Singularity Loci of Planar Parallel Manipulators with Revolute Joints

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Abstract: This paper addresses the problem of determining the singularity loci of 3-DOF planar parallel manipulators with revolute joints for a constant orientation of the mobile platform. The case when the base joints are actuated is of primary concern and is proved here to lead to singularity loci represented by curves of degree 42—an improved measure with respect to the one already given in the literature. A particular case is studied when two of the platform joints are coincident which allows the singularities to be studied geometrically. The case when the intermediate joints are actuated is also considered. On the basis of the presented study, important observations are made on the nature of the singularity loci for both types of actuation.

1 Introduction

Undoubtedly, the most common architecture for a 3-DOF planar parallel manipulator (*PPM*) is the 3-$\underline{R}RR$ one. The reason is mostly practical—these PPMs are easiest to build. The fact that the actuators are fixed to the base allows the use of inexpensive DC drives and reduces the weight of the mobile equipment. In addition, the links can be made of thin rods and even be curved (Schönherr, 1998) to decrease significantly the link interference. Finally, revolute joints have virtually no mechanical limits, which together with the previously mentioned feature, maximizes significantly the workspace of 3-$\underline{R}RR$ PPMs.

Another possible architecture for a 3-DOF PPMs is the 3-$R\underline{R}R$ one which is kinematically equivalent to the 3-$R\underline{P}R$ one. Both architectures, however, have their actuators as part of the mobile equipment which decreases the manipulator's workspace and speed. As we will see in this paper, while the singularity loci of 3-$R\underline{R}R$ (3-$R\underline{P}R$) PPMs are simple quadratic curves, the singularity loci of 3-$\underline{R}RR$ PPMs are generally impossible to determine and represent analytically. What is more, to the best of our knowledge, there has been only one article discussing their singularity loci.

The organization of this paper is as follows. In the next section, we determine the singularity loci of 3-$\underline{R}RR$ PPMs for a constant orientation and prove that they are represented by a polynomial of degree 42. Then, in Section 3, we consider a simplified 3-$\underline{R}RR$ PPM design in which two of the platform joints are coincident which leads to singularity loci being four circles and a sextic. Examples are then shown in Section 4 where we also discuss briefly the singularity loci of 3-$R\underline{R}R$ PPMs and then generalize the properties of the singularity loci for both types of actuation. Finally, Section 5 presents the conclusions.

2 Singularity Loci of 3-$\underline{R}RR$ PPMs

Referring to Fig. 1, let us select a reference frame fixed to the base, called the *base frame*, with center O and axes x and y. Similarly, let us select a reference frame fixed to the mobile platform, called the *mobile frame*, with center at a point C and axes x' and y'. Let us then denote the centers of the base joints by O_i, of the intermediate joints by A_i, and of the mobile platform joints by B_i (in this paper, $i = 1, 2, 3$). The vectors along lines OO_i, O_iA_i, O_iB_i, and CB_i are respectively denoted by $\mathbf{o}_i$, $\mathbf{u}_i$, $\mathbf{r}_i$, and $\mathbf{s}_i$. Throughout this paper, we will add the superscript $'$ to a vector when the latter is expressed in the mobile frame, and no superscript when the vector is expressed in the base frame.

The mobile platform's position is defined by vector $\mathbf{v} = [x,\, y]^T$, while its orientation is described by angle ϕ which is the angle between the base x-axis and the mobile x'-axis. The two coordinates of point C expressed in the base frame, x and y, and the angle ϕ constitute the so-called *generalized coordinates*, contained in the vector $\mathbf{q} = [x,\, y,\, \phi]^T$. The latter defines completely the *pose* (the position and orientation) of the mobile platform.

We will refer to the link connecting points O_i and A_i as *proximal link* i and to the link connecting points A_i and B_i as *distal link* i. Let all proximal links be of equal length ℓ_1 and all distal links be of equal length ℓ_2. The angle between proximal link i and the base x-axis will be denoted by θ_i and will be referred to as *articular coordinate* i with $\boldsymbol{\Theta} = [\theta_1,\ \theta_2,\ \theta_3]^T$.

We will now consider the task of computing the set of articular coordinates from the set of generalized coordinates, referred to as the *inverse kinematic problem*. Geometrically, for serial chain i, the problem can be seen as the one of finding the intersection point(s) between a circle of radius ℓ_1 and center O_i and a circle of radius ℓ_2 and center B_i. Clearly, depending on the position of point B_i, this problem may have two real solutions, a single one, or none at all. If $\ell_1 = \ell_2$, the problem may even have an infinite number of solutions.

Let the unit vector along distal link i be denoted by $\mathbf{n}_i$. Therefore, we have

$$\ell_2\mathbf{n}_i = \mathbf{v} + \mathbf{R}\,\mathbf{s}'_i - \mathbf{u}_i - \mathbf{o}_i, \tag{1}$$

where $\mathbf{R}$ is the rotation matrix defined by angle ϕ. Squaring both sides of eq. 1 gives us

$$\ell_2^2 = (\mathbf{v} + \mathbf{R}\,\mathbf{s}'_i - \mathbf{u}_i - \mathbf{o}_i)^T(\mathbf{v} + \mathbf{R}\,\mathbf{s}'_i - \mathbf{u}_i - \mathbf{o}_i), \tag{2}$$

$$\ell_2^2 = ||\mathbf{r}_i||^2 + \ell_1^2 - 2\mathbf{r}_i^T\mathbf{u}_i. \tag{3}$$

In addition, we have $\mathbf{u}_i = \ell_1\,[\cos\theta_i,\ \sin\theta_i]^T$ and

$$\mathbf{r}_i = \begin{bmatrix} x + \cos\phi\, x'_{B_i} - \sin\phi\, y'_{B_i} - x_{O_i} \\ y + \sin\phi\, x'_{B_i} - \cos\phi\, y'_{B_i} - y_{O_i} \end{bmatrix} \equiv \begin{bmatrix} x + a_i \\ y + b_i \end{bmatrix}, \tag{4}$$

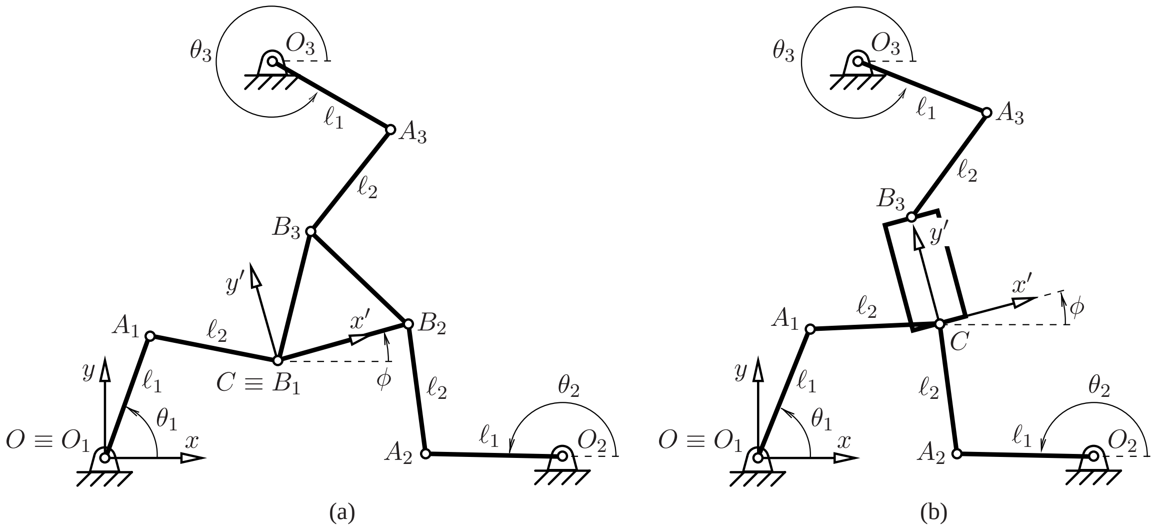

Figure 1: (a) A general and (b) a special 3-DOF 3-$\underline{R}RR$ planar parallel manipulator.

where x'_{B_i} and y'_{B_i} are the coordinates of point B_i in the mobile frame, and x_{O_i} and y_{O_i} are the coordinates of point O_i in the base frame. Furthermore, as seen from the last definition, a_i and b_i are constants for a given orientation of the mobile platform. Therefore, eq. 3 may be written as

$$\cos\theta_i\,(x+a_i)+\sin\theta_i\,(y+b_i)=\frac{(x+a_i)^2+(y+b_i)^2+\ell_1^2-\ell_2^2}{2\ell_1}\equiv p_i. \tag{5}$$

In order to have a real solution to the above equation, the following inequality should hold true:

$$(x+a_i)^2+(y+b_i)^2-p_i^2\equiv\Gamma_i\geq 0 \tag{6}$$

Unless $\Gamma_i=0$, there exist two real solutions to eq. 5, determined uniquely from:

$$\sin\theta_i=\frac{p_i(y+b_i)+(x+a_i)\delta_i\sqrt{\Gamma_i}}{\rho_i},\quad \cos\theta_i=\frac{p_i(x+a_i)-(y+b_i)\delta_i\sqrt{\Gamma_i}}{\rho_i}, \tag{7}$$

where $\delta_i=\pm 1$ is the so-called *branch index* that occupies a significant place in this study, and $\rho_i=||\mathbf{r}_i||^2=(x+a_i)^2+(y+b_i)^2$. Note, that for each serial chain, there exist two possible branches, and, hence, for the whole PPM, there exist eight *branch sets*. Note also that eq. 7 is not valid when $\rho_i=0$, which may occur only if $\ell_1=\ell_2$ and $B_i\equiv O_i$.

Having resolved the inverse kinematic problem, we may now proceed to obtaining the Jacobian matrices by differentiating eq. 2 with respect to time, leading to

$$\ell_2\mathbf{n}_i^T\left(\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}+\dot{\phi}\,\mathbf{E}\,\mathbf{s}_i-\ell_1\dot{\theta}_i\begin{bmatrix}-\sin\theta_i\\ \cos\theta_i\end{bmatrix}\right)=0, \tag{8}$$

where $\mathbf{E}$ is the orthogonal rotation matrix for $\phi=\pi/2$ and

$$\mathbf{s}_i=\begin{bmatrix}a_i+x_{O_i}\\ b_i+y_{O_i}\end{bmatrix}\equiv\begin{bmatrix}c_i\\ d_i\end{bmatrix}. \tag{9}$$

Equation 8 may be written in the following vector form:

$$\left[\ell_2\mathbf{n}_i^T,\;\; \ell_2\mathbf{n}_i^T\mathbf{E}\,\mathbf{s}_i\right]\dot{\mathbf{q}} - \ell_1\ell_2\mathbf{n}_i^T\begin{bmatrix} -\sin\theta_i \\ \cos\theta_i \end{bmatrix}\dot{\theta}_i = 0, \tag{10}$$

where

$$\ell_2\mathbf{n}_i = \begin{bmatrix} x + a_i - \ell_1\cos\theta_i \\ y + b_i - \ell_1\sin\theta_i \end{bmatrix}. \tag{11}$$

Using the expressions from eq. 7, we may easily simplify eq. 10 to

$$\left[\ell_2\mathbf{n}_i^T,\;\; \ell_2\mathbf{n}_i^T\mathbf{E}\,\mathbf{s}_i\right]\dot{\mathbf{q}} + \ell_1\delta_i\sqrt{\Gamma_i}\,\dot{\theta}_i = 0. \tag{12}$$

Finally, the velocity equations can be written in matrix form as:

$$\begin{bmatrix} \ell_2\mathbf{n}_1^T & \ell_2\mathbf{n}_1^T\mathbf{E}\,\mathbf{s}_1 \\ \ell_2\mathbf{n}_2^T & \ell_2\mathbf{n}_2^T\mathbf{E}\,\mathbf{s}_2 \\ \ell_2\mathbf{n}_3^T & \ell_2\mathbf{n}_3^T\mathbf{E}\,\mathbf{s}_3 \end{bmatrix}\dot{\mathbf{q}} + \ell_1\begin{bmatrix} \delta_1\sqrt{\Gamma_1} & 0 & 0 \\ 0 & \delta_2\sqrt{\Gamma_2} & 0 \\ 0 & 0 & \delta_3\sqrt{\Gamma_3} \end{bmatrix}\dot{\mathbf{\Theta}} = \mathbf{J}_q\dot{\mathbf{q}} + \ell_1\mathbf{J}_\Theta\dot{\mathbf{\Theta}} = \mathbf{0}, \tag{13}$$

where $\mathbf{J}_q$ and $\mathbf{J}_\Theta$ are 3×3 Jacobian matrices.

Thus, two different types of singularities may occur (Gosselin and Angeles, 1990). In Type I, $\det(\mathbf{J}_\Theta) = 0$, which happens when $\Gamma_i = 0$. In Type II, $\det(\mathbf{J}_q) = 0$, which happens when *the three lines associated with the distal links intersect at a single point or are all parallel*. For completeness, let us only mention that there also exist singularities of Type III, which may occur when $\ell_1 = \ell_2$ and $B_i \equiv O_i$.

It is the second type of singularities that is the subject of this study, though a major attention will also be paid to the relationship between all types. Particularly, our study will be on the corresponding singularity loci for a constant orientation of the mobile platform.

The singularity loci of Type I for a constant orientation are the boundaries of three annular regions, the *vertex spaces*, each defined by the inequality $\Gamma_i \geq 0$. The constant-orientation workspace is the intersection of these vertex spaces. The singularity loci of Type III (in case $\ell_1 = \ell_2$) are simply the centers of the vertex spaces with coordinates $(-a_i,\; -b_i)$. The determination of singularity loci of Type II is, however, a cumbersome task due to the existence of the radicals $\sqrt{\Gamma_i}$. Eliminating the radicals leads to a polynomial of high degree *corresponding to all eight branch sets*.

To our best knowledge, the only work attempting to resolve this problem has been reported by Gosselin and Wang (1997). In that work, the authors have concluded that the resulting polynomial is of degree 64 in y and 48 in x even though they have only considered a simplified 3-*$\underline{R}$RR* PPM design with collinear base and platform joints. Their remark that the reason for the high degree is the fact that the singularity loci are for all branch sets and not only for a single one as well as the high degree itself has motivated our research.

Indeed, in the next two sub-sections, we will describe the procedure used by us to prove that the degree is in fact 42 using the computer algebra system Maple*. Without loss of generality, we select $O \equiv O_1$ and $C \equiv B_1$. This implies that $a_1 = b_1 = c_1 = d_1 = 0$. In addition we select the base x-axis to pass through point O_2 which implies that $d_2 = b_2$. Note, however, that in the parameterization that we use, selecting the mobile platform axis to pass through point B_2 does not imply any significant simplification.

*The programs are available from `http://www.parallemic.org/Reviews/Review001.html`

2.1 Special Case: Proximal and Distal Links of Equal Lengths

Let us first consider the special case when $\ell_1 = \ell_2$ despite the fact that it leads to singularities of Type III. The reason is that in this case, we may obtain the polynomial in symbolic form. We render our problem dimensionless and set $\ell_1 = \ell_2 = 1$. Once the Jacobian $\mathbf{J}_q$ is expressed in x, y, and the parameters a_i, b_i, c_i, d_i, and ρ_i, we follow the procedure described below:

S1. Substitute the expressions $\delta_i\sqrt{\Gamma_i}$ with the parameters Δ_i.

S2. Obtain $det(\mathbf{J}_q)$. The denominator of this determinant is $8\rho_1\rho_2\rho_3$. Indeed, $\mathbf{J}_q$ is not defined when $B_i \equiv O_i$, i.e., when $\rho_i = 0$. Eliminating these possibilities, we consider further only the numerator, $\mathcal{E}$. This numerator is a function of x and y that cannot be generally factored and contains the three radicals $\delta_i\sqrt{\Gamma_i}$ (actually the parameters Δ_i) and the parameters ρ_i. Note, that this is the only expression that corresponds to the singularity loci for the given branch set.

S3. Eliminate the radical in $\sqrt{\Gamma_1}$. Rewrite $\mathcal{E}$ in the form $\mathcal{C}_1\Delta_1 = \mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ do not contain Δ_i. Next, raise to square leading to $\mathcal{C}_1^2\Gamma_1 = \mathcal{C}_2^2$. Both Γ_1 and $\mathcal{C}_2^2$ are multiples of ρ_1 which can, therefore, be canceled. At this step, our new expression, $\mathcal{E}_1 = \mathcal{C}_1^2(\Gamma_1/\rho_1) - \mathcal{C}_2^2/\rho_1$, does not contain δ_1 and, hence, corresponds to two branch sets.

S4. Split $\mathcal{E}_1$ and substitute the terms Δ_2^2 and Δ_3^2 with respectively Γ_2 and Γ_3. Note, that if we attempt to perform this substitution directly in $\mathcal{E}_1$, the resulting expression becomes too large to allow to be further handled symbolically. Thus, $\mathcal{E}_1$ is written in parts as

$$\mathcal{E}_1 = \mathcal{E}_{1,1} + \mathcal{E}_{1,2}\Delta_3 + \mathcal{E}_{1,3}\Delta_2 + \mathcal{E}_{1,4}\Delta_2\Delta_3 + \mathcal{E}_{1,5} + \mathcal{E}_{1,6} + \mathcal{E}_{1,7}\Delta_2 + \mathcal{E}_{1,8}\Delta_3 + \mathcal{E}_{1,9}$$

$$\begin{array}{lll} \mathcal{E}_{1,1} = \mathcal{C}_{0,0}/(\rho_2\rho_3) & \mathcal{E}_{1,4} = \big(\mathcal{C}_{1,1}/(\rho_2\rho_3)\big) & \mathcal{E}_{1,7} = \big(\mathcal{C}_{1,2}/\rho_2\big)(\Gamma_3/\rho_3) \\ \mathcal{E}_{1,2} = \big(\mathcal{C}_{0,1}/(\rho_2\rho_3)\big) & \mathcal{E}_{1,5} = \big(\mathcal{C}_{0,2}/\rho_2\big)\Gamma_3/\rho_3 & \mathcal{E}_{1,8} = \big(\mathcal{C}_{2,1}/\rho_3\big)(\Gamma_2/\rho_2) \\ \mathcal{E}_{1,3} = \big(\mathcal{C}_{1,0}/(\rho_2\rho_3)\big) & \mathcal{E}_{1,6} = \big(\mathcal{C}_{2,0}/\rho_3\big)\Gamma_2/\rho_2 & \mathcal{E}_{1,9} = \mathcal{C}_{2,2}(\Gamma_2/\rho_2)(\Gamma_3/\rho_3) \end{array}$$

where $\mathcal{C}_{j,k}$ $(j, k = 0, 1, 2)$ are coefficients that do not contain Δ_2 or Δ_3. Furthermore, all the divisions can be performed exactly.

S5. Eliminate the radical in $\sqrt{\Gamma_2}$:

$$\mathcal{E}_2 = \mathcal{E}_{2,1}^2\Gamma_2 + \mathcal{E}_{2,2}^2\Gamma_2\Gamma_3 + 2\mathcal{E}_{2,1}\mathcal{E}_{2,2}\Gamma_2\Delta_3 - \mathcal{E}_{2,3}^2 - \mathcal{E}_{2,4}^2\Gamma_3 - 2\mathcal{E}_{2,3}\mathcal{E}_{2,4}\Delta_3$$

$$\mathcal{E}_{2,1} = \mathcal{E}_{1,3} + \mathcal{E}_{1,7}, \quad \mathcal{E}_{2,2} = \mathcal{E}_{1,4}, \quad \mathcal{E}_{2,3} = \mathcal{E}_{1,1} + \mathcal{E}_{1,5} + \mathcal{E}_{1,6} + \mathcal{E}_{1,9}, \quad \mathcal{E}_{2,4} = \mathcal{E}_{1,2} + \mathcal{E}_{1,8}$$

The new expression $\mathcal{E}_2$ does not contain δ_1 or δ_2 and, hence, corresponds to the singularity loci of four branch sets.

S6. Eliminate the radical in $\sqrt{\Gamma_3}$:

$$\mathcal{E}_3 = (2\mathcal{E}_{2,1}\mathcal{E}_{2,2}\Gamma_2 - 2\mathcal{E}_{2,3}\mathcal{E}_{2,4})^2\Gamma_3 - (\mathcal{E}_{2,1}^2\Gamma_2 + \mathcal{E}_{2,2}^2\Gamma_2\Gamma_3 - \mathcal{E}_{2,3}^2 - \mathcal{E}_{2,4}^2\Gamma_3)^2$$

Finally, we substitute the expressions for ρ_i in $\mathcal{E}_3$, which becomes a polynomial in the variables x and y but cannot be expanded in symbolic form. However, it can quickly be verified to be of degree 48 using the Maple command `coeff(`$\mathcal{E}_3$`,(x,y))`. Furthermore, if we use the same command to extract and simplify all the coefficients of $\mathcal{E}_3$ corresponding to the terms of degree greater than 42, we can observe that they are all zero. In addition, the coefficients of the terms of degree less than 8 are also zero which makes $\mathcal{E}_3$ a *fewnomial* of degree 42.

2.2 General Case

In the case when $\ell_1 \neq \ell_2$, we set only $\ell_1 = 1$ and follow a much simplified procedure. Firstly, we assign *random integer values* to the coefficients a_i, b_i, c_i, and d_i since the procedure *cannot be performed symbolically*. Then, we eliminate the radical in $\sqrt{\Gamma_1}$ and divide the resulting expression $\mathcal{E}_1$ by ρ_1. Note that in this case Γ_i is not a multiple of ρ_i. Next, we substitute the terms Δ_2^2 and Δ_3^2 with respectively Γ_2 and Γ_3. Then, we eliminate the radical in $\sqrt{\Gamma_2}$, divide the resulting expression $\mathcal{E}_2$ by ρ_2^2, and substitute Δ_3^2 with Γ_3. Finally, we eliminate the radical in $\sqrt{\Gamma_3}$ and divide the resulting expression $\mathcal{E}_3$ by ρ_3^4. The polynomial $\mathcal{E}_3$ is again of degree 42, but this time the coefficients of all possible terms are generally non-zero (except for the odd-power terms of degree 42).

3 Singularity Loci of 3-*R̲RR* PPMs with Coincident Platform Joints

In general, the polynomial $\mathcal{E}_3$ cannot be factored. Special designs such as base and platform being equilateral triangles or collinear do not lead to simplified results. One particular case, however, simplifies greatly that polynomial and allows the singularity loci to be geometrically described. This case also brings insight into the complex relationship between branches and singularity loci.

The particular case of interest occurs simply when two platform joints are coincident, e.g., when $B_1 \equiv B_2$ (Fig. 1b). Using the parameterization introduced previously, this case implies that $a_1 = b_1 = c_1 = d_1 = b_2 = c_2 = d_2 = 0$. Now, we can either use the approach described in the last section or observe the following. As mentioned before, singularities of Type II occur when the lines associated with the distal links intersect at one point or are all parallel. Since two of the lines always intersect at point C, we have only two possible cases:

Case 1: Points C, B_3, and A_3 are collinear.

This case implies that $\mathbf{n}_3 = \pm[-\sin\phi,\ \cos\phi]^T$. Hence, the corresponding singularity loci consists of two circles of radius ℓ_1 and centers with coordinates $\mathbf{o}_3 + \mathbf{n}_3(\ell_2 + ||\mathbf{s}_3||)$ with the following algebraic equations:

$$\left(x + a_3 \pm \ell_2 c_3/\sqrt{c_3^2 + d_3^2}\right)^2 + \left(y + b_3 \pm \ell_2 d_3/\sqrt{c_3^2 + d_3^2}\right)^2 = \ell_1^2. \tag{14}$$

The following important observation can now be made. Each of the two circles is separated by lines parallel to $\mathbf{n}_3$ into semicircles corresponding to the two possible branches of chain 3.

Case 2: Points A_1, C, and A_2 are collinear.

This case has two subcases. In the first one, points A_1 and A_2 coincide which may occur at two locations symmetric with respect to line O_1O_2. Thus, the corresponding singularity loci are two circles given by the following equations:

$$(x + a_2/2)^2 + \left(y \pm \sqrt{\ell_1^2 - a_2^2/4}\right)^2 = \ell_2^2. \tag{15}$$

Again, each of the circles is divided into two semicircles by the lines O_1A_1 and O_2A_2 distinguished respectively by the branch indices δ_1 and δ_2. Thus, each circle is divided into four arcs corresponding to four different pairs of branch sets.

The second subcase occurs when $A_1 \not\equiv A_2$. Thus, the singularity loci are the coupler curve of the four-bar mechanism $O_1A_1A_2O_2$, defined by the following sextic (Hunt, 1978):

$$(x+a_2/2)^2(x^2+a_2x+\ell_2^2-\ell_1^2+y^2)^2+y^2(x^2+a_2x+a_2^2/2+\ell_2^2-\ell_1^2+y^2)^2-\ell_2^2a_2^2y^2=0. \quad (16)$$

This sextic, will also be divided in four parts, each corresponding to a branch subset defined by δ_1 and δ_2. Note that the sextic is symmetric with respect to the y-axis and the line $x=-a_2/2$. Indeed, the sextic can be represented by the following parametric equation:

$$x = \pm\varrho\cos\vartheta - a_2/2, \quad y = \pm\varrho\sin\vartheta \qquad (17)$$

where

$$\varrho = \frac{1}{2}\sqrt{4(\ell_1^2-\ell_2^2)-a_2^2(1-2\cos^2\vartheta) \pm 2a_2\sin\vartheta\sqrt{4\ell_2^2-a_2^2\cos^2\vartheta}}, \quad \text{for} \ \ 0 \le \vartheta \le \pi/2, \qquad (18)$$

where ϱ is the distance between point C and the center of line O_1O_2, referred to as point O_c, and ϑ is the angle between the x-axis and the line O_cC.

Note, however, that the sextic described by eq. 16 *always* has a solution at $(-a_2/2,\ 0)$, which in some cases may be an isolated point that is actually outside the constant-orientation workspace, while *if* eq. 18 has this point as a solution, the point is *not* isolated.

In conclusion, for a given branch set, we have two semi-circles defined by δ_3, and a pair (symmetric with respect to line O_1O_2) of circular arcs and arcs from a sextic defined by δ_1 and δ_2. All of these geometrical curves are parts of geometric objects defined by parametric equations and constrained by limits on the parameters that can easily be computed.

4 Examples and Discussions

Based on a simple discretization approach, we have obtained the singularity loci corresponding to all branch sets for two 3-*<u>R</u>RR* PPM designs and represented them in Figs. 2 and 3. In these figures, the singularity loci of Type II are plotted in continuous line, while the boundaries of the constant-orientation workspaces are drawn in dashed line.

Following the approach presented in the previous section, we have also obtained the singularity loci corresponding to all branch sets for a given 3-*<u>R</u>RR* PPM design with two coincident platform joints and represented them in Fig. 4. In this figure we have plotted in dashed line the complete vertex spaces. In all three figures, the points marked with the star symbol represent the centers of the vertex spaces. Also, note that the figures are not drawn to the same scale. More examples are available at the same web address as the Maple programs.

Finally, for the case of 3-*R<u>R</u>R* PPMs, kinematically equivalent to 3-*R<u>P</u>R* PPMs, we refer the reader to (Sefrioui and Gosselin, 1995). Briefly, we have the same vertex spaces and constant-orientation workspace but the singularity loci of Type II are simple *quadratic curves* or the whole workspace. However, the singularity loci are branch-independent and are the same for all eight branch sets. Indeed, the singularities of 3-*R<u>R</u>R* PPMs depend on the lines O_iB_i. Thus, the singularity loci of Type II are completely independent from the singularity loci of Type I, i.e., the vertex spaces. In addition, the singularity loci are greatly simplified if the PPM has a symmetric geometry.

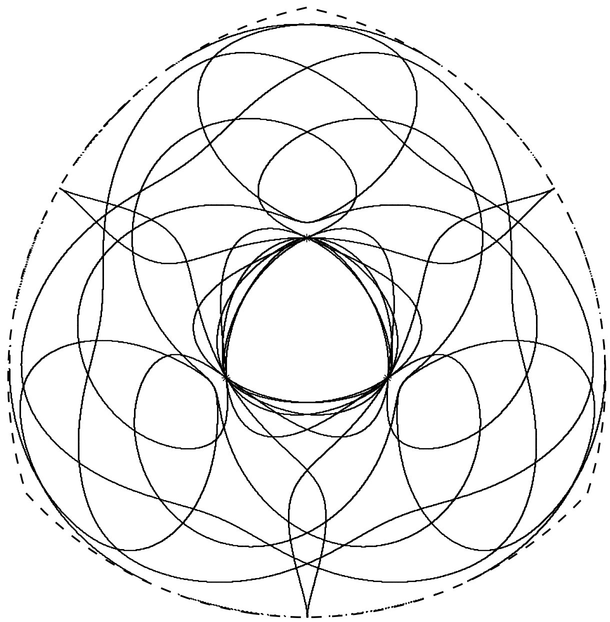

Figure 2: Singularity loci corresponding to all branch sets ($\ell_1 = \ell_2 = 1$, $a_2 = -0.852$, $b_2 = 0$, $c_2 = 0.889$, $d_2 = 0$, $a_3 = -0.426$, $b_3 = -0.738$, $c_3 = 0.444$, $d_3 = 0.770$).

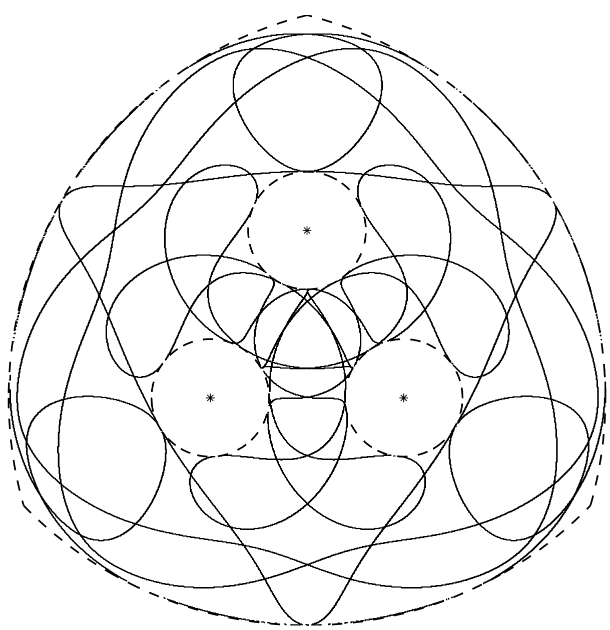

Figure 3: Singularity loci corresponding to all branch sets ($\ell_1 = 1$, $\ell_2 = 1.350$, $a_2 = -1.150$, $b_2 = 0$, $c_2 = 1.200$, $d_2 = 0$, $a_3 = -0.575$, $b_3 = -0.996$, $c_3 = 0.600$, $d_3 = 1.039$).

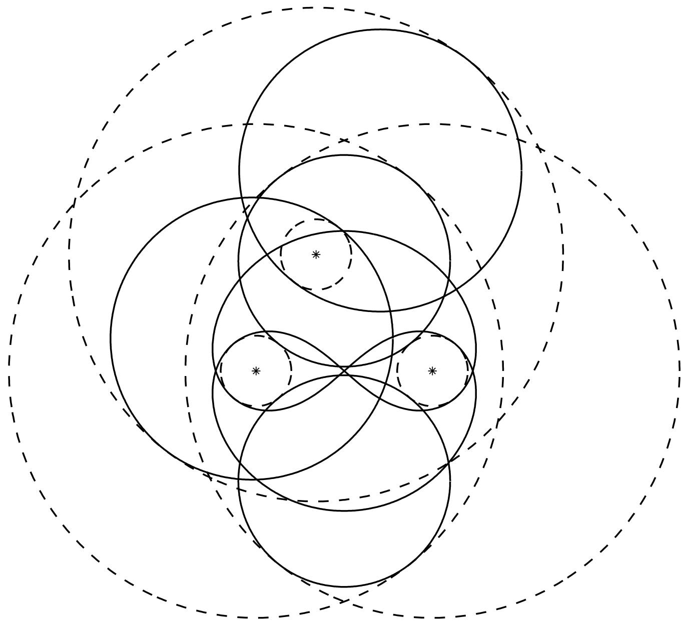

Figure 4: Singularity loci corresponding to all branch sets ($\ell_1 = 1$, $\ell_2 = 0.750$, $a_2 = -1.250$, $a_3 = -0.425$, $b_3 = -0.825$, $c_3 = 0.325$, $d_3 = 0.425$).

Based on the detailed study of examples like the ones presented herein and on the procedure for obtaining the polynomial of degree 42, we may summarize the following list of observations for the singularity loci of 3-$\underline{R}RR$ PPMs:

- No polynomial exists representing the singularity loci of Type II for a given branch set. The corresponding expression contains at least one radical.
- The singularity loci of Type II are always inside the vertex spaces and if a point of contact exists, then they are either tangential or normal to a vertex space at that point.
- At the points of contact, a change of a branch index occurs. Indeed, Fig. 4 illustrates how each curve (a circle or the sextic) is separated into arcs corresponding to the different branches by the points at which the curves are tangent to the vertex spaces.
- If the singularity loci of Type II extend outside the constant-orientation workspace, then there is a factorization in the polynomial of degree 42.
- The singularity loci of Type II for a given branch set always divide the workspace into separate regions, i.e., they are either closed curves or end at the workspace boundaries.
- The singularity loci of Type III (when $\ell_1 = \ell_2$), i.e., the centers of the vertex spaces, are part of the singularity loci of Type II.

5 Conclusions

In this paper, we make important theoretical considerations regarding the singularity loci of planar parallel manipulators with revolute joints. We show that the polynomial representing the singularity loci of Type II for all branches is of degree 42. We also point out a simplified manipulator design for which the singularity loci are simple geometric entities defined by parametric equations. While we do not propose any general method for the determination of the singularity loci for a given branch set, we present discussions that are important in the design of such parallel manipulators. For example, designers might consider trajectory planning with branch change in order to follow trajectories free of singularities of Type II that would otherwise be impossible for a single branch set. The change of branch can be accomplished, for example, by using some mechanical or electrical switch device placed at each intermediate revolute joint that automatically switches when the corresponding serial chain is fully stretched.

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