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Wrench-Feasible Workspace of Parallel Cable-Driven Mechanisms

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***Abstract*— The wrench-feasible workspace (WFW) of a parallel cable-driven mechanism is the set of poses of its mobile platform for which the cables can balance any wrench in a specified set of wrenches, such that the tension in each cable remains within a prescribed range. The WFW is fundamental since it takes into account both the requirement of non-negative cable tension and the requirement of a maximum admissible cable tension. This paper addresses the problem of the determination of the WFW of n-degree-of-freedom parallel mechanisms driven by more than n cables. Interval analysis based methods that allow to determine if a given n-dimensional box is fully included in the WFW are presented. Moreover, these methods are also able to approximate the WFW up to a chosen accuracy. The resulting approximation consists of a set of n-dimensional boxes such that each box of the set is fully included in the WFW.**

I. Introduction

Among the various factors that may limit the workspace of parallel cable-driven mechanisms, the inability of the cables to push on the mobile platform is an important and challenging one. This problem has been the subject of several studies and, recently, efficient methods that allow to determine the workspace of parallel cable-driven mechanisms have been proposed. These methods determine the workspace by delineating its boundary. In the case of planar mechanisms, this type of geometric method allows to determine the workspace defined as the set of poses of the mobile platform for which a given wrench [1], [2] or any wrench [3] can be generated with tension forces in the cables. They can also be extended to the determination of such workspaces for six-degree-of-freedom (DOF) mechanisms driven by $m > 6$ cables [4], [5]. However, these geometric determinations are limited to three-dimensional (3D) workspaces such as the workspace of a three-DOF planar mechanism or the constant-orientation workspace of a six-DOF mechanism. Consequently, in the case of six-DOF mechanisms, only a partial picture of the workspace is obtained since three of the six pose variables must be fixed for the workspace to be computed. Another limitation of these methods lies in the difficulty to take into account the requirement of a minimum and a maximum tension in each cable [6], [7] (bounds on the allowed tensions), especially for six-DOF mechanisms driven by more than six cables. Moreover, as a tool to design parallel cable-driven mechanisms, geometric methods seems to be limited to "trial-and-error" approaches.

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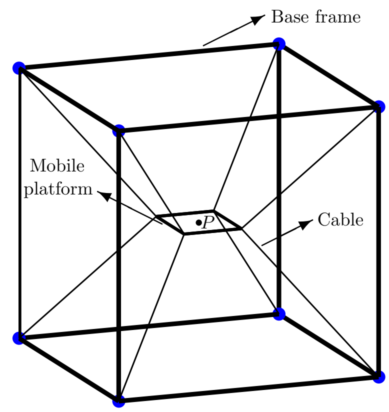

Fig. 1. A parallel cable-driven mechanism.

The determination of the workspace of some parallel mechanisms with rigid links, such as the Gough-Stewart platform, is a well-known problem to which geometric methods have been applied [8]. Again, these methods are limited to 3D workspaces since they aim at obtaining a graphical representation of the boundary of the workspace. A numerical approach based on interval analysis overcomes this limitation allowing the determination of 6D workspaces [9]. Interval based methods have another interesting feature: the ability to take into account the uncertainties, for instance the manufacturing tolerances, inherent to any real system. Moreover, the development of interval analysis methods is a step toward the possibility to work within the optimal design framework introduced in [10], [11]. For these reasons, this paper focus on the application of interval analysis methods to the determination of the wrench-feasible workspace (WFW) of parallel cable-driven mechanisms. Finally, note that a drawback of interval analysis based approaches is the high computation time that may be involved. Hence, care must be taken in the detailed choice of the methods and in their implementation so as to ensure a tractable computation time.

II. Definitions

A. Wrench-Feasible Workspace

The relationship between the tensions in the cables and the wrench $\mathbf{f}$ applied by the cables on a reference point P of the platform is given by [12], [13]

$$\mathbf{W}\boldsymbol{\tau} = \mathbf{f}, \quad (1)$$

where $\boldsymbol{\tau} = [\tau_1, \ldots, \tau_m]^T$ is the vector of cable tensions and $\mathbf{W}$ the wrench matrix. If n and m denote, respectively, the

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number of DOF of the mobile platform and the number of cables, $\mathbf{W}$ is a $n \times m$ matrix that depends on the pose $\mathbf{X}$ of the mobile platform where $\mathbf{X}$ is a column vector of dimension n. For instance, in the case of a 6-DOF mechanism, the first three components of $\mathbf{X}$ are the coordinates x, y and z of the reference point P whereas its last three components can be three Euler angles that define the orientation of the mobile platform with respect to the base frame. Note that $\mathbf{W} = -\mathbf{J}^{-T}$ where $\mathbf{J}^{-1}$ is the so-called inverse jacobian matrix [8] whose line vectors are Plücker coordinates of the cable lines. Then, let the required set of wrenches $\mathcal{W}_{req}$ be the set of wrenches that the cables must apply at the reference point P of the mobile platform and let $\mathcal{T}$ be the m-dimensional box wherein the cable tension vectors must lie. Usually, the box $\mathcal{T}$ of allowed cable tensions is to be defined as

$$\mathcal{T} = \{\boldsymbol{\tau} \mid \tau_i \in [\tau_{min}, \tau_{max}] \ \forall\, 1 \le i \le m\} \tag{2}$$

where τ_{min} and τ_{max} are two positive scalars such that $\tau_{min} < \tau_{max}$. A maximum tension τ_{max} is necessary in order to take into account the maximum torque of each actuator or the maximum tension that a cable can withstand. The minimum tension τ_{min} allows to ensure that none of the cables will be slack, a situation that may cause control problems. In (2), for the sake of clarity, the limiting tensions τ_{min} and τ_{max} have been assumed to be the same for the m cables of the mechanism. Without affecting the method introduced in this paper, different τ_{min} and τ_{max} can be chosen for each cable. Now, in order to deal with the influence on the workspace of the unilateral nature of the forces applied by the cables, following [6], the WFW is defined as follows.

Definition *The WFW is the set of poses of the mobile platform for which, for any wrench* $\mathbf{f}$ *in* $\mathcal{W}_{req}$*, there exists a vector of cable tensions* $\boldsymbol{\tau}$ *in* $\mathcal{T}$ *such that* $\mathbf{W}\boldsymbol{\tau} = \mathbf{f}$.
The system of linear equations $\mathbf{W}\boldsymbol{\tau} = \mathbf{f}$ is said to be *feasible* if it admits a solution $\boldsymbol{\tau}$ in $\mathcal{T}$.

B. Interval Evaluation, Interval Vectors and Interval Matrices

An interval $x^{\mathcal{I}}$ is a set of real numbers defined by

$$x^{\mathcal{I}} = [\underline{x}, \overline{x}] = \{x \mid \underline{x} \le x \le \overline{x}\} \tag{3}$$

where $\underline{x} \le \overline{x}$. A fundamental feature of interval analysis is the interval evaluation of a function. Let us consider a real function $f(x)$. The classic rules of addition, multiplication, etc. of real numbers can be redefined to allow addition, multiplication, etc. of intervals [14]. Based on these basic operations and on the interval evaluation of basic algebraic and transcendental functions such as x^2, sin and cos, almost any real function $f(x)$ can be evaluated for an interval $x^{\mathcal{I}}$ yielding an interval $f^{\mathcal{I}} = f^{\mathcal{I}}(x^{\mathcal{I}})$ which encloses the image of $x^{\mathcal{I}}$ under f (denoted $f(x^{\mathcal{I}})$), i.e.,

$$f(x^{\mathcal{I}}) = \left\{f(x) \mid x \in x^{\mathcal{I}}\right\} \subset f^{\mathcal{I}}. \tag{4}$$

The reverse inclusion does not hold in general and $f^{\mathcal{I}}$ overestimates $f(x^{\mathcal{I}})$ introducing pessimism in the evaluation. Without going into details, there exists several means to improve the interval evaluation of a function such as the use of the derivative of the function or of its Taylor series expansion [14], [15]. Likewise, the interval evaluation of a function $f(\mathbf{x})$ of several variables $\mathbf{x} = (x_1, \ldots, x_n)$ yields an interval $f^{\mathcal{I}}$ that contains the image $f(\mathbf{x}^{\mathcal{I}})$ of $\mathbf{x}^{\mathcal{I}}$ under f.

An m-dimensional interval vector $\mathbf{x}^{\mathcal{I}}$ is a vector whose components $x_i^{\mathcal{I}}$ are intervals. $\mathbf{x}^{\mathcal{I}}$ is essentially a set of vectors which has the shape of a box in $\mathbb{R}^m$. Likewise, a $n \times m$ interval matrix $\mathbf{A}^{\mathcal{I}}$ is a matrix whose components are intervals. The multiplication of an interval matrix $\mathbf{A}^{\mathcal{I}}$ by an interval vector $\mathbf{x}^{\mathcal{I}}$ yields an interval vector $\mathbf{A}^{\mathcal{I}}\mathbf{x}^{\mathcal{I}}$ such that [15]

$$\forall\, \mathbf{A} \in \mathbf{A}^{\mathcal{I}}, \forall\, \mathbf{x} \in \mathbf{x}^{\mathcal{I}},\ \mathbf{A}\mathbf{x} \in \mathbf{A}^{\mathcal{I}}\mathbf{x}^{\mathcal{I}} \tag{5}$$

In the sense of (5), $\mathbf{A}^{\mathcal{I}}\mathbf{x}^{\mathcal{I}}$ is a box enclosure of the set

$$\left\{\mathbf{A}\mathbf{x} \mid \mathbf{A} \in \mathbf{A}^{\mathcal{I}} \text{ and } \mathbf{x} \in \mathbf{x}^{\mathcal{I}}\right\} \tag{6}$$

which is itself generally not a box. Hence, there exists vectors $\mathbf{v} \in \mathbf{A}^{\mathcal{I}}\mathbf{x}^{\mathcal{I}}$ such that

$$\forall\, \mathbf{A} \in \mathbf{A}^{\mathcal{I}}, \forall\, \mathbf{x} \in \mathbf{x}^{\mathcal{I}},\ \mathbf{A}\mathbf{x} \neq \mathbf{v}. \tag{7}$$

III. Interval Analysis

A. Interval Wrench Matrix

Consider a box $\mathcal{B}$ of poses $\mathbf{X}$ of the mobile platform. $\mathcal{B}$ can be identified to an n-dimensional interval vector. Now, the wrench matrix $\mathbf{W}$ is pose dependent since each of its component w_{ij} is a function of the pose $\mathbf{X}$, $w_{ij} = w_{ij}(\mathbf{X})$. By means of an interval evaluation, each w_{ij} can be evaluated for the box $\mathcal{B}$ yielding an interval $w_{ij}^{\mathcal{I}}$ such that for each pose $\mathbf{X}$ in $\mathcal{B}$, $w_{ij}(\mathbf{X}) \in w_{ij}^{\mathcal{I}}$ ($w_{ij}(\mathcal{B}) \subset w_{ij}^{\mathcal{I}}$). Then, the $n \times m$ interval matrix $\mathbf{W}^{\mathcal{I}}$ whose components are the intervals $w_{ij}^{\mathcal{I}}$ is called the interval wrench matrix and it has the following property

$$\text{for all } \mathbf{X} \in \mathcal{B},\ \mathbf{W} \in \mathbf{W}^{\mathcal{I}}. \tag{8}$$

Thus, the interval wrench matrix $\mathbf{W}^{\mathcal{I}}$ overestimates the set of wrench matrices

$$\{\mathbf{W}(\mathbf{X}) \mid \mathbf{X} \in \mathcal{B}\} \tag{9}$$

by enclosing it within a set that can be thought of as a box in $\mathbb{R}^{nm}$ whereas (9) has an unknown complex shape since each w_{ij} is a nonlinear function of the pose $\mathbf{X}$. Consequently, it is worth noting that there exists matrices $\mathbf{W}_{\mathcal{O}} \in \mathbf{W}^{\mathcal{I}}$ which are not wrench matrices ($\forall \mathbf{X} \in \mathcal{B}$, $\mathbf{W}_{\mathcal{O}} \neq \mathbf{W}(\mathbf{X})$). This is known as the wrapping effect [15]. In the sequel, the notation $\mathbf{W}^{\mathcal{I}} = \mathbf{W}^{\mathcal{I}}(\mathcal{B})$ means that $\mathbf{W}^{\mathcal{I}}$ is obtained by interval evaluating $\mathbf{W}$ for the box $\mathcal{B}$.

For instance, let us consider a simple point-mass two-DOF mechanism driven by three cables as shown in Fig. 2. The wrench matrix of this mechanism is the 2×3 matrix $\mathbf{W} = [\mathbf{d}_1\ \mathbf{d}_2\ \mathbf{d}_3]$ where $\mathbf{d}_i = (1/\rho_i)\overrightarrow{PA_i}$. For the box of poses $\mathcal{B} = [[1.5, 2.5], [0.4, 1.8]]^T$, each component of each $\mathbf{d}_i$ is interval evaluated yielding the following $\mathbf{W}^{\mathcal{I}}$

$$\begin{bmatrix} [-2.986, -0.239] & [0.344, 2.334] & [-0.715, 0.715] \\ [-2.886, -0.0239] & [-1.734, 0.134] & [0.324, 3.0] \end{bmatrix}. \tag{10}$$

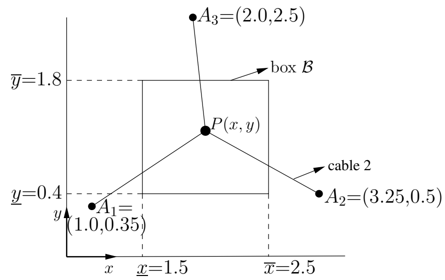

Fig. 2. A point-mass cable-driven mechanism and its box workspace.

It can be verified that for *any* position $\mathbf{X}$ of P lying in $\mathcal{B}$, the associated wrench matrix $\mathbf{W} = \mathbf{W}(\mathbf{X})$ lies within $\mathbf{W}^{\mathcal{I}}$. Finally, note that, although the matrix

$$\begin{bmatrix} -1 & 1.5 & 0 \\ -0.5 & 0 & 1.6 \end{bmatrix} \quad (11)$$

belongs to $\mathbf{W}^{\mathcal{I}}$, it does not correspond to any wrench matrix obtained for $\mathbf{X}$ in $\mathcal{B}$.

B. Interval Linear System

Since, in this paper, interval analysis is our main tool, henceforth, the required set of wrenches $\mathcal{W}_{req}$ is a box, i.e., an interval vector denoted by $\mathbf{f}^{\mathcal{I}}$.

The so-called system of interval linear equations

$$\mathbf{W}^{\mathcal{I}}\boldsymbol{\tau} = \mathbf{f}^{\mathcal{I}} \quad (12)$$

is said to have a solution in $\mathcal{T}$, i.e., is feasible, if and only if

$$\forall\, \mathbf{W} \in \mathbf{W}^{\mathcal{I}}, \forall\, \mathbf{f} \in \mathbf{f}^{\mathcal{I}}, \exists\, \boldsymbol{\tau} \in \mathcal{T} \text{ such that } \mathbf{W}\boldsymbol{\tau} = \mathbf{f}. \quad (13)$$

With $\mathbf{W}^{\mathcal{I}} = \mathbf{W}^{\mathcal{I}}(\mathcal{B})$, the feasibility of (12), i.e. (13), is a sufficient condition for a box $\mathcal{B}$ to be fully included in the WFW. Indeed, (8) and (13) imply that

$$\forall\, \mathbf{X} \in \mathcal{B}, \forall\, \mathbf{f} \in \mathbf{f}^{\mathcal{I}}, \exists\, \boldsymbol{\tau} \in \mathcal{T} \text{ such that } \mathbf{W}\boldsymbol{\tau} = \mathbf{f}. \quad (14)$$

In words, for any pose $\mathbf{X}$ in $\mathcal{B}$, $\mathbf{W} = \mathbf{W}(\mathbf{X})$ belongs to $\mathbf{W}^{\mathcal{I}}$ by (8) and thus, according to (13), for any wrench $\mathbf{f} \in \mathbf{f}^{\mathcal{I}}$ to be generated at the mobile platform, there exists a vector of cable tensions $\boldsymbol{\tau}$ in the allowed set $\mathcal{T}$ that can generate $\mathbf{f}$ ($\mathbf{W}\boldsymbol{\tau} = \mathbf{f}$). Hence, $\mathbf{X}$ belongs to the WFW. It shall be noted that, due to the overestimation of (9) by $\mathbf{W}^{\mathcal{I}}$, (13) *is a sufficient but not a necessary condition for a box $\mathcal{B}$ to be included in the WFW.*

C. The Square Case

When a parallel mechanism is driven by a number of cables equal to its number of DOF ($m = n$), the wrench matrix $\mathbf{W}$ is square and the system of interval linear equations (12) is a set of square systems of linear equations. In this case, well-known methods of interval analysis can be applied in order to find a box $\mathcal{T}_e$ enclosing the so-called solution set of (12) which is defined by

$$\left\{\boldsymbol{\tau} \mid \exists\, \mathbf{W} \in \mathbf{W}^{\mathcal{I}},\, \exists\, \mathbf{f} \in \mathbf{f}^{\mathcal{I}} \text{ such that } \mathbf{W}\boldsymbol{\tau} = \mathbf{f}\right\}. \quad (15)$$

Then, if $\mathcal{T}_e \subset \mathcal{T}$, assuming that none of the matrices $\mathbf{W}$ in $\mathbf{W}^{\mathcal{I}}$ is singular, (13) is true and $\mathcal{W}$ is included in the WFW. More details on square systems of interval linear equations — applied to the problem of the accuracy of parallel robots — can be found in [16]. Now, when the mobile platform of a parallel cable-driven mechanism is to be fully constrained, the number of cables must be greater than the number of DOF ($m > n$). In this case, the wrench matrix $\mathbf{W}$ is a rectangular matrix that has more columns than rows and (12) is a (underconstrained) system of interval linear equations whose solution set (15) is unbounded [17]. Hence, this solution set can obviously not be enclosed in a box and testing whether (13) is true requires another tool.

IV. Tools

The goal of this section is to point out a theorem that allows to test whether (13) holds. This theorem is based on a theorem due to Rohn (Theorem 1.16 of [18]). Beforehand, a set of ”vertex” matrices of an interval matrix must be introduced. Finally, a method to test whether a box workspace $\mathcal{W}$ is fully outside the WFW is presented.

A. A Set of Vertex Matrices of an Interval Matrix

Let $\mathbf{A}^{\mathcal{I}}$ be a $n \times m$ interval matrix whose components are the intervals $A^{\mathcal{I}}_{ij} = [\underline{A^{\mathcal{I}}_{ij}}, \overline{A^{\mathcal{I}}_{ij}}]$. A set of 2^n “vertex” matrices of $\mathbf{A}^{\mathcal{I}}$, denoted $\{\mathbf{A}_\mathbf{y}\}$, is of particular interest to us. In order to define $\{\mathbf{A}_\mathbf{y}\}$, let Y_n be the set of n-dimensional vectors $\mathbf{y}$ whose components y_i, $1 \le i \le n$, are equal either to 1 or to -1. We associate with each of the 2^n $\mathbf{y}$ in Y_n the vertex matrix $\mathbf{A}_\mathbf{y}$ such that the components ${A_\mathbf{y}}_{ij}$ of its ith line are defined by ${A_\mathbf{y}}_{ij} = \overline{A^{\mathcal{I}}_{ij}}$ if $y_i = -1$ or ${A_\mathbf{y}}_{ij} = \underline{A^{\mathcal{I}}_{ij}}$ if $y_i = 1$. $\{\mathbf{A}_\mathbf{y}\}$ is the set of the 2^n vertex matrices $\mathbf{A}_\mathbf{y}$. Note that the ith component of $\mathbf{y}$ defines completely the ith line of $\mathbf{A}_\mathbf{y}$ and that the components of $\mathbf{A}_\mathbf{y}$ can be computed by means of

$${A_\mathbf{y}}_{ij} = \underline{A^{\mathcal{I}}_{ij}} + (\overline{A^{\mathcal{I}}_{ij}} - \underline{A^{\mathcal{I}}_{ij}})(1 - y_i)/2. \quad (16)$$

For instance, with $\mathbf{y} = [1, -1]^T \in Y_2$, the vertex matrix $\mathbf{W}_\mathbf{y}$ of the interval matrix $\mathbf{W}^{\mathcal{I}}$ shown in (10) is equal to

$$\begin{bmatrix} -2.986 & 0.344 & -0.715 \\ -0.0239 & 0.134 & 3.0 \end{bmatrix}. \quad (17)$$

Let us also define the set $\{\mathbf{b}_\mathbf{y}\}$ of “vertex” vectors of an interval vector $\mathbf{b}^{\mathcal{I}}$ as the set of the 2^n vectors $\mathbf{b}_\mathbf{y}$, $\mathbf{y}$ in Y_n, whose components ${b_\mathbf{y}}_i$ ($1 \le i \le n$) are given by ${b_\mathbf{y}}_i = \underline{b^{\mathcal{I}}_i}$ if $y_i = -1$ or ${b_\mathbf{y}}_i = \overline{b^{\mathcal{I}}_i}$ if $y_i = 1$. The interval $[\underline{b^{\mathcal{I}}_i}, \overline{b^{\mathcal{I}}_i}]$ is the ith component of $\mathbf{b}^{\mathcal{I}}$. The components of $\mathbf{b}_\mathbf{y}$ can be computed by the following equation

$${b_\mathbf{y}}_i = \underline{b^{\mathcal{I}}_i} + (\overline{b^{\mathcal{I}}_i} - \underline{b^{\mathcal{I}}_i})(1 + y_i)/2. \quad (18)$$

B. A Useful Theorem

Note that the system of interval linear equations (12) is a shorthand notation for the infinite set of systems of linear equations

$$\left\{\mathbf{W}\boldsymbol{\tau} = \mathbf{f} \mid \mathbf{W} \in \mathbf{W}^{\mathcal{I}} \text{ and } \mathbf{f} \in \mathbf{f}^{\mathcal{I}}\right\}. \quad (19)$$

The following theorem states that the infinitely many systems (19) are all feasible, i.e., (13) is true, if and only if finitely many of them are feasible.

Theorem 1 (Rohn) *The system of interval linear equations $\mathbf{W}^{\mathcal{I}}\boldsymbol{\tau} = \mathbf{f}^{\mathcal{I}}$ is feasible if and only if the 2^n systems of linear equations $\mathbf{W}_{\mathbf{y}}\boldsymbol{\tau} = \mathbf{f}_{\mathbf{y}}$, $\mathbf{y} \in Y_n$, are feasible.*

The necessary condition is trivial whereas the reader interested in the proof of the sufficient condition is referred to [18]. Since (13) is a sufficient condition for a box $\mathcal{B}$ to be included in the WFW, this very useful theorem allows to conclude that $\mathcal{B}$ is fully inside the WFW whenever each of the 2^n systems $\mathbf{W}_{\mathbf{y}}\boldsymbol{\tau} = \mathbf{f}_{\mathbf{y}}$, $\mathbf{y} \in Y_n$, is feasible where $\mathbf{W}_{\mathbf{y}}$ and $\mathbf{f}_{\mathbf{y}}$ are the vertex matrix of $\mathbf{W}^{\mathcal{I}}$ and the vertex vector of $\mathbf{f}^{\mathcal{I}}$ defined by (16) and (18), respectively.

The feasibility of a system $\mathbf{W}_{\mathbf{y}}\boldsymbol{\tau} = \mathbf{f}_{\mathbf{y}}$ is a well-known problem in linear programming (LP) and it can be tested by means of the simplex method applied to the general LP problem [19] (chapter 8)

$$\text{minimize} \quad \mathbf{c}^T\boldsymbol{\tau} \quad \text{subject to} \quad \mathbf{W}_{\mathbf{y}}\boldsymbol{\tau} = \mathbf{f}_{\mathbf{y}}, \ \ \boldsymbol{\tau} \in \mathcal{T} \tag{20}$$

for which a trivial linear form $\mathbf{c} = \mathbf{0}$ can be considered since only the feasibility of the problem is to be determined.

In brief, by means of Theorem 1 and of the simplex method, a procedure denoted $\texttt{Feasibility}(\mathbf{W}^{\mathcal{I}}, \mathbf{f}^{\mathcal{I}}, \mathcal{T})$ can be written. This procedure returns 1 if Theorem 1 is true and 0 otherwise. It requires at most 2^n call to the simplex method. Note that the method introduced in [20] allows to guarantee the correctness of the solution determined by the simplex method whose computations may be affected by numerical round-off errors.

C. A Test to Discard a Box $\mathcal{B}$

As defined by (2), $\mathcal{T}$ is an interval vector. Thus, the interval wrench matrix $\mathbf{W}^{\mathcal{I}}$ can be multiplied by $\mathcal{T}$ yielding an interval vector $\mathbf{W}^{\mathcal{I}}\mathcal{T}$ such that (cf section II-B)

$$\forall\, \mathbf{W} \in \mathbf{W}^{\mathcal{I}}, \forall\, \boldsymbol{\tau} \in \mathcal{T}, \ \mathbf{W}\boldsymbol{\tau} \in \mathbf{W}^{\mathcal{I}}\mathcal{T}. \tag{21}$$

Then, with $\mathbf{W}^{\mathcal{I}} = \mathbf{W}^{\mathcal{I}}(\mathcal{B})$,

$$\mathbf{f}^{\mathcal{I}} \not\subset \mathbf{W}^{\mathcal{I}}\mathcal{T} \tag{22}$$

is a sufficient condition for $\mathcal{B}$ to be fully outside the WFW. Indeed, according to (22), there exists a wrench $\mathbf{f}_0 \in \mathbf{f}^{\mathcal{I}}$ such that $\mathbf{f}_0 \notin \mathbf{W}^{\mathcal{I}}\mathcal{T}$. For any pose $\mathbf{X}$ in $\mathcal{B}$, $\mathbf{W} = \mathbf{W}(\mathbf{X})$ lies within $\mathbf{W}^{\mathcal{I}}$ and, hence, (21) implies that for all $\boldsymbol{\tau} \in \mathcal{T}$, $\mathbf{W}\boldsymbol{\tau} \neq \mathbf{f}_0$. Then, for any pose $\mathbf{X}$ in $\mathcal{B}$, the wrench $\mathbf{f}_0$ of the required set of wrenches $\mathbf{f}^{\mathcal{I}}$ is infeasible proving that $\mathcal{B}$ is fully outside the WFW.

Thus, the computation of the interval vector $\mathbf{W}^{\mathcal{I}}\mathcal{T}$ and the comparison of $\mathbf{W}^{\mathcal{I}}\mathcal{T}$ and $\mathbf{f}^{\mathcal{I}}$ by means of (22) provide a procedure $\texttt{Out}()$ whose arguments are the interval wrench matrix $\mathbf{W}^{\mathcal{I}}$ together with $\mathbf{f}^{\mathcal{I}}$ and $\mathcal{T}$. This procedure returns 1 if $\mathcal{B}$ is completely outside the WFW according to the sufficient condition (22), otherwise it returns 0.

Again, since $\mathbf{W}^{\mathcal{I}}\mathcal{T}$ is a box that overestimates the set $\{\mathbf{W}\boldsymbol{\tau} \ \mid \ \mathbf{W} \in \mathbf{W}^{\mathcal{I}}, \ \boldsymbol{\tau} \in \mathcal{T}\}$, (22) is *sufficient but not necessary* for $\mathcal{B}$ to be fully outside the WFW.

V. Determination of the WFW

The generic structure of an algorithm based on interval analysis [8], [16] can now be applied, on one hand, to test whether a given box workspace $\mathcal{W}$ is fully included in the WFW and, on the other hand, to approximate as a set of n-dimensional boxes $\mathcal{B}_i$ the part of the WFW lying within a given box $\mathcal{B}$.

A. Test of a Box Workspace $\mathcal{W}$

In order to test whether a prescribed box workspace $\mathcal{W}$ is fully inside the WFW, a list $\mathcal{L}$ of n-dimensional boxes $\mathcal{B}_i$ is managed by the following algorithm. The number of boxes in the list $\mathcal{L}$ is denoted by p.

1) $i = 1$, $\mathcal{B}_1 = \mathcal{W}$, $\mathcal{L} = \{\mathcal{B}_1\}$, $p = 1$.
2) if $i > p$ then `RETURN 1`
3) $\mathbf{W}^{\mathcal{I}} = \texttt{Compute\_WrenchMatrix}(\mathcal{B}_i)$
4) if $\texttt{Out}(\mathbf{W}^{\mathcal{I}}, \mathbf{f}^{\mathcal{I}}, \mathcal{T}) = 1$ then `RETURN -1`
5) if $\texttt{Feasibility}(\mathbf{W}^{\mathcal{I}}, \mathbf{f}^{\mathcal{I}}, \mathcal{T}) = 1$ then $i = i + 1$, go to step 2
6) else, $\texttt{Feasibility}(\mathbf{W}^{\mathcal{I}}, \mathbf{f}^{\mathcal{I}}, \mathcal{T}) = 0$,
   a) if the widths of all the interval components of $\mathcal{B}_i$ are lower than ϵ then `RETURN 0`
   b) else, $\texttt{Bisect}(\mathcal{B}_i)$ and put the resulting two new boxes in $\mathcal{L}$, $p = p + 2$, $i = i + 1$, go to step 2

At step 1, the list $\mathcal{L}$ is initialized with the box workspace $\mathcal{W}$. At step 3, the interval wrench matrix $\mathbf{W}^{\mathcal{I}} = \mathbf{W}^{\mathcal{I}}(\mathcal{B}_i)$ is computed. When the corresponding system of interval linear equations (12) is feasible (step 5), all the poses in $\mathcal{B}_i$ belong to the WFW and the algorithm proceeds to the next box $\mathcal{B}_i$. On the contrary, when the procedure $\texttt{Out}()$ returns 1 (step 4), the box $\mathcal{B}_i \subset \mathcal{W}$ is fully outside the WFW and the algorithm returns -1 since $\mathcal{W}$ cannot be included in the WFW. At step 6, (12) is not feasible but, at the same time, (22) is false ($\mathbf{f}^{\mathcal{I}} \subset \mathbf{W}^{\mathcal{I}}\mathcal{T}$). Then, two situations can occur. At step 6 a, the box $\mathcal{B}_i$ is deemed too small to be bisected and the algorithm returns 0 since it cannot guarantee, on one hand, that all the poses in $\mathcal{B}_i$ — and, hence, that all the poses in $\mathcal{W}$ — lie within the WFW and, on the other hand, that all the poses in $\mathcal{B}_i$ — and, hence, that some poses in $\mathcal{W}$ — are outside the WFW. At step 6 b, $\mathcal{B}_i$ is bisected yielding two new boxes of smaller size which are put in the list $\mathcal{L}$. When the algorithm returns 1, the box workspace $\mathcal{W}$ is fully included in the WFW whereas a value of -1 means that $\mathcal{W}$ is not included in the WFW. When 0 is returned, either $\mathcal{W}$ is not fully included in the WFW or the threshold ϵ is too large for the algorithm to be able to conclude.

As the bisection process reduces the size of the boxes $\mathcal{B}_i$, the enclosure $\mathbf{W}^{\mathcal{I}}$ of the set of wrench matrices (9) becomes sharper. Now, if $\mathcal{B}_i$ is a box included in the WFW, the procedure $\texttt{Feasibility}()$ may not be able to conclude that $\mathcal{B}_i$ belongs to the WFW due to the overestimation of (9) inherent in the computation $\mathbf{W}^{\mathcal{I}} = \mathbf{W}^{\mathcal{I}}(\mathcal{B}_i)$. But, after the bisection of $\mathcal{B}_i$ which yields two smaller boxes $\mathcal{B}_i^1$ and $\mathcal{B}_i^2$, the overestimations involved in the computations $\mathbf{W}^{\mathcal{I}}(\mathcal{B}_i^1)$ and $\mathbf{W}^{\mathcal{I}}(\mathcal{B}_i^2)$ are somehow lesser and the algorithm might be able to conclude that $\mathcal{B}_i$ is included in the WFW by verifying

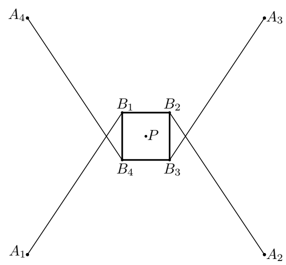

Fig. 3. A 3-DOF planar parallel mechanism driven by four cables.

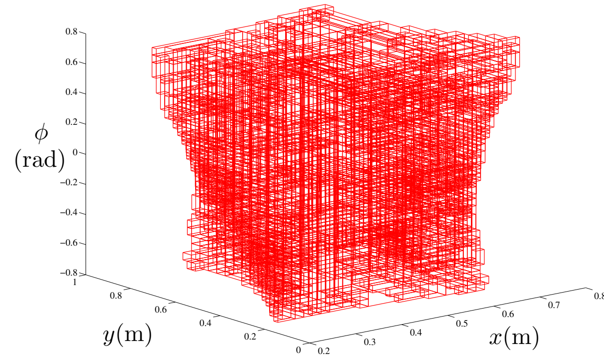

Fig. 4. The WFW as a set of boxes.

that both $\mathcal{B}_i^1$ and $\mathcal{B}_i^2$ are included in the WFW. This is the key property that makes the algorithm work.

B. The WFW as a Set of Boxes

Let us consider a large box $\mathcal{B}$. For example, $\mathcal{B}$ can be the frame on which the actuated reels of the mechanism are mounted. The algorithm presented in section V-A can be modified so as to approximate the part of the WFW lying within $\mathcal{B}$ by a set of n-dimensional boxes $\mathcal{B}_i$. To this end, at step 1, the list $\mathcal{L}$ is initialized with the box $\mathcal{B}$ instead of $\mathcal{W}$. At step 5, when a box $\mathcal{B}_i$ of $\mathcal{L}$ is found to be included in the WFW, this box $\mathcal{B}_i$ is stored in a list $\mathcal{A}$ before proceeding to the next box of $\mathcal{L}$ by going back to step 2. Finally, at steps 4 and 6 a, when a box $\mathcal{B}_i$ is, respectively, fully outside the WFW or too small to be bisected, instead of stopping the algorithm by returning -1 or 0, $i = i + 1$ and the next box of $\mathcal{L}$ is considered. This modified algorithm outputs the list $\mathcal{A}$ whose boxes provide an approximation of the part of the WFW included in the initial box $\mathcal{B}$. The quality of the approximation is highly dependent on the value of the threshold ϵ.

VI. Examples

A. A 3-DOF Planar Mechanism

Let us consider the 3-DOF planar parallel cable-driven mechanism shown in Fig. 3. The base and the mobile platform are squares of side length 1m and 0.2m, respectively. The cables attached at points B_i of the platform are crossed as shown in Fig. 3. The position of the platform is defined by the position (x, y) of its reference point P whereas its orientation is given by angle ϕ. The WFW to be determined is defined by τ_{min}=1N, τ_{max}=54N and $\mathbf{f}^{\mathcal{I}}$=$[[-10, 10], [-10, 10], [-0.5, 0.5]]^T$ where the first two components of $\mathbf{f}^{\mathcal{I}}$ are interval of forces F_x and F_y (N) and the third component is an interval of moments M_z (N.m).

The algorithm presented in V-A shows that the following box workspace

$$\mathcal{W} = \begin{bmatrix} x^{\mathcal{I}} \\ y^{\mathcal{I}} \\ \phi^{\mathcal{I}} \end{bmatrix} = \begin{bmatrix} [0.3, 0.7]\ (m) \\ [0.3, 0.7]\ (m) \\ [-0.7854, 0.7854]\ (rad) \end{bmatrix} \tag{23}$$

is fully included in the WFW. The algorithm presented in V-B allows to obtain an approximation of the part of the WFW lying within the box $\mathcal{B}$=$[[-0.1, 1.1], [-0.1, 1.1], [-0.7854, 0.7854]]$ as shown in Fig. 4. Note that, at step 6a, two different ϵ have been used, one ϵ_{pos} (positions x and y) for the first two components of $\mathcal{B}$ and the other one ϵ_{ori} for the third component of $\mathcal{B}$ (orientation). Fig. 4 shows the boxes of the output list $\mathcal{A}$ for ϵ_{pos}=0.02 and ϵ_{ori}=0.05. Finally, a constant-orientation cross section of the WFW for $\phi = \pi/4$ (box $\mathcal{B}$=$[[-0.1, 1.1], [-0.1, 1.1], [0.7854, 0.7854]]$) is shown in Fig. 5 together with the boundary of the constant-orientation wrench-closure workspace (WCW) [3]. Note that the WCW is a special instance of the WFW for which τ_{max} tends to infinity, τ_{min} is equal to 0 and the required set of wrenches $\mathbf{f}^{\mathcal{I}}$ is the whole space of wrenches.

B. A 6-DOF Mechanism

Fig. 6 shows a six-DOF parallel mechanism driven by eight cables together with a box. Let us consider the WFW defined by τ_{min}=1N, τ_{max}=540N and $\mathbf{f}^{\mathcal{I}}$=$[[-10, 10], [-10, 10], [-10, 10], [-0.5, 0.5], [-0.5, 0.5], [-0.5, 0.5]]^T$ where the first three components of $\mathbf{f}^{\mathcal{I}}$ correspond to forces (N) and the last three to moments (N.m). The algorithm of section V-A allows to determine that the box shown in Fig. 6 is included in the WFW for any orientation of the mobile platform such that $\phi \in [-0.2618, 0.2618]$ (rad), $\theta \in [-0.2618, 0.2618]$ (rad) and $\psi = 0$ where ϕ, θ and ψ are three Euler angles (XYZ convention).

C. Computation Time

Our implentation in C++ uses the interval arithmetic of the BIAS/Profil C++ library and the simplex of the GNU Linear Programming Kit (GLPK). On a DELL Precision 380 PC (3,6 Ghz), the algorithm introduces in section V-A finds in 67s that the box (23) is included in the WFW (with ϵ_{pos}=ϵ_{ori}=0.0005) whereas the time required to compute the set of boxes shown in Fig. 4 is 44s (ϵ_{pos}=0.02, ϵ_{ori}=0.05). In section VI-B, the computation time is 133s (ϵ_{pos}=0.01, ϵ_{ori}=0.05).

The basic method presented in this paper can be improved in several ways. For instance, in the procedure `Feasibility`$(\mathbf{W}^{\mathcal{I}}, \mathbf{f}^{\mathcal{I}}, \mathcal{T})$, the interval evaluation $\mathbf{W}^{\mathcal{I}} = \mathbf{W}^{\mathcal{I}}(\mathcal{B})$ of the wrench matrix $\mathbf{W}$ can be replaced by the

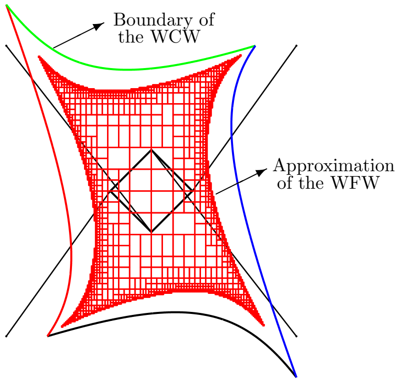

Fig. 5. Constant-orientation WFW (set of boxes) and wrench-closure workspace (geometric determination of the boundary).

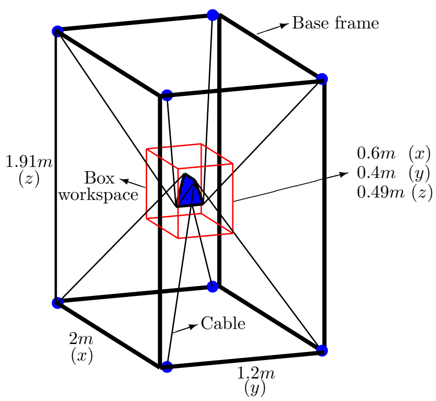

Fig. 6. A six-DOF parallel mechanism driven by eight cables.

interval evaluation $\mathbf{W}_q^{\mathcal{I}} = \mathbf{W}_q^{\mathcal{I}}(\mathcal{B})$ of the matrix $\mathbf{W}_q$ whose expression is obtained from that of $\mathbf{W}$ by removing the denominators ρ_i which appears in each column of $\mathbf{W}$ (ρ_i denotes the length of cable i, cf [12]). Then, if the allowed set of tension $\mathcal{T}$ is replaced by

$$\mathcal{T}_q = \left\{ \boldsymbol{\tau}_q \mid \tau_{q_i} \in [\tau_{min}/\underline{\rho_i}, \tau_{max}/\overline{\rho_i}] \ \forall \, 1 \leq i \leq m \right\} \quad (24)$$

where $\underline{\rho_i}$ and $\overline{\rho_i}$ are, respectively, the lower and upper bounds of the interval evaluation of ρ_i for the box $\mathcal{B}$, it can be proved that $\texttt{Feasibility}(\mathbf{W}_q^{\mathcal{I}}, \mathbf{f}^{\mathcal{I}}, \mathcal{T}_q) = 1$ is a sufficient condition for the box $\mathcal{B}$ to be included in the WFW. This modification has a very positive effect on the computation time. For instance, the computation time needed in section VI-B is reduced from 133s to 18s. Note that, for a given box $\mathcal{B}$, this modification is possible only if

$$\tau_{min}/\underline{\rho_i} \leq \tau_{max}/\overline{\rho_i} \iff \tau_{min}/\tau_{max} \leq \underline{\rho_i}/\overline{\rho_i}. \quad (25)$$

VII. Conclusions

The determination of the WFW is a difficult problem. In this paper, it was shown how methods based on interval analysis allow to deal with this issue. Indeed, an efficient algorithm that solves the practical problem of determining whether a given box workspace is fully included in the WFW was proposed together with a variant of this algorithm that provides an approximation of the WFW up to a given accuracy. In the case of n-dof mechanisms driven by n or more than n cables — especially when $n = 6$, — geometric methods that determines the boundary of the WFW are currently unused certainly due to the complexity of the analytic description of the boundary. Hence, the proposed algorithms provide useful alternatives to the only currently available determination methods, that is, to the "brute force" time-consuming discretization methods consisting in testing a cloud of points.

The tools presented in this paper are basics and can be improved in several ways so as to decrease the computational time which may be high. Moreover, these tools form a necessary basis to the application to parallel cable-driven mechanisms of an optimal design framework based on interval analysis presented in [10] and [11].

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