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UnCertainties`: A Package for Interval Analysis

Applications to Equation Solving and Global Optimization under Uncertainty

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In this talk, we present and describe in details the under development Uncertainties` *Mathematica* package, a collection of data structures and functions implementing the most advanced algorithms of Interval Analysis for solving systems of equations and for global optimization.

```
<< UnCertainties`
```

Certified Computations with Uncertainties

```
RumpFunc[x_, y_] := (1335/4 - x^2) y^6 +
   x^2 (11 x^2 y^2 - 121 y^4 - 2) + (11/2) y^8 + x/(2 y)

RumpFunc[77 617, 33 096];
N[%, 21]

Manipulate[RumpFunc[SetPrecision[77 617., p],
   SetPrecision[33 096., p]], {p, 10, 60, 1}]

RumpFuncN[x_, y_] := (1335/4 - x^2) y^6 +
   x^2 (11 x^2 y^2 - 121 y^4 - 2) + (5.5) y^8 + x/(2 y)

RumpFuncN[77 617, 33 096]

RumpFuncN[Interval[77 617.], Interval[33 096.]]
```

Interval and Boxes

Representation

Intervals

```
Interval[0]

Interval[0.]

x = Interval[1.];
x[[1, 2]] - x[[1, 1]]

Interval[{-1, 2}]
```

Boxes : Vectors of Intervals
Intervals Matrices

Interval Arithmetic

Arithmetic on Intervals

```
Interval[{-1, 2}] + Interval[{1, 3}]

Interval[{-1, 2}] * Interval[{1, 3}]

Interval[{-1, 2}] - Interval[{1, 3}]

Interval[{-1, 2}] - Interval[{1, 2}]

Interval[{-1, 2}] / Interval[{1, 3}]

Interval[{1, 3}] / Interval[{-1, 2}]

Exp[Interval[{1., 3.}]]

Sin[Interval[{1., 3.}]]

Sqrt[Interval[{-1., 9.}]]
```

Arithmetic on Boxes

```
{Interval[{-1, 2}], Interval[{1, 3}]} +
  {Interval[{0, 3}], Interval[{-1, 0}]}

{Interval[{-1, 2}], Interval[{1, 3}]} -
  {Interval[{-1, 2}], Interval[{1, 3}]}

Interval[{-1, 2}] {Interval[{-1, 2}], Interval[{1, 3}]}

{Interval[{-1, 2}], Interval[{1, 3}]}.
  {Interval[{-1, 2}], Interval[{1, 3}]}
```

Arithmetic on Matrices

Miscellanies

Properties

```
Lower[Interval[{-1, 2}]]
Lower[{Interval[{-1, 2}], Interval[{1, 3}]}]
```

```
Upper[Interval[{-1, 2}]]
Upper[{Interval[{-1, 2}], Interval[{1, 3}]}]
```

```
MidPoint[Interval[{-1, 2}]]
MidPoint[{Interval[{-1, 2}], Interval[{1, 3}]}]
```

```
Width[Interval[{-1, 2}]]
Width[{Interval[{-1, 2}], Interval[{1, 3}]}]
```

Set Operations on Intervals

```
IntervalIntersection[Interval[{-2, 3}], Interval[{1, 4}]]
```

```
IntervalIntersection[Interval[{-2, 3}], Interval[{4, 7}]]
```

```
IntervalUnion[Interval[{1, 3}], Interval[{2, 4}]]
```

```
IntervalUnion[Interval[{1, 2}], Interval[{3, 4}]]
```

```
IntervalHull[IntervalUnion[Interval[{1, 2}], Interval[{3, 4}]]]
```

```
IntervalHull[Interval[{1, 2}], Interval[{3, 4}]]
```

```
IntervalMemberQ[Interval[{1, 2}], 1.2]
```

Set Operations on Boxes

```
BoxIntersection[{Interval[{-1, 2}], Interval[{1, 3}]},
  {Interval[{0, 3}], Interval[{-1, 2}]}]
```

```
IntervalHull[{Interval[{-1, 2}], Interval[{1, 3}]},
  {Interval[{0, 3}], Interval[{-1, 0}]}]
```

Visualization

```
View[Interval[{-1, 2}]]
```

```
View[{Interval[{-1, 2}], Interval[{1, 3}]}]
```

```
View[{Interval[{-1, 2}], Interval[{1, 3}]},
  {Interval[{0, 3}], Interval[{-1, 2}]}]
```

```
View[{Interval[{-1, 2}], Interval[{1, 3}], Interval[{0, 1}]}]
```

```
View[{Interval[{-1, 2}], Interval[{1, 3}], Interval[{0, 1}]},
  {Interval[{0, 3}], Interval[{-1, 2}], Interval[{0.5, 2}]}]
```

Evaluation

```
f1[x_] := x (x + 1)
f2[x_] := x * x + x
f3[x_] := x^2 + x
f4[x_] := (x + 1/2)^2 - 1/4

f1[Interval[{-1, 1}]]
f2[Interval[{-1, 1}]]
f3[Interval[{-1, 1}]]
f4[Interval[{-1, 1}]]

g1[x_, y_] := (x - y) / (x + y)
g2[x_, y_] := 1 - (2/(1 + x/y))
Simplify[g1[x, y] - g2[x, y]]

g1[Interval[{-1, 2}], Interval[{3, 5}]]
g2[Interval[{-1, 2}], Interval[{3, 5}]]

Clear[f1, f2, f3, f4, g1, g2]
```

Simple Evaluations

```
NaturalEval[x^2 - x + 1, {x, Interval[{-1, 1}]}]

MidPointEval[x^2 - x + 1, {x, Interval[{-1, 1}]}]

RandomEval[x^2 - x + 1, {x, Interval[{-1, 1}]}]
```

with Symbolic Pre-Processing

Horner Form for Polynomials
minimizing the number of occurences

Taylor Forms

first order

$$[f]([x]) = f(m) + \left[\frac{\partial f}{\partial_x}\right]([x])([x] - m) \tag{1}$$

```
CenteredEval[p1, {x, y, z}, b]

b1 = {Interval[{1.9, 2}], Interval[{0.9, 1}], Interval[{1.4, 1.5}]};
b2 = {Interval[{1.9, 1.91}],
    Interval[{0.9, 0.91}], Interval[{1.49, 1.5}]};
```

```
NaturalEval[p1, {x, y, z}, b1]
CenteredEval[p1, {x, y, z}, b1]
```

```
NaturalEval[p1, {x, y, z}, b2]
CenteredEval[p1, {x, y, z}, b2]
```

second order

$$[f]([x]) = f(m) + (x - m)\frac{\partial f}{\partial_x}(x) + \frac{([x] - m)}{2}\left[\frac{\partial^2 f}{\partial_x \partial_x}\right]([x]) \quad (2)$$

```
TaylorEval[p1, {x, y, z}, b]
TaylorEval[p1, {x, y, z}, b1]
TaylorEval[p1, {x, y, z}, b2]
```

Solving by Global Search and Bisection

1 Dimension

```
Plot[(3 Cos[x] - Log[x]), {x, 0.01, 50}]

expr /. x → 1.5

expr = 3 Cos[x] - Log[x]

IEval[expr, {x, Interval[{0.01, 50}]}] // N

xl = Bisect[Interval[{-100., 50}]]

xl = Flatten[Map[Bisect,
    Pick[xl, Map[IntervalZeroQ[N[IEval[expr, {x, #}]]] &, xl]]]]

Map[N[IEval[expr, {x, #}]] &, xl]

Map[IntervalZeroQ[N[IEval[expr, {x, #}]]] &, xl]

BisectionSolve[3 Cos[x] == Log[x], {x, Interval[{0.01, 50}]}]
```

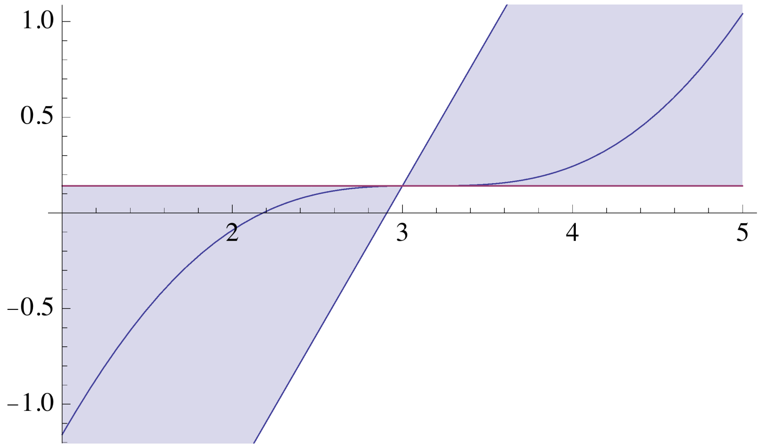
More

```
BisectionSolve[{Exp[x - 2] == y, y^2 == x},
  {{x, Interval[{-100000, 100000}]},
   {y, Interval[{-100000, 100000}]}}] // N

BisectionSolve[{x + y == 2, x - 3 y + z == 3, x - y + z == 0},
  {{x, Interval[{-1000, 1000}]}, {y, Interval[{-1000, 1000}]},
   {z, Interval[{-1000, 1000}]}}] // N
```

Solving by Filtering

Newton Exemple




```
f =.
f[x_] := Sin[x] + x - 3
m = MidPoint[Interval[{1., 5.}]]
i1 = IntervalIntersection[
    Interval[{1., 5.}], m - f[m] / f'[Interval[{1., 5.}]]]
m = MidPoint[i1]
i1 = IntervalIntersection[i1, m - f[m] / f'[i1]]
NewtonSolve[Sin[x] + x - 3, {x, Interval[{1., 5.}]}]
NewtonSolve[3 Cos[x] - Log[x], {x, Interval[{0.01, 50}]}]
```

Optimization

```
Plot[(3 Cos[x] - Log[x] + Sin[x/3]), {x, 10, 50}]
IMaximize[3 Cos[x] - Log[x] + Sin[x/3], {x, Interval[{10, 50.}]}]
```

Perspectives

To be ready for the next *Mathematica* User's Conference with a complete implementation of the following.

Interval Analysis Methods

Linear Case

Gauss Elimination

Gauss Seidel

Krawczyk

Constraint Programming Methods

2B Filtering

3B Filtering