-type 3 and 1 type 1

This mechanism is presented in figure 11. If we con-

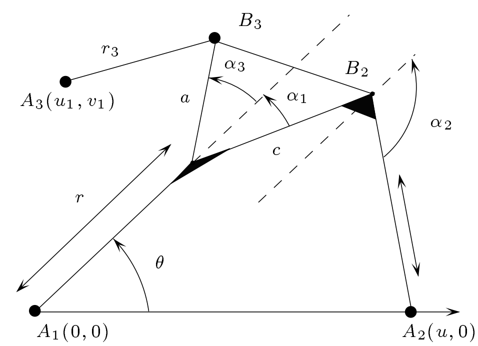

Figure 11: The 2-type 3-1-type 1 mechanism

sider the mechanism $A_1B_1B_3B_2A_2$ it can be shown that B_3 describes a coupler curve of order 4 with full circularity (i.e. 2). Consequently as the solutions of the direct kinematics problem for B_3 are obtained by intersection this curve with the circle centered at A_3 the number of real intersection points will be at most 4.

Using the notation of figure 6 we write first that A_2 belongs to the line coming from the platform:

$$(r\cos\theta + c\cos(\theta+\alpha_1) - u)\sin(\theta+\alpha_2) - \\ (r\sin\theta + c\sin(\theta+\alpha_1))\cos(\theta+\alpha_2) = 0 \quad (12)$$

Then we write that point B_3 should be at distance r_3 from A_3:

$$(r\cos\theta + a\cos(\theta+\alpha_3) - u_1)^2 + \\ (r\sin\theta + a\sin(\theta+\alpha_3) - v_1)^2 = r_3^2 \quad (13)$$

We use the linear equation (12) to compute r which is then back substituted in equation (13), now an equation in the sine and cosine of θ. We get then a fourth-order polynomial in $T = tan(\theta/2)$.

2.3.6 2-type 3 and 1 type 2

This mechanism is presented in figure 12. As the possible positions for B_3 are obtained as the intersection of the coupler curve of $A_1B_1B_3B_2A_2$ with the line going through A_3 from chain 3 we will clearly get at most four intersection points.

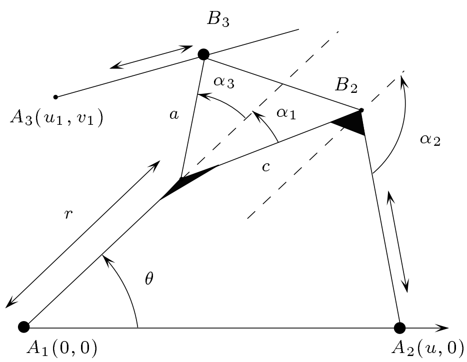

Figure 12: The 2-type 3-1-type 2 mechanism

Equation (12) is still valid. We write now that point B_3 belongs to the line:

$$(r\cos\theta + a\cos(\theta+\alpha_3) - u_1)v_y - \\ (r\sin\theta + a\sin(\theta+\alpha_3) - v_1)v_x = 0 \quad (14)$$

We use the linear equation (12) to compute r which is then back substituted in equation (13), now an equation in the sine and cosine of θ. We get then a fourth-order polynomial in $T = tan(\theta/2)$.

2.4 Type 1-2-3

The last case is obtained when all three chains are of different type (figure 13). The coupler curve of the $A_1B_1B_3B_2A_2$ mechanism is a sextic with full circularity. The possible positions for B_3 are obtained as the intersection of the coupler curve with the line from chain 3 going through A_3: we will clearly get at most six intersection points.

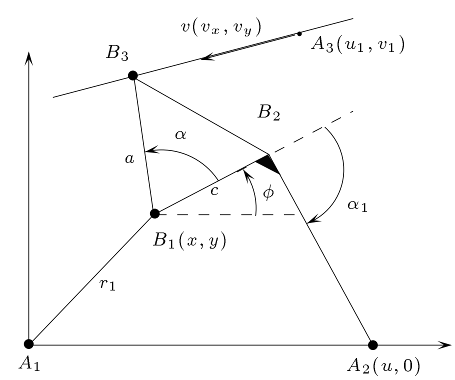

Figure 13: The type 1-2-3 mechanism

To compute the solutions we write first that B_1 is at distance r_1 from A_1:

$$x^2 + y^2 = r_1^2 \quad (15)$$

Then we write that point B_3 should be on the line going through $A_3(u_1, v_1)$ with unit vector $\mathbf{v}(v_x, v_y)$:

$$(x + a\cos(\phi+\alpha) - u_1)v_y - (y + a\sin(\phi+\alpha) - v_1)v_x = 0 \quad (16)$$

Finally we write that the vectors $(\mathbf{A_2B_2}, \mathbf{v_2})$ have the same direction:

$$(x + c\cos\phi - u)\sin(\phi+\alpha_1) - (y + c\sin\phi)\cos(\phi+\alpha_1) = 0 \quad (17)$$

Equations (16),(17) are linear in x, y, after solving this system their values are back substituted in equation (15). This equation is now in the sine and cosine of ϕ. We get then a sixth-order polynomial in $T = tan(\phi/2)$.

3 Summary

Table 3 summarizes the results by giving the number of solution of the forward kinematics for all the possible combinations of the three basic chains.

1-1-1	2-2-2	3-3-3	1-1-2	1-1-3
6	2	2	6	6
2-2-1	2-2-3	3-3-1	3-3-2	1-2-3
4	4	4	4	6

Table 3: Summary of the maximum number of solutions for all the possible combinations

4 Conclusion

We have examined all the possible planar fully parallel robots and have shown that from the direct kinematics view point all the possible kinematics chains can be reduced to a member of a set of three basic chains. Then the direct kinematics has been solved for all the possible combinations of these basic chains. Consequently we have obtained the solution of the forward kinematics for all possible planar parallel robots. With this approach it will be possible to study more in details all the other features of this type of robots.

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