# Some properties of the Irvine cable model and their use for the kinematic analysis of cable-driven parallel robots

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**Abstract.** Cable model has a strong influence on the complexity of the kinematic analysis of cable-driven parallel robots (CDPR). The most complete model relies on Irvine equation that takes into account both the elasticity and the deformation of the cable due to its own mass and has be shown to be very realistic. This model is complex, non algebraic and numerically ill-conditioned, thereby leading to difficulties when using it in a kinematic analysis involving several cables. We exhibit some properties of this model that may drastically improve the analysis computation time when used in kinematic studies.

**Keywords:** cable-driven parallel robots, cable model, sagging cables

## 1 Introduction

In this paper we will consider the Irvine sagging cable model that has been proposed for elastic and deformable cable with mass [3] and that has been shown to be in very good agreement with experimental results [9]. This model assumes that the cable lies in a vertical plane, the *cable plane*, and is therefore a 2D model. A reference frame is defined in this plane with its origin at $A_i$, one of the extremity of the cable. The coordinates of the other cable extremity $B_i$ are $(x_b \geq 0, z_b)$ and we will assume that $B_i$ is below $A_i$ so that $z_b \leq 0$ (Assumption 1). Vertical and horizontal forces $F_z, F_x > 0$ are exerted on the cable at point $B_i$ (figure 1).

For a cable with length at rest $L_0$ the coordinates of $B$ are given by:

$$x_b = F_x(\frac{L_0}{EA_0} + \frac{sinh^{-1}(F_z) - sinh^{-1}(F_z - \frac{\mu g L_0}{F_x})}{\mu g}) \tag{1}$$

$$z_b = \frac{F_z}{EA_0} - \mu g L_0^2/2 + \frac{\sqrt{F_x^2 + F_z^2} - \sqrt{F_x^2 + (F_z - \mu g L_0)^2}}{\mu g} \tag{2}$$

where $E$ is the Young modulus of the cable material, $A_0$ the cable cross-section area and $\mu$ the cable linear density.

Studying the properties of the Irvine cable model is justified by its use in the kinematic analysis of cable-driven parallel robots (CDPR). The model will evidently play an essential role for the inverse and direct kinematics (IK and DK)

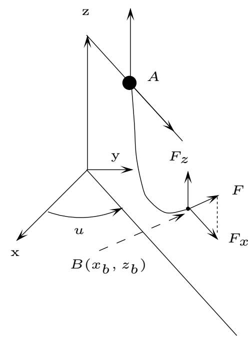


**Fig. 1.** Notation for a sagging cable

which in turn plays a role in workspace and singularity analysis. This model also influences the static analysis whose purpose is to determine the tension in the cables [2]. Previous works have focused on the analysis of equations (1,2) whose unknowns are $x_b, z_b, L_0, F_x, F_z$, especially assuming that 3 of these 5 unknowns have a fixed value [7] in which case the solution is unique but has to be determined numerically (case 1). Other works have addressed the even more complex case of the IK of CDPR in which $n$ cables are attached to a rigid body in a known pose (hence the cable plane and the $x_b, z_b$ of each cable are known) with the purpose of determining the $L_0$. Here we have a system of $2n$ equations (1,2) with $3n$ unknowns but the mechanical equilibrium of the platform imposes 6 additional equations. If $n = 6$ we end-up with a square system of equations [4],[8],[10] (case 2). For the IK solving authors have used optimization or have assumed that the solution is sufficiently close to the rigid leg case one which is therefore used as initial guess for a solving based on the Newton scheme. However these methods cannot guarantee to find the solution in case 1 or all solutions in case 2 where there may be multiple solutions. The problem is even more complex for the DK of CDPR: in that case the kinematic constraints is always a square system that has usually multiple solutions. We have addressed these issues in previous publications using as solving method an interval analysis-based approach that is guaranteed to provide all solutions provided that the unknowns may be bounded [5, 6]. However the efficiency of this approach is heavily dependent on a careful modeling and analysis of the equations at hand, transforming a problem that is almost intractable to one that may be solved in a few seconds.

Interval analysis is based on *interval evaluation* of a function $f$ in the unknowns $\{x_1, x_2, \ldots x_n\}$ that are supposed to be bounded i.e. for each $x_i$ we have $x_i \in [\underline{x_i}, \overline{x_i}]$. Such bounds defines a *box* in the $n$-dimensional space of the unknowns. Being given such a box $\mathcal{B}$ the interval evaluation $\hat{f}$ of $f$ over $\mathcal{B}$ is an interval $[\underline{f}, \overline{f}]$ such that for any point $X$ in $\mathcal{B}$ we have $\underline{f} \leq f(X) \leq \overline{f}$. In other words $\underline{f}$ is either equal to or a minorant of the minimum $f_{min}$ of $f$ over $\mathcal{B}$ while $\overline{f}$ is equal to or a majorant of the maximum $f_{max}$ of $f$ over $\mathcal{B}$ . The interval evaluation of $f$ is relatively easy to obtain as soon as $f$ is expressed in terms of classical mathematical functions using the *natural evaluation* which basically consist in replacing the operators by interval equivalents. For example interval evaluation of the Irvine equations may be obtained by natural evaluation. However the efficiency of interval algorithms is drastically dependent upon the *tightness* of the interval evaluation: the closer $\underline{f}, \overline{f}$ are to $f_{min}$, $f_{max}$, the faster will be the algorithm. An interval evaluation will be denoted *tight* if $\hat{f} = [f_{min}, f_{max}]$. But the natural evaluation may lead to large under or overestimation of the minimum and maximum as soon as there are multiple occurrences of the unknowns in $f$ (it may be proven that if there is only a single occurrence of each unknown in $f$, then $\hat{f}$ is tight, up to round-off errors). The tightness will improve when the widths of the intervals for the unknowns decrease but an efficient way to improve the tightness of the evaluation is to consider the derivatives of $f$ and their own interval evaluation. Let $f_i$ be the derivative of $f$ with respect to $x_i$ and let $[\underline{f_i}, \overline{f_i}]$ be its interval evaluation over $\mathcal{B}$. If $\underline{f_i} > 0$ or $\overline{f_i} < 0$, then $f$ is monotonic with respect to $x_i$. Consequently $\hat{f}$ may be obtained as $[\text{Min}\hat{f}(\mathcal{B}_i), \text{Max}\hat{f}(\mathcal{B}_i)]$ where $\mathcal{B}_i$) are the boxes that are derived from $\mathcal{B}$ with $x_i$ set to $\underline{x_i}$ or $\overline{x_i}$. Note that this process has to be applied recursively. Indeed assume that for some $i$ we have $\underline{f_i} < 0, \overline{f_i} > 0$ while for $j > i$ $f$ is monotonic with respect to $x_j$ so that $\hat{f}$ will be obtained using $\mathcal{B}_j$. But the interval evaluation of $f_i$ for $\mathcal{B}_j$ may differ from the one obtained for $\mathcal{B}$ as $x_j$ has now a fixed value instead of the interval value that has been previously used for the interval evaluation of $f_i$. Using this process we may tighten the interval evaluation of $f$ up to the point where $\hat{f} = [f_m in, f_m ax]$ if $f$ is such that all $f_i, i \in [1, n]$ are positive or negative.

We present in the next sections some interesting properties of the Irvine equations that can be used for analysis or solving purposes.

## 2 Properties of the Irvine equations

A preliminary property will play an important role: we have assumed that $B$ has an altitude that is equal or lower to the one of $A$ with the direct consequence that $F_z \leq \mu g L_0/2$.

### 2.1 Derivatives of the Irvine equations

The sign of the derivatives of the Irvine equations may be obtained with interval evaluation but it is interesting to determine beforehand if they may be inherently monotonic.

Under assumption 1 we may establish the sign of derivatives of equations (1),(2) that will be presented without proof as they are trivial. We have

$$\frac{\partial z_b}{\partial L_0} < 0 \quad \frac{\partial z_b}{\partial F_z} > 0 \quad \frac{\partial z_b}{\partial F_x} > 0 \tag{3}$$

As all derivatives of $z_b$ have a constant sign, then its interval evaluation for interval values for $F_x, F_z, L_0$ will always be tight and can be computed efficiently using only floating point operators. This may have an impact on the IK solving in which $z_b$ has a fixed value: If $\hat{z_b} \cap z_b = \emptyset$, then (2) has no solution for the current $F_x, F_z, L_0$ box. We have also:

$$\frac{\partial x_b}{\partial L_0} > 0 \quad \frac{\partial x_b}{\partial F_z} > 0 \quad \frac{\partial x_b}{\partial F_x} > 0 \tag{4}$$

It may also be interesting to consider the distance $D = x_b^2 + z_b^2$ between $A$ and $B$. We have $\partial D/\partial L_0 > 0$ but no general monotonicity can be obtained with respect to $F_x, F_z$.

Let $F_x, F_z, L_0$ being bounded i.e. $F_x \in [\underline{F_x}, \overline{F_x}]$, $F_z \in [\underline{F_z}, \overline{F_z}]$, $L_0 \in [\underline{L_0}, \overline{L_0}]$. Let us assume that $z_b$ is fixed and consider the equation $f(L_0, F_z, F_x) - z_b = 0$. Using the implicit value theorem it may be shown that the solution of this equation satisfy

$$\frac{\partial F_x}{\partial L_0} > 0 \quad \frac{\partial F_x}{\partial F_z} < 0$$

so that $F_x$ is restricted to lie in the interval $[\underline{F'_x}, \overline{F'_x}]$ where $\underline{F'_x}$ is the solution of (2) obtained for $L_0 = \overline{L_0}$, $F_z = \underline{F_z}$ and $\overline{F'_x}$ is the solution of (2) obtained for $L_0 = \underline{L_0}$, $F_z = \overline{F_z}$. The range for $F_x$ may therefore be calculated as $[\underline{F_x}, \overline{F_x}] \cap [\underline{F'_x}, \overline{F'_x}]$ and the equation has no solution if this intersection is empty. More generally if we consider (2) when 2 of the unknowns are fixed and denotes by $S$ its solution in the last unknown we get $[\underline{L'_0}, \overline{L'_0}] = [S(\overline{F_x}, \overline{F_z}), S(\underline{F_x}, \underline{F_z})]$ and $[\underline{F'_z}, \overline{F'_z}] = [(S(\underline{F_x}, \overline{L_0}), S(\overline{F_x}, \underline{L_0})]$.

## 2.2 New forms for the Irvine equation

A property of interval analysis is that two mathematically equivalent forms of $f$ may have different interval evaluations. For example $f^1 = x^2 + 2x + 1$ and $f^2 = (x+1)^2$ are equivalent but $\hat{f^2}$ will be tight with only one occurrence of $x$ while $\hat{f^1}$ will not if $\underline{x} < 0$. Therefore interval analysis algorithm are partly based on heuristics that compute various interval evaluations of the same $f$ expressed in different ways and returning $\hat{f}$ as their intersection. We present in this section various new relationships between the quantities appearing in the Irvine equations. They are usually expressed in implicit form $G(x_b, z_b, F_x, F_z, L_0) = 0$ (and the interval evaluation of $G$ may allow one to discard boxes in an interval analysis algorithm but it may also happen that the analysis of $G$ provides bounds for one variable being given interval values for the other unknowns.

### Using the $z_b$ equation

**$F_x$ as function of $z_b, F_z, L_0$** Let

$$a^2 = F_x^2 + F_z^2 \qquad b^2 = F_x^2 + (F_z - \mu g L_0)^2 \quad a^2 - b^2 = \mu g L_0(2F_z - \mu g L_0)$$

then $z_b$ may be written as

$$z_b = \frac{(a-b)}{\mu g}(\frac{(a+b)}{2EA_0} + 1) = \frac{L_0(2F_z - \mu g L_0)}{2EA_0} + \frac{a-b}{\mu g} \quad (5)$$

Let us assume now that $z_b, F_x, L_0$ are given so that (2) has only $F_x$ as unknown. Our objective is to get an expression of this unknown. Let us define

$$a^2 = F_x^2 + F_z^2 \qquad b^2 = F_x^2 + (F_z - \mu g L_0)^2 \qquad U = \frac{L_0(F_z - \mu g L_0/2)}{EA_0} - z_b$$

so that equation (2) may be written as

$$U + \frac{(a-b)}{\mu g} = 0 \quad (6)$$

We have also

$$a^2 - b^2 = 2F_z \mu g L_0 - (\mu g L_0)^2 = V = (a+b)(a-b) = (a+b)(-U\mu g)$$

from which we get

$$b = -\frac{V}{U\mu g} - a$$

Reporting $b$ in (6) leads to

$$2a = -U\mu g - \frac{V}{U\mu g} = W \quad (7)$$

Note that $U, V$ are not function of $F_x$ so that $W$ is expressed only as a function of $F_z, L_0$. As $a^2 = (W/2)^2 = F_x^2 + F_z^2$ we get

$$F_x^2 = (W/2)^2 - F_z^2 \quad (8)$$

where the right-hand term is a function of $F_z, L_0$ only. This equation provides $F_x$ if $z_b, L_0, F_z$ are fixed. Let's assume now that $F_z$ has an interval value and consider $P = F_x^2 = (W/2)^2 - F_z^2$. Our problem is to determine the value of $F_z$ so that $P > 0$. The polynomial $P$ is of degree 4 in $F_z$ and factors out in 4 terms that are linear in $F_z$. The root of $P$ are

$$s_1 = \frac{\mu g L_0}{2} + \frac{\mu g A_0 E z_b}{2A_0E + \mu g L_0} \qquad s_2 = \frac{\mu g L_0}{2} + (z_b - L_0)\frac{A_0 E}{L_0}$$

and

$$s_3 = \frac{\mu g L_0}{2} + (z_b + L_0)\frac{A_0 E}{L_0} \qquad s_4 = \frac{\mu g L_0}{2} - \frac{\mu g A_0 E z_b}{2A_0E - \mu g L_0}$$

If we assume $2A_0E > \mu g L_0$ then the roots in $F_z$ are ordered as $s_2, s_1(< \mu g L_0/2), s_4(> \mu g L_0/2), s_3$ and $P$ will be positive if $F_z \in [s_2, s_1]$. If $2A_0E < \mu g L_0$ then the roots are ordered as $s_2, s_4(< \mu g L_0/2), s_1(< \mu g L_0/2), s_3$. Therefore there are 2 possible ranges for $F_z$ leading to a positive $P$: $[s_2, s_4]$, $[s_1, \mu g L_0/2]$

**$L_0$ as function of $z_b, F_z, F_x$** We are now interested in determining $L_0$ when $F_x, F_z, z_b$ are fixed. Let $U_1 = \sqrt{F_x^2 + F_z^2}$, $U_2 = \mu g F_z/EA_0$, $U_3 = (\mu g)^2/(2EA_0)$ and $U_4 = -\mu g z_b + U_1$. Equation (2) may be written as

$$U_2 L_0 - U_3 L_0^2 + U_4 = \sqrt{F_x^2 + (F_z - \mu g L_0)^2} \tag{9}$$

Squaring the previous equation leads to

$$P_s = (U_2 L_0 - U_3 L_0^2 + U_4)^2 - (F_x^2 + (F_z - \mu g L_0)^2) = 0 \tag{10}$$

As $U_1, U_2, U_3, U_4$ are not function of $L_0$ this equation is a fourth order polynomial in $L_0$. Using the Sturm sequences it is possible to show that $P_s$ has only 2 roots in the range $[0, \infty]$. But it may be seen that (10) leads to 2 possibilities for (9) namely

$$U_2 L_0 - U_3 L_0^2 + U_4 = \pm\sqrt{F_x^2 + (F_z - \mu g L_0)^2}$$

The negative version leads to $z_b > 0$ which is not valid under assumption 1 and therefore solving $P_s$ (whose roots may be obtained in analytical form) leads to a single solution for $L_0$.

**$F_z$ as function of $z_b, F_x, L_0$** We consider determining $F_z$ for given $L_0, F_x, z_b$. Equation (8) is a 4th order polynomial $Q$ in $F_z$ with the constraint that $W > 0$. Using Budan-Fourier theorem [1] it is possible to show that $Q$ has 0 or 2 roots in the range $]-\infty, \mu g L_0/2]$ but only one this root will lead to a positive $W > 0$. The analysis of the sign of $W$ is complex but it may be shown that if $EA_0 \gg \mu g L_0$, then $F_z$ must belong to the range $[\mu g L_0/2 + EA_0 z_b/L_0, \mu g L_0/2 - \mu g z_b^2/(2L_0)]$.

### Using the $x_b$ and $z_b$ equations

**$F_z$ as function of $x_b, z_b, F_x, L_0$** First we will consider the calculation of $F_z$ being given $F_x, L_0, x_b, z_b$. We define

$$u = \frac{F_z}{F_x} \quad v = \frac{F_z - \mu g L_0}{F_x}$$

so that equation (1) may be written as

$$(\frac{x_b}{F_x} - \frac{L_0}{EA_0})\mu g = sinh^{-1}(u) - sinh^{-1}(v) \tag{11}$$

We define $H_1 = x_b/F_x - L_0/(EA_0)$ and considering that $sinh^{-1}(u) - sinh^{-1}(v) = sinh^{-1}(u\sqrt{1+v^2} - v\sqrt{1+u^2})$ and taking the hyperbolic sine of both terms of equation (11) we obtain:

$$H = sinh(H_1)\mu g = u\sqrt{1+v^2} - v\sqrt{1+u^2} \tag{12}$$

We have already defined $a^2 = F_x^2 + F_z^2$, $b^2 = F_x^2 + (F_z - \mu g L_0)^2$ so that $a^2/F_x^2 = 1 + u^2$ and $b^2/F_x^2 = 1 + v^2$. Equation (12) may therefore be written as:

$$F_x H = ub - va \tag{13}$$

Note that the left-hand term of this equation is not a function of $F_z$. We have already established in section 2.2 the values of $a, b$ as function of $z_b, L_0, F_z$ while $u, v$ are functions of $F_x, L_0, F_z$. Hence the right-hand term of (13) is a function of $z_b, L_0, F_x, F_z$. This function is a third order polynomial $P_3$ in $F_z$. By using the Sturm sequence [1] and the constraint $a > 0$ it is possible to show that $P_3$ has a single real root in the range $]-\infty, \mu g L_0/2]$.

**$zb$ as a function of $z_b, F_x, F_z, L_0$** Equation (2) provides a mean of calculating $z_b$ when $F_x, F_z, L_0$ are known but does not involve $x_b$. We provide here another form that involves $x_b$. Using the notation and result of section 2.2 we get:

$$z_b = \frac{L_0}{F_z}(\sqrt{F_x^2+F_z^2} - \frac{F_x^2}{\mu g L_0} sinh(\mu g(\frac{x_b}{F_x} - \frac{L_0}{EA_0}))) + \frac{L_0(F_x - \mu g L_0/2)}{EA_0} \quad (14)$$

Note that we may also obtain a bound on the cable tension $\sqrt{F_x^2+F_z^2}$ at $A$ as

$$\sqrt{F_x^2+F_z^2} = \frac{F_x^2}{\mu g L_0} sinh(\mu g(\frac{x_b}{F_x} - \frac{L_0}{EA_0})) + F_z(\frac{z_b}{L_0} + \frac{\mu g L_0}{2EA_0} - \frac{F_z}{EA_0}) \quad (15)$$

### 2.3 Using the cable tangents

Sensors may provide measurements of the cable tangents $v = (F_z - \mu g L_0)/F_x$ at $A$ and $u = F_z/F_x$ at $B$. Under the assumption that $u, v$ are known we get

$$F_x = \frac{\mu g L_0}{(u-v)} \qquad F_z = uF_x \qquad F_x^2 + F_z^2 = (\frac{\mu g L_0}{(u-v)})^2(1+u^2) \quad (16)$$

A trivial transformation of (2) leads to:

$$\mu\, g{L_0}^2(u+v) + 2\, A_0\, EL_0(\sqrt{u^2+1} - \sqrt{v^2+1}) + 2\, z_b\, EA_0(v-u) = 0 \quad (17)$$

which is a quadratic polynomial in $L_0$ whose coefficients are functions of $u, v, x_b$. It is easy to show that this polynomial have a single positive root. Now equation (1) may be written as

$$F_x(\frac{L_0}{EA_0} + \frac{(sinh^{-1}(u) - sinh^{-1}(v))}{\mu g}) - x_b = 0 \quad (18)$$

As we have $F_x = \mu g L_0/(u-v)$ this equation may be transformed in a second order polynomial in $L_0$ whose coefficients are functions of $u, v, x_b$. Here again it is easy to show that this polynomial has at most one positive root.

As $F_z = uF_x$ and $L_0 = (F_x(u-v))/(\mu g)$ equations (11), (18) are polynomials in $F_x$ with coefficients that are function of $u, v$. The resultant of these equations in $F_x$ establishes a polynomial relationship between $x_b, z_b$ which is a quadric, more precisely a *parabola* which is written as

$$(Ax_b + Cz_b)^2 + Dx_b + Fz_b = 0$$

with $R_1 = (sinh^{-1}(u) - sinh^{-1}(v))/(\mu g)$, $R_2 = \sqrt{1+u^2} - \sqrt{1+v^2}$, $A = \sqrt{\mu g}(u-v)(u+v)$, $C = -2\sqrt{\mu g}(u-v)$, $D = 2EA_0(u-v)(R_1\mu g(u+v) - 2R_2)R_2$, $F = -2\mu g EA_0(u-v)(R_1\mu g(u+v) - 2R_2)R_1$. Note that if $EA_0 \gg \mu g L_0$, then $A, C$ are small and $D, F$ very large so that the parabola is very close to a line.

## 3 Conclusion

We have presented in this paper various results regarding the the Irvine equations that may be useful both for the analysis and solving of kinematic equations that rely on this cable model as they establish a more general view of the underlying structure of this model. We have already implemented some of these results in our CDPR IK and DK solver with a strong influence on the solving time. Still open issues on the CDPR with sagging cables such as workspace and singularity analysis have to be investigated with this new look on the Irvine equations.

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