Symbolic computation for the Determination of the Minimal direct kinematics Polynomial and the Singular configurations of parallel manipulators

J-P. Merlet
INRIA Centre de Sophia-Antipolis
2004 Route des Lucioles
06565 Valbonne Cedex , France

November 27, 2013

Abstract

In this paper we will address the problems of parallel manipulator's direct kinematics (i.e. find the position and orientation of the mobile plate as a function of the articular coordinates) and the determination of their singular configurations. We will show how symbolic computation has been used to determine the minimal degree of a polynomial formulation of the direct kinematic problem together with this polynomial and to calculate all the necessary and sufficient conditions which are to be satisfied by the position and orientation of the mobile plate to get a singular configuration.

We consider a 6 d.o.f. manipulator in the case where the mobile plate is a triangle and the links lengths are time-varying. Using geometrical considerations and symbolic computation we are able to show that an upper-bound of the maximum number of assembly-modes and thus the maximum number of solutions of the direct kinematics problem, is 16. We describe then how symbolic computation has been used to reduce the direct kinematics problem as the solution of a sixteenth order polynomial in one variable. Using a numerical procedure we show that this polynomial may have 16 real roots and we exhibit one example for which the maximum number of assembly mode is reached.

Then we consider shortly the problem of singular configuration which can be reduced to determine some special geometric conditions to be fulfilled by the link's lines. We show how a geometric package can be used to determine necessary and sufficient conditions for the position and orientation of the mobile plate so that the manipulator is in a singular configuration.

1 Introduction

The direct kinematics problem of a manipulator can be stated in the following manner : the articular coordinates being known is it possible to find the generalized coordinates of the effector? The inverse kinematics problem for parallel manipulator is easily solved : in general each articular coordinate can be expressed as a non-linear function of the generalized coordinates [1], [2]. Thus we have to solve a system of non-linear equations to determine the solution of the direct kinematics problem.

Solving a system of non-linear equations is a difficult task and a numerical resolution is tedious. Furthermore we have no a-priori information about the uniqueness of the solution (except in the case of singular configurations [3]).

Nanua and Waldron [4] have initiated a new approach to this problem. They reduce the resolution of the system of non-linear equations to the one of a polynomial in one variable. The number of assembly-modes of the manipulator (i.e. the number of way one can assemble the manipulator with fixed articular coordinates) is clearly related to the degree of this polynomial: it cannot be greater than this degree.

In the case of the manipulator called the TSSM [5], [6] (Triangular Symmetric Simplified Manipulator, see figure 1) these authors show that the direct kinematics problem may be expressed as solution of a polynomial in one variable, which degree is 24. Charentus and Renaud [7][8] have studied the same manipulator, in the case where the mobile plate is an equilateral triangle. They have shown that the degree of the polynomial can be reduced to 16. Hunt [9] has proposed a conjecture for the same manipulator which states that the number of assembly-modes cannot be greater than 16. In a first part we will prove the conjecture of Hunt and calculate the polynomial for the TSSM in the general case. This is done by showing that the TSSM is similar to another mechanism called the *equivalent mechanism* of the TSSM for which we can establish a polynomial and an upper-bound of the maximum number of assembly mode (called the *UBAM* for brief in the following sections). We will present then a configuration for which there is 16 assembly-modes.

2 The TSSM

2.1 Equivalent mechanism

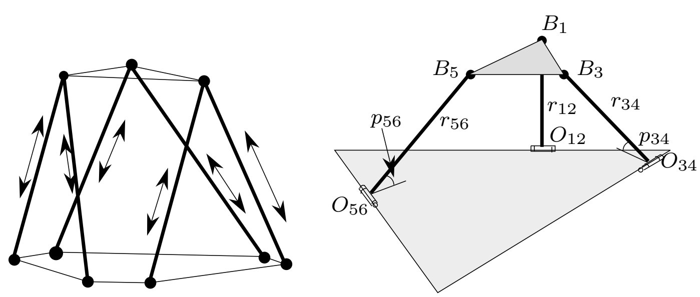

Figure 1: The TSSM parallel manipulator (side and top view)

The TSSM (figure 1) is a 6 d.o.f. parallel manipulator in which a mobile plate is connected to a fixed base through 6 articulated links, each link being connected both at the base and the mobile plate through ball-and-socket and universal joints. By controlling the links lengths we are able to control the position and orientation of the mobile plate [5], [10], [11], [12]. We define a reference frame (O, x, y, z) where O is located in the plane of the base, z is perpendicular to the base, y is the symetry axis of the base and x is deduced from y, z. The length of link i will be denoted by ρ_i, the coordinates of the articulation point on the base of segment i (denoted by A_i) in the reference frame are $(xa_i, ya_i, 0)$. Others notations are defined in figure 2.

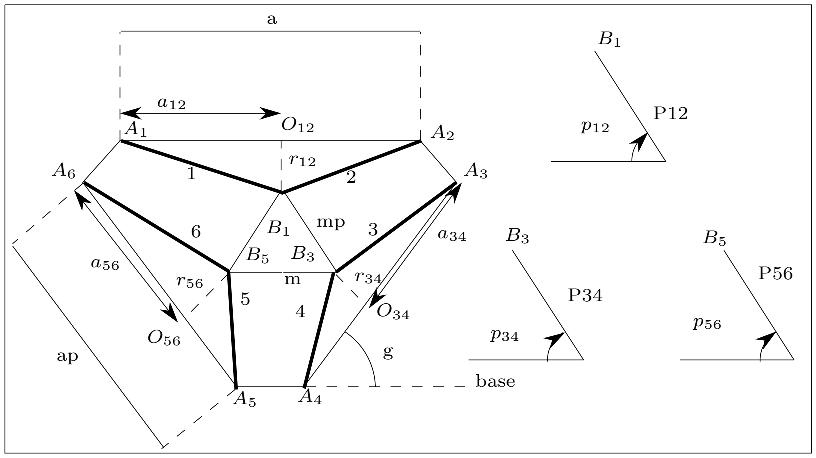

Figure 2: Notation (the TSSM is represented in top view)

For fixed links lengths the articulation points B_1, B_3, B_5 of the mobile plate are able to describe circles centered in O_{12}, O_{34}, O_{56} whose radius are r_{12}, r_{34}, r_{56}. The coordinates of the center O_{ij} are denoted by $(x_{O_{ij}}, (y_{O_{ij}}, 0)$. The characteristics of these circles can be determined using only the knowledge of the links lengths. Thus the TSSM is equivalent to a mechanism constituted of three links articulated on revolute joints and connected to the mobile plate (figure 3) for which the articular coordinates are the angles p_{12}, p_{34}, p_{56}. This mechanism is called the *equivalent mechanism* of the TSSM.

2.2 Minimal degree of the TSSM polynomial

Hunt [9] has conjectured that an UBAM of the TSSM is 16 by using the following method: if we dismantle one of the link of the equivalent mechanism of the TSSM we get a RSSR mechanism (figure 4). It is known [13] that point B of this mechanism describes a sixteenth order surface, the *RSSR spin surface*. In order to find the possible configurations of mobile plate we have to intersect this surface with the circle described by the extremity of the dismantled link. A sixteenth order surface is intersected by a circle in no more than 32 points. Working on the conjecture that the RSSR spin-surface contains the imaginary spherical circle eight times Hunt deduces that at least 16 points are imaginary, and therefore there is at most 16 assembly-modes for the TSSM.

Thus to demonstrate this conjecture we have to determine the circularity of the RSSR spin-surface.

Let X, Y, Z denote the coordinates of B. The coordinates of the articulation points B_1, B_3 can be expressed as a function of the unknown angles p_{12}, p_{34}. Thus we are able to write three equations relating the known distance between the articulation points to the unknowns p_{12}, p_{34}, X, Y, Z. We have:

$$a_{12} = \frac{\rho_1^2 + a^2 - \rho_2^2}{2a} \qquad a = 2xa_2 \tag{1}$$

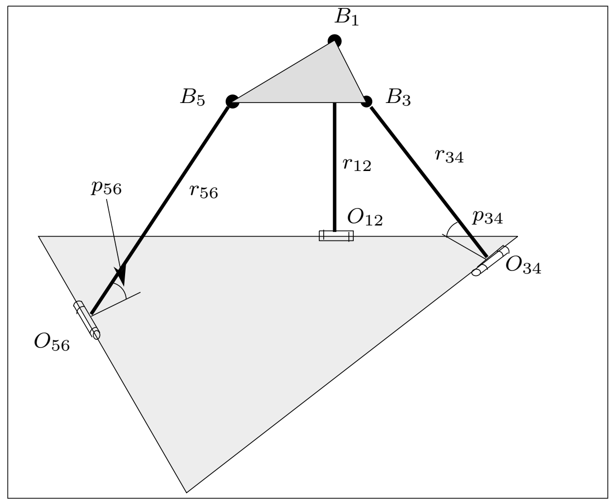

Figure 3: Equivalent mechanism of the TSSM

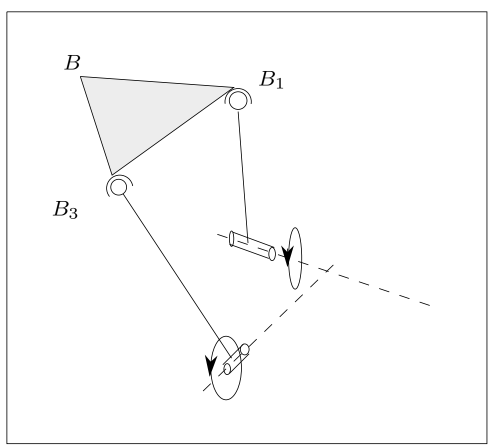

Figure 4: The RSSR mechanism obtained when one link of the equivalent mechanism of the TSSM is dismantled

$$x_{O_{12}} = -xa_2 + a_{12} = \frac{\rho_1^2-\rho_2^2}{4xa_2} \quad y_{O_{12}} = ya_2 \quad r_{12}^2 = \rho_1^2 - a_{12}^2 \tag{2}$$

If n_{ij} denote the unit vector between O_{ij} and the corresponding articulation points we get :

$$n_{12} = -\cos(p_{12})\mathbf{j} + \sin(p_{12})\mathbf{k} \tag{3}$$

In the same way:

$$a_{34} = \frac{\rho_3^2+ap^2-\rho_4^2}{2ap} \tag{4}$$

$$x_{O_{34}} = xa_3 - a_{34}\cos(g) \quad y_{O_{34}} = ya_3 - a_{34}\sin(g) \quad r_{34}^2 = \rho_3^2 - a_{34}^2 \tag{5}$$

$$n_{34} = -\cos(p_{34})\sin(g)\mathbf{i} + \cos(p_{34})\cos(g)\mathbf{j} + \sin(p_{34})\mathbf{k} \tag{6}$$

We may write then

$$\mathbf{OB_1} = \mathbf{OO_{12}} + r_{12}\mathbf{n_{12}} \tag{7}$$

$$\mathbf{OB_3} = \mathbf{OO_{34}} + r_{34}\mathbf{n_{34}} \tag{8}$$

We are able to express the norm of the vectors $\mathbf{B_1B_3}$, $\mathbf{B_1B}$, $\mathbf{B_3B}$ i.e. the distances between the articulation points of the mobile whose values are mp and m, m. This yields to the following three equations :

$$||B_1B_3||^2 - mp^2 = 0 \quad ||B_1B||^2 - m^2 = 0 \quad ||B_3B||^2 - m^2 = 0 \tag{9}$$

From this point every further calculation have been done using Maple for the obvious reason that doing them by hand will be rather tedious although the calculations are not really complicated. The above equations can be written as :

$$E_1\cos(p_{12}) + E_2\sin(p_{12}) + E_3 = 0 \tag{10}$$

$$F_1\cos(p_{34}) + F_2\sin(p_{34}) + F_3 = 0 \tag{11}$$

$$K_{11}\sin(p_{34})\sin(p_{12}) + (K_{21}\cos(p_{34}) + K_{22})\cos(p_{12}) + K_{32}\cos(p_{34}) + K_{33} = 0 \tag{12}$$

where the E_i, F_j coefficients does not depend upon the angles but only upon the three coordinates of B. In fact they are polynomials in X, Y, Z and E_{ij}, F_{ij} will denote some coefficient of the corresponding polynomial, which definition can be found in [14]. As for the coefficients K_{ij} they are fully defined by the geometry of the mechanism. Equations 10,12 are linear in term of $\sin(p_{12}), \cos(p_{12})$. We solve this linear system and write the equation $\cos(p_{12})^2+\sin(p_{12})^2 = 1$ which yields:

$$(N_1 - N_2)\cos(p_{34})^2 + N_3\sin(p_{34})\cos(p_{34}) + N_4\sin(p_{34}) + N_5\cos(p_{34}) + N_6 + N_2 = 0 \tag{13}$$

where the coefficients N_i are only function of the coefficients E_j, K_{lm}. These coefficients can be further expanded by writing:

$$N_i = \sum N_{ij}E_k$$

where the coefficients N_{ij} does not contain any term E_l. These coefficients are not presented here but can be found in [14]. Then $\sin(p_{34})$ is determined using equation 11. If we put this value in equation 13 and write $\sin(p_{34})^2 + \cos(p_{34})^2 = 1$ we get two equations:

$$I_1\cos(p_{34})^2 + I_2\cos(p_{34}) + I_3 = 0 \tag{14}$$

$$H_1\cos(p_{34})^2 + H_2\cos(p_{34}) + H_3 = 0 \tag{15}$$

where the coefficients of I_i, H_j are function only of the coordinates of B. The orders of these coefficients are 4, 4, 4, 2, 3, 4. The equations 14, 15 yield to :

$$\begin{vmatrix} |I_1 H_2| & |I_1 H_3| \\ |I_1 H_3| & |I_2 H_3| \end{vmatrix} = 0 \tag{16}$$

where

$$|I_i H_j| = I_i H_j - I_j H_i \tag{17}$$

Using this method we get a sixteenth order polynomial. Its higher degree term is:

$$F_{21}^4 (Y^2 + X^2 + Z^2)^8 (N_{13} - N_{21})^2 \tag{18}$$

Although the value of N_{13}, N_{21}, F_{21} have not been presented here it can be shown that if $(N_{13} - N_{21})$ is equal to zero the mobile plate is reduced to a line. As for F_{21} it cannot be equal to zero. Therefore the circularity of the RSSR spin-surface is 8 and the conjecture of Hunt is verified. Thus there is at most 16 assembly-modes for the TSSM.

2.3 Description of the symbolic computation of the algorithm

The previous calculations has been made with the use of the symbolic computation program Maple. Although this program is quite powerful a direct implementation of this algorithm is not possible due to the size of the various coefficients which appear during the computation. Thus we use a decomposition method.

The first step of the method is to compute the coefficients E, F of equations 10, 11. In a second step we compute the coefficients K of equation 12. From now on we use only the abbreviated form of these equations. This form enables to calculate in the third step the coefficients N of equation 13. In a fourth step we use the abbreviated form of equations 10, 11 and 13. From equation 11 we calculate the value of $\sin(p_{34})$ and write the equation $\sin(p_{34})^2 + \cos(p_{34})^2 = 1$ which yield to equation 15. We put then the value of $\sin(p_{34})$ in equation 13 to get equation 14. Then we consider the complete form of the coefficient E, F, resulting from the first step and write these coefficients as polynomial in X, Y, Z. This yield to an abbreviated form of these coefficients. Then coefficients N are expressed as polynomial in term of the coefficients E. From the abbreviated form of the coefficients E, F, N we are able to compute the closed-form of the coefficients I, H which yield to the coefficients $|I_i H_j|$. At this step we are able to determine which term of equation 16 will yield to the higher term which is $|I_1 H_3|^2$. We compute then the higher coefficient which yield to equation 18.

2.4 Determination of the polynomial

The principle of the determination of the polynomial is similar to the one used for the determination of the circularity of the RSSR spin surface.

By introducing the unknown angle p_{56} we are able to express the norm of the vectors $\mathbf{B_1 B_3}$, $\mathbf{B_1 B_5}$, $\mathbf{B_3 B_5}$ i.e. the distances between the articulation points of the mobile whose values are mp, mp et m as a function of the angles p_{12}, p_{34}, p_{56}. This yields to the following three equations :

$$||B_1 B_3||^2 - mp^2 = 0 \quad ||B_1 B_5||^2 - mp^2 = 0 \quad ||B_3 B_5||^2 - m^2 = 0 \tag{19}$$

It is possible to show [14] that these equations may be written as :

$$K_{11}\sin(p_{34})\sin(p_{12}) + (K_{21}\cos(p_{34}) + K_{22})\cos(p_{12}) + K_{32}\cos(p_{34}) + K_{33} = 0 \quad (20)$$
$$L_{11}\sin(p_{56})\sin(p_{12}) + (L_{21}\cos(p_{56}) + L_{22})\cos(p_{12}) + L_{32}\cos(p_{56}) + L_{33} = 0 \quad (21)$$
$$M_{11}\sin(p_{34})\sin(p_{56}) + (M_{21}\cos(p_{34}) + M_{22})\cos(p_{56}) + M_{32}\cos(p_{34}) + M_{33} = 0 \quad (22)$$

where the coefficients K, L, M does not depend upon the angles p_{12}, p_{34}, p_{56}. We may notice that for a given set p_{12}, p_{34}, p_{56}, solution of these equations, the set $-p_{12}, -p_{34}, -p_{56}$ is also a solution. This mean simply that for a given position of the mobile plate the symmetrical position with respect to the base has the same links lengths.

Noticing that equations 20, 21 are linear in term of $\sin(p_{12}), \cos(p_{12})$ we solve this linear system and write the equation $\cos(p_{12})^2 + \sin(p_{12})^2 = 1$ which has the following form:

$$(N_1 - N_2)\cos(p_{56})^2 + N_5\sin(p_{56}) + (N_3\sin(p_{56}) + N_4)\cos(p_{56}) + N_2 + N_6 = 0 \quad (23)$$

From equation 22 we get the value of $\sin(p_{56})$:

$$\sin(p_{56}) = -\frac{(M_{21}\cos(p_{34}) + M_{22})\cos(p_{56}) + M_{32}\cos(p_{34}) + M_{33}}{M_{11}\sin(p_{34})} \quad (24)$$

From now on the process is similar to the one proposed by Nanua and Waldron. Charentus and Renaud [7][8] (in the case where the mobile plate is equilateral) have noticed that the coefficients N_3, N_5 can be written as :

$$N_3 = N_3'\sin(p_{34}) \qquad\qquad N_5 = N_5'\sin(p_{34})$$

The term $\sin(p_{34})$ being present in the denominator of $\sin(p_{56})$ we get a simplification when we use the value of $\sin(p_{56})$ in equation 23.

Thus equation 23 is written as :

$$I_1\cos(p_{56})^2 + I_2\cos(p_{56}) + I_3 = 0 \quad (25)$$

Using the equation $\sin(p_{56})^2 + \cos(p_{56})^2 = 1$ we get from equation 24:

$$H_1\cos(p_{56})^2 + H_2\cos(p_{56}) + H_3 = 0 \quad (26)$$

The I_i, H_j coefficients are second order polynomial in $\cos(p_{34})$ only. We express then a similar condition as in 16 and get a eighth order polynomial in $\cos(p_{34})$. If we define $x = tan(\frac{p_{34}}{2})$ we have $\cos(p_{34}) = \frac{1-x^2}{1+x^2}$, and we get from equation 16 a sixteenth order polynomial in x. These terms are fourth order polynomials in $\cos(p_{34})$. Therefore equation 16 In fact the odd power of x in this polynomial have zero as coefficient. Thus we have to solve only a eighth order polynomial (this means that if p_{34} is solution of the polynomial $-p_{34}$ is also a solution, as known from the beginning). From the determination of p_{34} it is easy to determine the value of p_{12}, p_{56} by following the process described by Nanua. Similar methods have been designed to compute a polynomial formulation of various parallel manipulators (see [14], [15]). The symbolic computation program which calculate this polynomial uses the same decomposition principle as the program designed to calculate the circularity of the RSSR spin-surface. This program generate automaticaly a C-program to compute the value of the coefficients of the polynomial.

2.5 Example

We present here an example of a TSSM with 16 assembly-modes (i.e. the polynomial has 16 real roots). The nominal position of the mobile plate is: $x_0 = y_0 = 0, z_0 = 20, \psi = -10, \theta = -5, \phi = 10$ where ψ, θ, ϕ are the Euler's angles in degree. The equivalent configurations for which the mobile plate is over the base are given in Table 1. We show in figure 5 the eight positions of the mobile plate for which the mobile plate is over the base (the 8 others configurations are the symmetrics with respect to the base of the drawn configurations).

3 Singular configurations

Singular configurations of a TSSM are obtained when the rank of its inverse jacobian matrix J^{-1} is less than 6. Although we have an analytic formulation of this matrix it is quite impossible to determine the roots of its determinant. In order to solve this problem we have first to notice that the lines of this matrix correspond to the Plücker coordinates of the lines associated to each link. Thus the degeneracy of the matrix J^{-1} corresponds to a linear dependence of the Plücker vectors or, in other term, to a linear dependence of lines in space. Geometric conditions on the lines which yield to such a dependence are well known from Grassmann geometry (see [15], [3]): for example 4 Plücker vectors are linearly dependent if the associated line have a common point (therefore the variety spanned by these four lines has a rank of 3). Therefore for a set of n lines ($n \leq 6$) there is geometric constraints on the lines such that the variety spanned by these lines is of rank $n-1$. For the TSSM we have only to consider the set of 4, 5 and 6 lines which can degenerate to variety of rank 3 (*plane*), 4 (*congruence* which can be *degenerate* or *hyperbolic* according to the geometric conditions which are satisified) and 5 (*complex* which can be *special* or *general*).

These geometric constraints are mainly lines intersection conditions and coplanarity of lines.

From the position and orientation of the mobile plate we are able to calculate the position of one point of the link's lines (the articulation point on the mobile). Another point of these lines being known (the articulation point on the base) we are thus able to calculate the Plücker vector of each link's lines. The above geometric conditions are easily written if one knows the Plücker vectors of the lines.

We have developed a Maple package with which we can calculate the Plücker vector of a line from the coordinates of two points belonging to this line, find a condition on the Plücker coordinates so that two lines intersect or are coplanar. With this package we have been able to find all the conditions on the position and orientation of the mobile plate to get a singular configuration. For more details see [15] or [3].

The table 2 summarizes the results deduced from the Maple program. (in this table xa_i, ya_j denote the position of the articulation point on the base, x_0, y_0, z_0 the position of the center of the mobile plate and ψ, θ, ϕ the Euler's angle and A, B, C, D, F some function).

4 Conclusion

The direct kinematics problem is one of the most challenging problem of parallel manipulators. In a first part we show that there can be at most sixteen different configurations of the manipulator for a given set of links lengths. Then we have demonstrated that in the general case it will not be possible to find an analytical solution to this problem because it is equivalent to solve a polynomial which order is 16. The interest of the polynomial formulation of the direct kinematics problem is that it gives all the possible configurations of the manipulator. As for the numerical efficiency the resolution using the polynomial is very slow compared to others known methods based on an estimate of the solution [16] (we get at least a factor 10 on a Sun workstation). Thus this method can be useful only during the initialization process, where no estimate of the position of the mobile plate is known. Symbolic computation has been a very useful tool to determine the polynomial because the calculation are rather tedious. Our experience shows however that the programmer has to split its calculation to avoid memory overflow.

Symbolic computation has been also used to determine the singular configurations of this kind of closed-loop mechanism. There is evidently a lack of geometric packages in the existent tools. Thus it has been necessary to design a package to deal with Plücker vector and some basic geometrical properties such as intersection and coplanarity of lines.

Table 1: Position and orientation of the 16 assembly-modes of the TSSM

x_0	y_0	z_0	ψ	θ	ϕ
0.1099	-6.8071	15.1572	178.79	104.2473	-179.39757
0.0	0.0	20.0	170.000	4.9999	-170.0
2.8029	-4.6660	12.7407	55.3895	89.1782	136.1996
1.3617	4.9038	17.3824	-106.3317	149.9318	58.9676
0.1606	5.3765	17.1868	-170.3808	164.0139	7.9545
-0.3525	-3.8663	11.9183	-12.5596	45.1107	-168.3013
-1.4134	4.8262	17.4299	102.6405	147.3844	-61.9768
-2.3355	-4.4679	12.5479	-50.8490	79.0396	-137.3533

plane	$\tan\psi = (ya_3 - ya_4)/(xa_3 + xa_4)$ $x_0 = A(\psi,\theta,\phi), y_0 = B(\psi,\theta,\phi), z_0 = C(\psi,\theta,\phi)$
degenerate congruence	$x_0 = A(\psi,\theta,\phi), y_0 = B(\psi,\theta,\phi), z_0 = C(\psi,\theta,\phi)$
hyperbolic congruence (first case)	$x_0 = A(z_0,\psi,\theta,\phi), y_0 = B(z_0,\psi,\theta,\phi)$
hyperbolic congruence (second case)	$y_0 = A(x_0,z_0,\psi,\theta,\phi) \quad , F(x_0,z_0,\psi,\theta,\phi) = 0$
general complex (first case)	$\theta = \phi = 0 \quad \psi = \pm\frac{\pi}{2}$
general complex (second case)	$\theta = \pm\frac{\pi}{2}$ or $\psi = \phi$ $A(x_0,y_0,\psi,\theta,\phi)z_0^3 + B(x_0,y_0,\psi,\theta,\phi)z_0^2 +$ $C(x_0,y_0,\psi,\theta,\phi)z_0 + D(x_0,y_0,\psi,\theta,\phi) = 0$
special complex	$x_0 = A(z_0,\psi,\theta,\phi), y_0 = B(z_0,\psi,\theta,\phi)$

Table 2: Relations between the position and orientation parameters to get a singular configuration

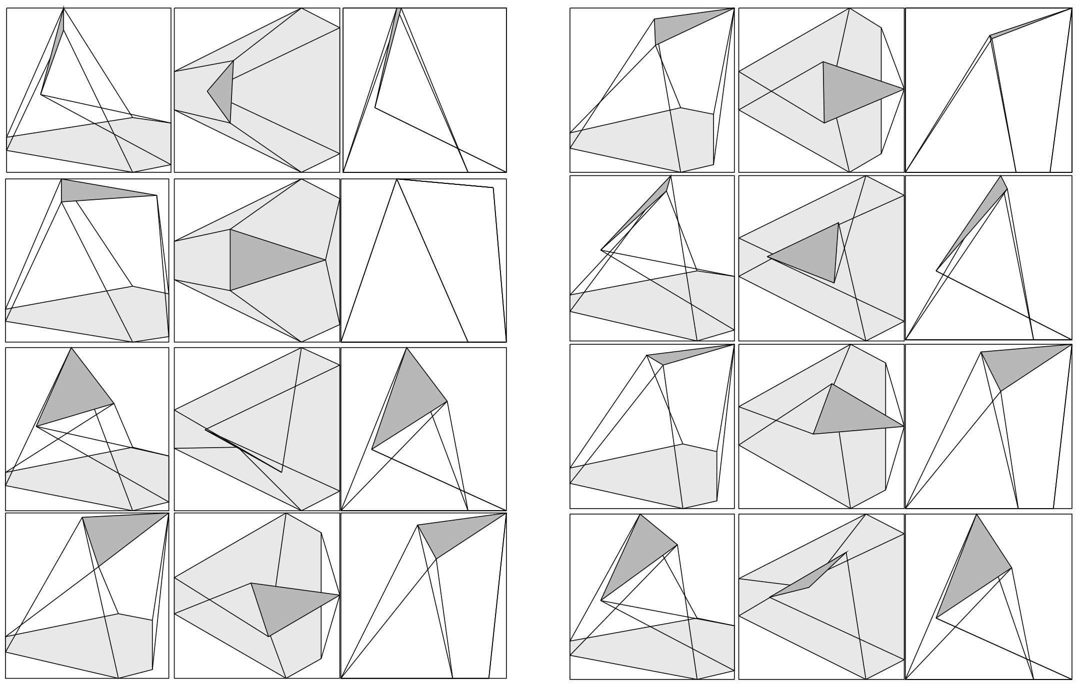

Figure 5: 8 over-the-base assembly-modes of the TSSM (perspective, top and side view)

References

[1] Merlet J-P. 1987 (March).Parallel Manipulator, Part 1: Theory, Design, Kinematics and Control. INRIA Research Report n°646.

[2] Gosselin. C. 1988. Kinematic analysis, optimization and programming of parallel robotic manipulators. Ph.D. thesis, McGill University, Montréal, Québec, Canada.

[3] Merlet J-P. 1989 (February). Parallel Manipulator, Part 2: Singular configurations and Grassmann geometry. INRIA Research Report, n°791.

[4] Nanua P., Waldron K.J. 1989 (May 14-19). Direct kinematic Solution of a Stewart Platform. *IEEE Int. Conf. on Robotics and Automation*, Scotsdale, Arizona, pp. 431-437.

[5] Fichter E.F. A Stewart platform based manipulator: general theory and practical construction.*The Int. J. of Robotics Research*5(2):157-181.

[6] Koliskor, 1986 (April 22-25). The l-coordinate approach to the industrial robot design. *V IFAC/IFIP/IMACS/IFORS Symposium.*, Suzdal, USSR, pp. 108-115.

[7] Charentus S., Renaud M. 1989 (July). Calcul du modèle géométrique direct de la plateforme de Stewart. Report n° 89260 LAAS, Toulouse, France.

[8] Charentus S., Renaud M. 1989 (June 19-21). Modelling and control of a modular, redundant robot manipulator. *1st Int. Symp. on Experimental Robotics*, Montréal, Canada.

[9] Hunt K.H. Structural kinematics of in Parallel Actuated Robot Arms . *Trans. of the ASME, J. of Mechanisms,Transmissions, and Automation in design* (105): 705-712.

[10] Reboulet C., Robert A. Hybrid control of a manipulator with an active compliant wrist. *Proc 3th ISRR*, Gouvieux, France, 7-11 Oct.1985, pp.76-80.

[11] Zamanov V.B, Sotirov Z.M. Structures and kinematics of parallel topology manipulating systems. *Proc.Int. Symp. on Design and Synthesis*, Tokyo, July 11-13 1984, pp.453-458.

[12] Mohamed M.G., Duffy J. A Direct Determination of the Instantaneous Kinematics of Fully Parallel Robot Manipulators. *Trans. of the ASME, J. of Mechanisms, Transmissions, and Automation in design*, Vol 107:226-229.

[13] Hunt K.H. 1978. *Kinematic geometry of mechanisms.* Oxford: Clarendon Press.

[14] Merlet J-P. 1989 (December). Manipulateurs parallèles, 4eme partie: Mode d'assemblage et cinématique directe sous forme polynomiale. INRIA Research Report, n°1135.

[15] Merlet J-P. 1989 (October). Parallel Manipulator, Part 2: Singular configurations of parallel manipulators and Grassmann geometry. The Int. J. of Robotics Research, Vol 8, n°5,pp. 45-56

[16] Merlet J-P. 1989 (March). Manipulateurs parallèles. 3eme partie : Applications. INRIA Research Report n°1003.