# An algorithm for the Forward kinematics of general parallel manipulators

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Abstract

Forward kinematics has been studied mainly for parallel manipulators with planar faces. In this case the size of the set of equations from the inverse kinematics can be reduced from 6 to 3 and this last set can be combined into a polynomial in one variable. But this method cannot be extented to general parallel manipulators without planar faces. We present here an algorithm for the forward kinematic of general manipulators and we show that the number of solutions is bounded by 352. We present a configuration with twelve solutions.

## 1 Introduction

Parallel manipulators present a great interest for many industrial applications due to their high positionning ability and high nominal load. Many applications has been presented in the past either for flight simulator [12] or as robotic devices [4], [3], for example with force-feedback control [11], [7], [8]. This kind of applications uses both inverse kinematics (which is in general straightforward) but also forward kinematics. The later is known to be a difficult problem from a long time [1]. The problem of the direct kinematics has been addressed for manipulators where the mobile plate is a triangle and each of the three articulation points on the mobile are shared by two links. In this case Hunt [5] conjectured that there will be at most sixteen solutions and this conjecture has been proved geometrically in [9]. Then it has been noticed that for a fixed set of links lengths each articulation point of the mobile plate moves on a circle which center and radius may be determined through the links lengths. Thus its position is fully defined by one angle. By expressing the position of the three articulation points of the mobile as a function of the 3 angles and writing that the distance between these points are known quantities one get three equations in the sines and cosines of the unknown angles.

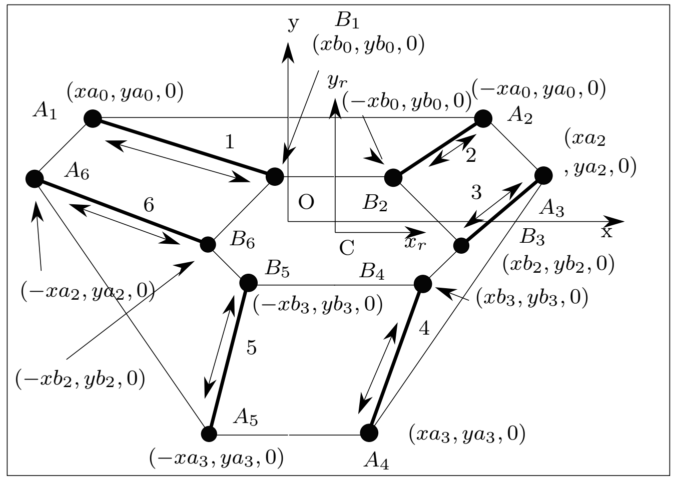


Figure 1: Top view of the considered manipulator

Nanua [10] has combined these equations to get a 24th order polynomial in one of the unknown.

Then it has been shown that the order of this polynomial can be reduced to 16, with only even power, if the lines normal to the circles and going through their centers lie in the same plane and a numerical procedure has enabled to find many configurations with sixteen assembly modes ([9], [2]). In the first work other cases has been considered for the position of the lines yielding to polynomial of order 20 and 24 and an analysis of various parallel manipulator has been performed. A very recent work [6] has proposed a method to find a polynomial of order 16 in every cases i.e. a minimal polynomial.

But this method cannot be applied if the manipulator's mobile plate is not a triangle. We will consider here the case where both plates are hexagons with a symmetry axis ($y$ and $y_r$ in Figure 1).

All the articulation points $A_i$ on the base are coplanar as well as the articulation points $B_i$ on the mobile. The links are numbered from 1 to 6 and we define a reference frame ($O$,x,y,z) where y is the symmetry axis of the fixed base. We then define a mobile frame $(C, x_r, y_r, z_r)$ for the mobile with $y_r$ the symmetry axis of the mobile plate. The coordinates of the articulations point $A_i$ in the reference frame are $(xa_i, ya_i, za_i)$ and for convenience the axis $z$ of the reference frame is chosen such that $za_i = 0$. In the same manner the coordinates of the articulations points $B_i$ in the mobile frame are $(xb_i, yb_i, 0)$. The position of the mobile plate is defined in the reference frame by the coordinates of point $C(x_0, y_0, z_0)$ and its orientation by three Euler's angles $\psi, \theta, \phi$ with the associated rotation matrix $R$ and $\rho_i$ will denote the length of link i. The subscripts will be omitted each time there cannot be any misunderstanding. A subscript $_r$ will denote that the coordinates are expressed in the mobile frame.

## 2 Relation between $C$ and $R$

We will consider here the expression of a links length $\rho$ as a function of the position and orientation of the mobile plate. We have:

$$\underline{AB} = \underline{AO} + \underline{OC} + \underline{CB} = \underline{AO} + \underline{OC} + R\underline{CB_r} \quad (1)$$

Therefore:

$$\begin{aligned} \rho^2 &= \underline{AO}\ \underline{AO}^T + \underline{CB_r}\ \underline{CB_r}^T + \\ &\quad 2(\underline{AO}^T + \underline{CB_r}^T R^T)\underline{OC} + \\ &\quad 2\underline{AO}^T R\underline{CB_r} + \underline{OC}\ \underline{OC}^T \end{aligned} \quad (2)$$

Let us denote $d_A$ the distance between $A$ and $O$ and $d_B$ the distance between $B$ and $C$. Thus we have:

$$\begin{aligned} \rho^2 &= d_A^2 + d_B^2 + 2(\underline{AO}^T + \underline{CB_r}^T R^T)\underline{OC} \\ &\quad + 2\underline{AO}^T R\underline{CB_r} + \underline{OC}\ \underline{OC}^T \end{aligned} \quad (3)$$

Let us consider now two links. We define:

$$U_{ij} = d_{A_i}^2 + d_{B_i}^2 - d_{A_j}^2 - d_{B_j}^2 \qquad \underline{W_{ij}} = \underline{A_iO}^T - \underline{A_jO}^T$$

$$\underline{T_{ij}} = \underline{CB_{i_r}}^T - \underline{CB_{j_r}}^T$$

$$S_{ij} = \underline{A_iO}^T R\underline{CB_{i_r}} - \underline{A_jO}^T R\underline{CB_{j_r}}$$

Therefore we have:

$$\rho_{ij} = \rho_i^2 - \rho_j^2 = U_{ij} + 2S_{ij} + 2(\underline{W_{ij}} + \underline{T_{ij}}^T R^T)\underline{OC} \quad (4)$$

This equation is linear in term of the coordinates of $C$. If we consider the three equations $\rho_{12}, \rho_{45}, \rho_{65}$ we get thus a linear system $\mathcal{S}$ in the three unknowns $x_0, y_0, z_0$. The resolution of this system enables to find the coordinates of $C$ as a function of $\psi, \theta, \phi$. An important remark is that if a set $(\psi, \theta, \phi)$ yields to a solution $(x_0, y_0, z_0)$ then the set $(\psi, -\theta, \phi)$ yields to the solution $(x_0, y_0, -z_0)$. The determinant $\Delta$ of the system is:

$$\begin{aligned} \Delta &= 32\sin\theta(xa_0xb_3 - xa_3xb_0) \\ &\quad (\sin\psi(yb_3 - yb_2) + \sin\phi(ya_3 - ya_2)) \end{aligned} \quad (5)$$

which will vanished if $\sin\theta = 0$, $\sin\phi = \sin\psi = 0$ or $\sin\psi(yb_3 - yb_2) = -\sin\phi(ya_3 - ya_2)$. We will suppose first that none of these conditions is fulfilled.

## 3 The $\phi$-curve

The system $\mathcal{S}$ being solved it may be shown that the two equations $\rho_{24}, \rho_{36}$ can be written as:

$$u_1\cos\theta + u_2 = 0 \qquad v_1\cos\theta + v_2 = 0 \quad (6)$$

where $u_1, u_2, v_1, v_2$ contain only terms in sine and cosine of $\psi, \phi$. From these two equations we get an equation in $\psi, \phi$ by writing the constraint $u_1v_2 - v_2u_1 = 0$:

$$\begin{aligned} u_1v_2 - v_2u_1 &= p_1\sin^3\psi + \\ &\quad p_2\cos^3\psi + (p_{31}\cos\psi + \\ &\quad p_{32})\sin^2\psi + (p_{41}\sin\psi + p_{42})\cos^2\psi + \\ &\quad \sin\psi(p_{51}\cos\psi + p_{52}) + p_6\cos\psi = 0 \end{aligned} \quad (7)$$

where the coefficients $p_i$ are independent from $\psi$. If we define $x = \tan\frac{\psi}{2}$ equation (7) is then a sixth order polynomial in $x$. Let us consider this equation for a given $\phi = \phi_s$. We get then at most 6 solutions in $\psi$, $\psi_{s_i}$. For each pair $\phi_s, \psi s_i$ equation (6) yields two solutions in $\theta$, $(\theta_s, -\theta_s)$. Thus for a given $\phi_s$ we get at most 6 pairs of possible solutions $(\phi_s, \psi s_i, \theta_s)$, $(\phi_s, \psi s_i, -\theta_s)$ which in turn yields to six pairs of solution for the coordinates of $C$, $(x_{0_1}, y_{0_1}, z_{0_1})$, $(x_{0_2}, y_{0_2}, z_{0_2})$. But by using the remark done during the resolution of the system $\mathcal{S}$ we know that $x_{0_2} = x_{0_1}, y_{0_2} = y_{0_1}, z_{0_2} = -z_{0_1}$. Thus the second set of solution represents simply the symmetrical configuration with respect to the fixed base of the first one. Thus for every $\phi$ we get at most a set of 6 possible solutions of the forward kinematics. They are only possible solutions because during the resolution we have used only fives equations among the set of 6 independent equations defined by (2) and therefore we have to verify if the links lengths associated to these solutions are identical to the initial set. In order to verify the validity of one solution we define a performance index $\mathcal{C}$:

$$\mathcal{C} = \sum_1^6 ||\rho_{s_i} - \rho_i|| \quad (8)$$

where $\rho_{s_i}$ denotes the length of link i for a possible solution. This index will vanished for each solution of the forward kinematics. By the use of a discretization of $\phi$

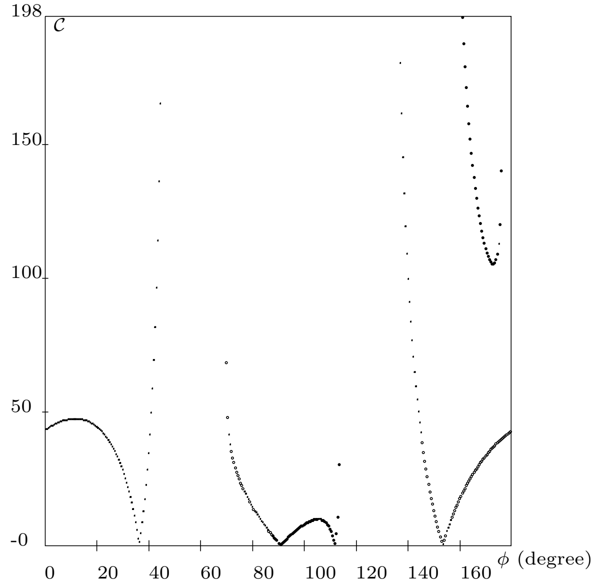


Figure 2: A typical $\phi$-curve ($\phi$ in degree)

we are able to draw a plotting of $\mathcal{C}$ as a function of $\phi$ which is called a *$\phi$-curve*. It can be shown that the discretization of $\phi$ has not to be done between $[0, 2\pi]$ but only between $[0, \pi]$ because any solution between $[\pi, 2\pi]$ will give identical solutions to those find in the interval $[0, \pi]$. By looking at the $\phi$-curve we can determine among the possible solutions those which have a performance index close to zero. These solutions are then fed to a least square algorithm which enables to get the exact solutions. For example if we consider the $\phi$-curve described in figure 2 it appears that 4 solutions have an index close to zero. Taking into account the symmetrical solutions we will thus have 8 solutions for this particular case to which we have to add the eventual solutions corresponding to the particular cases of the resolution of the system $\mathcal{S}$.

# 4 Particular cases

The previous section does not deal with particular cases for which the determinant of the system $\mathcal{S}$ vanishes.

First we consider the case where $\sin\theta = 0$. In this case $\rho_{12}$ is linear in term of $x_0$. Then $\rho_{34}$ becomes linear in term of $y_0$. By using this results $\rho_6^2$ can be written as $z_0^2 + c = 0$. Thus we get two possible solutions and have only to verify if the corresponding links lengths are the same as the original set.

If $\sin\psi(yb_3 - yb_2) = -\sin\phi(ya_3 - ya_2)$ then equation $\rho_{65}$ does not contain any term in $z_0$. It is possible to show that the $\phi$-curve will be identical if we choose any other equation instead of $\rho_{65}$. Thus the only change compared with the general case is that we use another equation to compute the value of $z_0$.

If $\sin\psi = \sin\phi = 0$ then $\rho_{12}$ is linear in term of $x_0$. Then $\rho_{24}, \rho_{65}$ are linear in term of $y_0, z_0$. We expand $\rho_1^2$ and if we define $x = \tan\frac{\theta}{2}$ we get a fourth order polynomial in $x$. Thus we get a four possible solutions and have only to verify if the corresponding links lengths are the same as the original set.

# 5 Maximum number of solutions

At this stage we don't have presented the problem of direct kinematics as a solution of a polynomial in one variable. To get such a polynomial we use the solutions of the system $\mathcal{S}$ to calculate the value of $\rho_1^2$. We get:

$$a_4\cos^4\theta + a_3\cos^3\theta + a_2\cos^2\theta + a_1\cos\theta + a_0 = 0 \quad (9)$$

where the coefficients $a_i$ are independent from $\theta$. If we consider now equation (6) we get:

$$\cos\theta = -\frac{u_2}{u_1} \quad (10)$$

If we define $x = \tan\frac{\psi}{2}$,$y = \tan\frac{\phi}{2}$ equation (9) is a polynomial in $x, y$ which order is 32. Using the same substitution equation (7) is a polynomial in $x, y$ which order is 11. By calculating the resultant of these two polynomials it would be theoretically possible to get a polynomial in one variable which order would be 32 x 12 =352. Thus the maximum number of solution for a SSM is 352.

# 6 Numerical example

The following algorithm has been implemented:
1) verify if the initial set of links lengths can satisfied the particular cases and find the corresponding configurations.
2) compute the performance index $\mathcal{C}$ for a discretization of $\phi$ in the range $[0, \pi]$ (a step of one degree seems to be sufficient).
3) if $\mathcal{C}$ is sufficiently low use the possible solution as an estimate for a least-square method.

This algorithm has been used for a manipulator with the following characteristics:

| number | xa | ya | xb | yb |
|---|---|---|---|---|
| 0 | -9.7 | 9.1 | -3 | 7.3 |
| 2 | 12.76 | 3.9 | 7.822 | -1.052 |
| 3 | 3 | -13 | 4.822 | -6.248 |

The initial set of links lengths is determined for the configuration $x_0 = -5$, $y_0 = 5$, $z_0 = 17$, $\psi = 0$, $\theta = 30$, $\phi = 0$. The six over-the-base solutions are given in table 1 and the corresponding configurations are presented in figure 3.

# 7 Conclusion

The proposed algorithm enables to find all the solutions of the forward kinematics problem for a general parallel manipulator. Although the computation time is rather important (about thirty seconds on a SUN 3-60 workstation) it is a first step toward a general resolution of the difficult forward kinematics problem. The upper bound of the number of solutions is probably overestimated: we plan to continue to work on the equations to decrease it.

| $x_0$ | $y_0$ | $z_0$ | $\psi$ | $\theta$ | $\phi$ |
|---|---|---|---|---|---|
| -5.0 | 5.0 | 17.0 | 0.0 | 30.0 | 0.0 |
| 4.864 | 3.2 | 14.606 | 323.627 | 95.32 | 36.371 |
| -10.993 | 1.78 | 12.329 | 206.593 | -77.993 | 153.406 |
| -5.0 | -7.648 | 11.288 | 0.0 | -118.179 | 0.0 |
| 5.502 | -4.708 | 8.39 | 68.13 | 127.378 | 111.871 |
| -4.693 | -2.020 | 5.186 | 88.941 | -82.951 | 91.057 |

Table 1: Solution of the forward kinematics problem (all the angles are in degree)

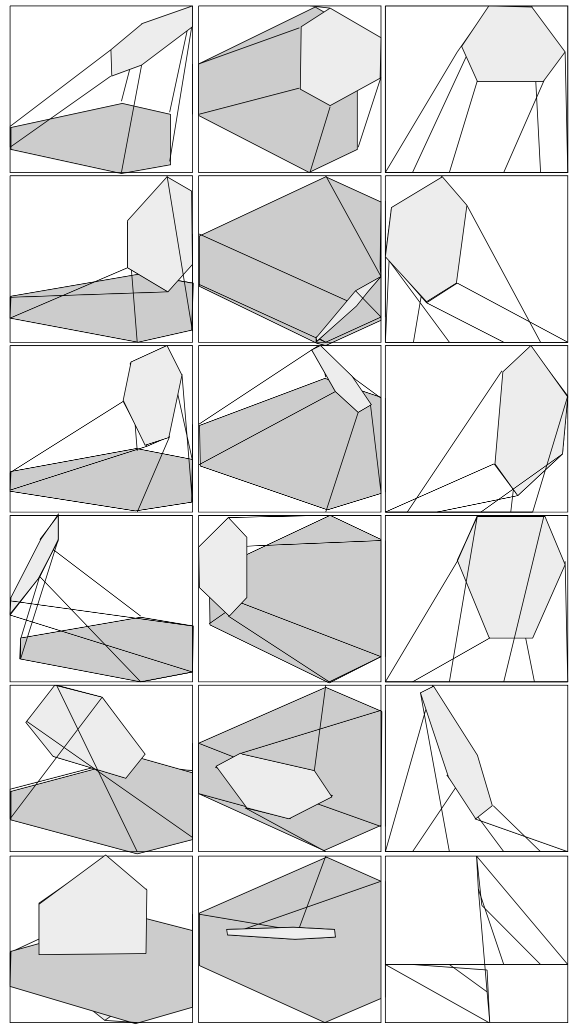

Figure 3: The 6 configurations for which the links lengths are identical (perspective, top and side view). We would get the 6 others solutions by putting the mobile plate in a symmetrical position with respect to the base.

# References


[1] Bricard R. "Mémoire sur la théorie de l'octaèdre articulé", Journal de Mathématiques pures et appliquées, Liouville, cinquième série, tome 3, 1897

[2] Charentus S., Renaud M. "Calcul du modèle géométrique direct de la plate-forme de Stewart", LAAS Report n° 89260, 1989, July, Toulouse, France.

[3] Gosselin. C. *Kinematic analysis, optimization and programming of parallel robotic manipulators*, Ph. D. thesis, McGill University, Montréal, Québec, Canada, 1988

[4] Hunt K.H. *Kinematic geometry of mechanisms*, Clarendon Press, Oxford, 1978

[5] Hunt K.H. "Structural kinematics of in Parallel Actuated Robot Arms". Trans. of the ASME, J. of Mechanisms,Transmissions, and Automation in design Vol 105: 705-712.

[6] Innocenti C.,Parenti-Castelli V., "Direct position analysis of the Stewart Platform Mechanism", Mechanism and Machine Theory, Vol. 25, n° 6, pp. 611-621, 1990.

[7] Merlet J-P. "Parallel manipulators, Part 1, Theory" Rapport de Recherche INRIA n° 646, March 1987

[8] Merlet J-P. "Force-feedback control of parallel manipulators" IEEE Int. Conf. on Robotics and Automation, Philadelphia,24-29 April 1988

[9] Merlet J-P. "Parallel manipulators, Part 4, Mode d'assemblage et cinématique directe sous forme polynomiale" INRIA Research Report n° 1135, December 1989

[10] Nanua P., Waldron K.J. "Direct kinematic Solution of a Stewart Platform", IEEE Int. Conf. on Robotics and Automation, Scotsdale, Arizona, May 14-19 1989,pp. 431-437.

[11] Reboulet C., Robert A. "Hybrid control of a manipulator with an active compliant wrist", 3th ISRR, Gouvieux,France, 7-11 Oct. 1985, pp. 76-80.

[12] Stewart D. "A platform with 6 degrees of freedom",*Proc. of the institution of mechanical engineers* 1965-66, Vol 180, part 1, n° 15, pp.371-386.