Elimination-Based Solution Method for the Forward Kinematics of the General Stewart-Gough Platform

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Abstract - In order that the elimination method can be efficiently used for the forward kinematics of the general Stewart-Gough platform, the computation burden should be reduced. In this paper, an elimination-based algorithm is proposed, which demands fairly less computation time than the existing algorithms. This is mainly due to the fact that it finally leads to the 15×15 Sylvester's matrix, which is relatively small in size, and the 40th-degree univariate equation is directly derived from the matrix. The algorithm is demonstrated by a numerical example.

1. Introduction

The general Stewart-Gough platform has six arbitrarily located joints in its base and moving platform. This fact provides the mechanism the highest potentiality to meet the desired performances, when considering the number of changeable design parameters. However, on the other hand, it makes the forward kinematics more difficult than any other types of the Stewart-Gough platforms. The forward kinematics of the Stewart-Gough platform is to find the poses of the moving platform for a given set of leg lengths. In the case of the general Stewart-Gough platform, the formulation of necessary kinematic conditions generates a set of highly nonlinear equations that has 40 solutions in the complex domain. Numerical iterative schemes with relevant initial estimates are applicable to this problem, but they do not guarantee the convergence to the actual solution of the current pose.

Another approach is to find all the possible configurations of the moving platform and then to select the actual solution out of them by proper criteria, such as the current assembly mode or the pose of the latest sampling time. As a method to obtain all the solutions of a nonlinear system, algebraic elimination method is a useful tool, which usually changes the initial set of equations into a univariate polynomial equation that can be readily solved by various efficient and available numerical algorithms.

Since the existence of 40 configurations of the general Stewart-Gough platform had been first demonstrated numerically by Raghavan [1], many researchers have applied elimination method to find all the solutions of the problem. Husty [2] produced a 40th-degree univariate equation by finding the greatest common divisor of the intermediate polynomials of degree 320, while Innocenti [3] derived it from the two 56th-degree univariate equations that are obtained from respective 45×45 matrices. Dhingra et al. [4] used the Gröbner-Sylvester hybrid method to obtain a 40th-degree polynomial from the 68×68 Sylvester's matrix formed by 68 equations of calculated Gröbner-basis. However, those applications of elimination theory are still not satisfactory from the viewpoint of computation time.

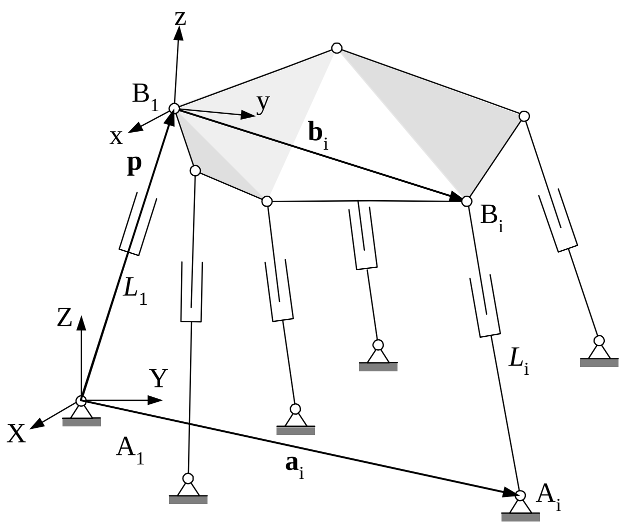

Figure 1. The general Stewart-Gough platform.

This paper presents an elimination-based algorithm for the forward kinematics of the general Stewart-Gough platform, which provides all the solutions in fairly less computation time than the existing algorithms. Furthermore, it directly leads to a 40th-degree univariate equation from a constructed 15×15 Sylvester's matrix without factoring out or deriving the greatest common divisor. The proposed algorithm has been programmed in C^{++} code using an extra data type of 30 significant digits for higher precision. A numerical example is provided to demonstrate the developed algorithm.

2. Kinematic Constraint Equations

Figure 1 shows a kinematic model of the general Stewart-Gough platform. The six inputs necessary to describe the position and orientation of the moving platform are the leg lengths controlled by each prismatic joint. The origins of the frames X-Y-Z and x-y-z are chosen coincident with the locations A_1 and B_1 respectively. Let $\mathbf{a}_i$ denote the position vector A_i in X-Y-Z frame, $\mathbf{b}_i$ denote the position vector B_i in x-y-z. $\mathbf{p}$ is the position vector of the origin of x-y-z with respect to X-Y-Z. With given leg lengths, the kinematic constraint equations corresponding to the conditions of constant length of each leg are as follows

$$(\mathbf{p}+\mathbf{R}\mathbf{b}_i-\mathbf{a}_i)^T(\mathbf{p}+\mathbf{R}\mathbf{b}_i-\mathbf{a}_i)=L_i^2, \qquad i=2,\dots,6 \tag{1}$$

$$\mathbf{p}^T\mathbf{p}=L_1^2 \tag{2}$$

where L_i is the i-th leg length and $\mathbf{R}$ is a rotational matrix. By Cayley's formula [5], $\mathbf{R}$ can be expressed as

$$\mathbf{R}=(\mathbf{I}-\mathbf{C})^{-1}(\mathbf{I}+\mathbf{C}) \tag{3}$$

where $\mathbf{I}$ is the 3×3 identity matrix and $\mathbf{C}$ is an arbitrary 3×3 skew symmetric matrix with three independent parameters, that can be

$$\mathbf{C} = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix} \tag{4}$$

Substituting (4) into (3) yields

$$\mathbf{R} = \Delta^{-1} \begin{bmatrix} 1+c_1^2-c_2^2-c_3^2 & 2(c_1c_2-c_3) & 2(c_3c_1+c_2) \\ 2(c_1c_2+c_3) & 1-c_1^2+c_2^2-c_3^2 & 2(c_2c_3-c_1) \\ 2(c_3c_1-c_2) & 2(c_2c_3+c_1) & 1-c_1^2-c_2^2+c_3^2 \end{bmatrix} \tag{5}$$

where $\Delta = 1+c_1^2+c_2^2+c_3^2$. With the relation (2), Eq. (1) can be rearranged as

$$2\mathbf{b}_\mathrm{i}^\mathrm{T}\mathbf{R}^\mathrm{T}\mathbf{p} - 2\mathbf{a}_\mathrm{i}^\mathrm{T}\mathbf{p} - 2\mathbf{a}_\mathrm{i}^\mathrm{T}\mathbf{R}\mathbf{b}_\mathrm{i} + \mathbf{a}_\mathrm{i}^\mathrm{T}\mathbf{a}_\mathrm{i} + \mathbf{b}_\mathrm{i}^\mathrm{T}\mathbf{b}_\mathrm{i} - L_\mathrm{i}^2 + L_1^2 = 0, \qquad \mathrm{i} = 2,...,6 \tag{6}$$

If we assemble the rotational parameters into the vector **c** = [c_1, c_2, c_3]$^\mathrm{T}$, the six equations (2) and (6) contain the translation vector **p** and the vector **c**. These variables should be computed to determine the poses of the moving platform for given six leg lengths.

For the convenience of the procedure in what follows, another translation vector **q** can be introduced as follows [3, 6]

$$\mathbf{q} = \mathbf{R}^\mathrm{T}\mathbf{p} \tag{7}$$

Accordingly, Eq. (6) can be rewritten as follows after divided by two

$$\mathbf{b}_\mathrm{i}^\mathrm{T}\mathbf{q} - \mathbf{a}_\mathrm{i}^\mathrm{T}\mathbf{p} - \mathbf{a}_\mathrm{i}^\mathrm{T}\mathbf{R}\mathbf{b}_\mathrm{i} + \frac{1}{2}\left(\mathbf{a}_\mathrm{i}^\mathrm{T}\mathbf{a}_\mathrm{i} + \mathbf{b}_\mathrm{i}^\mathrm{T}\mathbf{b}_\mathrm{i} - L_\mathrm{i}^2 + L_1^2\right) = 0, \qquad \mathrm{i} = 2,...,6 \tag{8}$$

From Eqs. (3) and (7), we obtain the following equation

$$(\mathbf{I}+\mathbf{C})\mathbf{q} - (\mathbf{I}-\mathbf{C})\mathbf{p} = \mathbf{0} \tag{9}$$

If we let **p**=[p_x, p_y, p_z]$^\mathrm{T}$ and **q**=[q_x, q_y, q_z]$^\mathrm{T}$, Eq. (9) can be written in scalar forms as follows

$$q_\mathrm{x} - c_3q_\mathrm{y} + c_2q_\mathrm{z} - p_\mathrm{x} - c_3p_\mathrm{y} + c_2p_\mathrm{z} = 0 \tag{10}$$

$$c_3q_\mathrm{x} + q_\mathrm{y} - c_1q_\mathrm{z} + c_3p_\mathrm{x} - p_\mathrm{y} - c_1p_\mathrm{z} = 0 \tag{11}$$

$$-c_2q_\mathrm{x} + c_1q_\mathrm{y} + q_\mathrm{z} - c_2p_\mathrm{x} + c_1p_\mathrm{y} - p_\mathrm{z} = 0 \tag{12}$$

Now, the vector **p**, **q**, and **c** are the unknowns of the nine equations (2), (8), and (10)-(12).

3. Elimination and Back-Substitution

3.1 Polynomials in c_1, c_2, and c_3 only

The translation parameters **p** and **q** in Eqs. (2), (8), and (10)-(12) can be eliminated leaving the polynomials in c_1, c_2, and c_3 only. Hereafter, the nine intermediate polynomials are introduced, from which 15 polynomials to constitute the 15×15 Sylvester’s matrix are derived.

Three polynomials Θ_i (i=1,2,3) and the substitution expressions for $c_1^{5-n}c_2^n$ (n = 0,...,5)

In matrix form, Eqs. (8) and (10)-(12) can be arranged as follows

$$\mathbf{Mu} = \begin{bmatrix} b_{21} & b_{22} & b_{23} & -a_{21} & -a_{22} & -a_{23} & F_2 \\ b_{31} & b_{32} & b_{33} & -a_{31} & -a_{32} & -a_{33} & F_3 \\ b_{41} & b_{42} & b_{43} & -a_{41} & -a_{42} & -a_{43} & F_4 \\ b_{51} & b_{52} & b_{53} & -a_{51} & -a_{52} & -a_{53} & F_5 \\ b_{61} & b_{62} & b_{63} & -a_{61} & -a_{62} & -a_{63} & F_6 \\ 1 & -c_3 & c_2 & -1 & -c_3 & c_2 & 0 \\ c_3 & 1 & -c_1 & c_3 & -1 & -c_1 & 0 \\ -c_2 & c_1 & 1 & -c_2 & c_1 & -1 & 0 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \\ q_z \\ p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \mathbf{0} \tag{13}$$

where

$$F_i = -\mathbf{a}_i^T \mathbf{R} \mathbf{b}_i + \frac{1}{2}\left(\mathbf{a}_i^T \mathbf{a}_i + \mathbf{b}_i^T \mathbf{b}_i - L_i^2 + L_1^2\right), \qquad i = 2,...,6 \tag{14}$$

and a_{ij} and b_{ij} (i=2,...,6, j=1,2,3) are the j-th component of vector $\mathbf{a}_i$ and $\mathbf{b}_i$. If we consider Eq. (13) as a linear system of six unknowns q_x, q_y, q_z, p_x, p_y, and p_z, the determinants of 7×7 minors of the 8×7 matrix **M** should be zero for the existence of a solution. If we apply this condition to the three square matrices formed by removing 6th, 7th, and 8th rows one by one from **M** in Eq. (13), the following three polynomials in c_1, c_2, and c_3 are obtained after rationalization

$$\Theta_1(c_1, c_2, c_3) \equiv \sum_{\substack{i=0,...,4;\, j,k=0,...,3 \\ i+j+k \le 4}} g_{1\text{-}ijk}\, c_1^i c_2^j c_3^k = \sum_{\substack{i=0,...,4;\, j=0,...,3 \\ i+j \le 4}} G_{1\text{-}ij}\, c_1^i c_2^j = 0 \tag{15}$$

$$\Theta_2(c_1, c_2, c_3) \equiv \sum_{\substack{i,k=0,...,3;\, j=0,...,4 \\ i+j+k \le 4}} g_{2\text{-}ijk}\, c_1^i c_2^j c_3^k = \sum_{\substack{i=0,...,3;\, j=0,...,4 \\ i+j \le 4}} G_{2\text{-}ij}\, c_1^i c_2^j = 0 \tag{16}$$

$$\Theta_3(c_1, c_2, c_3) \equiv \sum_{\substack{i,j=0,...,3;\, k=0,...,4 \\ i+j+k \le 4}} g_{3\text{-}ijk}\, c_1^i c_2^j c_3^k = \sum_{\substack{i,j=0,...,3 \\ i+j \le 3}} G_{3\text{-}ij}\, c_1^i c_2^j = 0 \tag{17}$$

where $g_{n\text{-}ijk}$ (n=1,2,3) are real constants depending on input data only, and the coefficients $G_{n\text{-}ij}$ (n=1,2,3) of the rewritten equations with respect to c_1 and c_2 are comprised of $g_{n\text{-}ijk}$ and c_3. As is seen, the polynomials (15)-(17) are all 4th-degree in c_1, c_2, and c_3, but there are no 4th-degree terms with respect to c_1 and c_2 in Eq. (17).

Using the six equations $\Theta_1 \times c_1 = 0$, $\Theta_2 \times c_1 = 0$, $\Theta_2 \times c_2 = 0$, $\Theta_3 \times c_1^2 = 0$, $\Theta_3 \times c_1 c_2 = 0$, and $\Theta_3 \times c_2^2 = 0$, we can make the following equation in matrix form

$$\begin{bmatrix} G_{1\text{-}40} & G_{1\text{-}31} & G_{1\text{-}22} & G_{1\text{-}13} & 0 & 0 \\ 0 & G_{2\text{-}31} & G_{2\text{-}22} & G_{2\text{-}13} & G_{2\text{-}04} & 0 \\ 0 & 0 & G_{2\text{-}31} & G_{2\text{-}22} & G_{2\text{-}13} & G_{2\text{-}04} \\ G_{3\text{-}30} & G_{3\text{-}21} & G_{3\text{-}12} & G_{3\text{-}03} & 0 & 0 \\ 0 & G_{3\text{-}30} & G_{3\text{-}21} & G_{3\text{-}12} & G_{3\text{-}03} & 0 \\ 0 & 0 & G_{3\text{-}30} & G_{3\text{-}21} & G_{3\text{-}12} & G_{3\text{-}03} \end{bmatrix} \begin{bmatrix} c_1^5 \\ c_1^4 c_2 \\ c_1^3 c_2^2 \\ c_1^2 c_2^3 \\ c_1 c_2^4 \\ c_2^5 \end{bmatrix} = - \begin{bmatrix} \overline{\Theta}_1 \times c_1 \\ \overline{\Theta}_2 \times c_1 \\ \overline{\Theta}_2 \times c_2 \\ \overline{\Theta}_3 \times c_1^2 \\ \overline{\Theta}_3 \times c_1 c_2 \\ \overline{\Theta}_3 \times c_2^2 \end{bmatrix} \tag{18}$$

where

$$\overline{\Theta}_1 = \sum_{\substack{i,j=0,...,3\\ i+j\le 3}} G_{1\text{-ij}}\, c_1^{i} c_2^{j}, \qquad \overline{\Theta}_2 = \sum_{\substack{i,j=0,...,3\\ i+j\le 3}} G_{2\text{-ij}}\, c_1^{i} c_2^{j}\,, \qquad \overline{\Theta}_3 = \sum_{\substack{i,j=0,1,2\\ i+j\le 2}} G_{3\text{-ij}}\, c_1^{i} c_2^{j} \tag{19}$$

Solving the system (18) symbolically with regarding all the power products $c_1^{5\text{-n}} c_2^{n}\,(n = 0,...,5)$ as linear unknowns, we obtain the following expressions

$$c_1^{5\text{-n}} c_2^{n} = \sum_{\substack{i,j=0,...,4\\ 1\le i+j\le 4}} U_{\text{n-ij}}\, c_1^{i} c_2^{j}, \qquad n = 0,...,5 \tag{20}$$

where

$$U_{\text{n-ij}}(c_3) = \sum_{k=0}^{5-i-j} u_{\text{n-ijk}}\, c_3^{k} \tag{21}$$

and all $u_{\text{n-ijk}}$ are real constants depending on the input data only.

Equation (20) provides a way to keep the degree of all equations obtained hereafter, within four with respect to c_1 and c_2.

Six polynomials Φ_i (i=1,...,6)

In addition to Eqs. (15)-(17), there should be one or more polynomials that reflect Eq. (2). Taking one of the three equations (10)-(12) in turn, together with the five equations of (8), we can make three linear systems and the first one, for example, is as follows

$$\begin{bmatrix} b_{21} & b_{22} & b_{23} & -a_{21} & -a_{22} & -a_{23} & F_2 \\ b_{31} & b_{32} & b_{33} & -a_{31} & -a_{32} & -a_{33} & F_3 \\ b_{41} & b_{42} & b_{43} & -a_{41} & -a_{42} & -a_{43} & F_4 \\ b_{51} & b_{52} & b_{53} & -a_{51} & -a_{52} & -a_{53} & F_5 \\ b_{61} & b_{62} & b_{63} & -a_{61} & -a_{62} & -a_{63} & F_6 \\ 1 & -c_3 & c_2 & -1 & -c_3 & c_2 & 0 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \\ q_z \\ p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \mathbf{0} \tag{22}$$

Solving the three systems respectively, we can determine as many symbolic solutions $\mathbf{p}_n$ (n=1,2,3) for $\mathbf{p}$ in terms of c_1, c_2, and c_3. If we set $\mathbf{p}_n = [\, p_{nx}, p_{ny}, p_{nz}\,]^T = [\, p_{n1}, p_{n2}, p_{n3}\,]^T$ for the convenience of expression, each component can be expressed in the following form

$$p_{nk} = \frac{N_{pnk}(c_1, c_2, c_3)}{D_{pn}(c_1, c_2, c_3)} = \frac{1}{D_{pn}}\left(N_{nk,0} + \sum_{i=0}^{1} N_{nk,1i} c_1^{1\text{-i}} c_2^{i} + \sum_{i=0}^{2} N_{nk,2i} c_1^{2\text{-i}} c_2^{i} + \sum_{i=0}^{3} m_{nk,3i} c_1^{3\text{-i}} c_2^{i} \right), \quad n,k = 1,2,3 \tag{23}$$

where

$$D_{pn} = (1 + c_1^2 + c_2^2 + c_3^2)\,(d_{n,0} + d_{n,1} c_1 + d_{n,2} c_2 + d_{n,3} c_3) \tag{24}$$

$$N_{nk,0} = \sum_{j=0}^{3} m_{nk,0j} c_3^{j}, \quad N_{nk,1i} = \sum_{j=0}^{2} m_{nk,1ij} c_3^{j}, \quad N_{nk,2i} = \sum_{j=0}^{1} m_{nk,2ij} c_3^{j} \tag{25}$$

and d's and m's with subscripts are real coefficients determined by input data only. Using the fact $a/b=c/d=(a+c)/(b+d)$, we can obtain three additional solutions $\mathbf{p}_n$ (n=4,5,6) for $\mathbf{p}$ as follows

$$\mathbf{p}_4 = \frac{1}{D_{p1}+D_{p2}}[N_{p11}+N_{p21}, \quad N_{p12}+N_{p22}, \quad N_{p13}+N_{p23}]^T \tag{26}$$

$$\mathbf{p}_5 = \frac{1}{D_{p1}+D_{p3}}[N_{p11}+N_{p31}, \quad N_{p12}+N_{p32}, \quad N_{p13}+N_{p33}]^T \tag{27}$$

$$\mathbf{p}_6 = \frac{1}{D_{p2}+D_{p3}}[N_{p21}+N_{p31}, \quad N_{p22}+N_{p32}, \quad N_{p23}+N_{p33}]^T \tag{28}$$

Now, substituting $\mathbf{p}_n$ (n=1,...,6) for $\mathbf{p}$ into Eq. (2) in turn, leads to the following six polynomials of degree 6 in c_1, c_2, and c_3 after rationalization

$$\Phi_n(c_1,c_2,c_3) \equiv \sum_{\substack{i,j,k=0,\ldots,6 \\ i+j+k\le 6}} h_{n\text{-}ijk}\, c_1^i c_2^j c_3^k = 0, \qquad n=1,\ldots,6 \tag{29}$$

where $h_{n\text{-}ijk}$ (n=1,...,6) are real constants depending on input data only.

3.2 Sylvester's matrix and a univariate polynomial in c_3

As aforementioned, considering the degree with respect to c_1 and c_2, we can transform all the equations of degree above 4 into the equations of degree within 4 with the aid of Eq. (20). That is, all the terms $c_1^{6-i}c_2^i$ $(i=0,\ldots,6)$ and $c_1^{5-i}c_2^i$ $(i=0,\ldots,5)$ of Eq. (29) can be successively removed, leaving the terms $c_1^i c_2^j$ $(i,j=0,\ldots,4,\ i+j\le 4)$ only. As a result, the equations can be written in the following form

$$\Phi'_n(c_1,c_2,c_3) \equiv \sum_{\substack{i,j=0,\ldots,4;k=0,\ldots,6 \\ i+j\le 4;i+j+k\le 6}} h'_{n\text{-}ijk}\, c_1^i c_2^j c_3^k = \sum_{\substack{i,j=0,\ldots,4 \\ i+j\le 4}} H_{n\text{-}ij}\, c_1^i c_2^j = 0, \qquad n=1,\ldots,6 \tag{30}$$

where $h'_{n\text{-}ijk}$ are real constants depending on input data only, and the coefficients $H_{n\text{-}ij}$ of the rewritten equations are comprised of $h'_{n\text{-}ijk}$ and c_3. If we do the same substitution again for the additional equations $\Phi'_1\times c_1=0$, $\Phi'_1\times c_2=0$, $\Phi'_2\times c_1=0$, $\Phi'_2\times c_2=0$, and $\Phi'_1\times c_1c_2=0$, those five equations can be written, in turn, in the following form

$$\Phi''_n(c_1,c_2,c_3) \equiv \sum_{\substack{i,j=0,\ldots,4;k=0,\ldots,6 \\ 1\le i+j\le 4;i+j+k\le 7}} h''_{n\text{-}ijk}\, c_1^i c_2^j c_3^k = \sum_{\substack{i,j=0,\ldots,4 \\ 1\le i+j\le 4}} H'_{n\text{-}ij}\, c_1^i c_2^j = 0, \qquad n=1,\ldots,4 \tag{31}$$

$$\Phi''_5(c_1,c_2,c_3) \equiv \sum_{\substack{i,j=0,\ldots,4;k=0,\ldots,7 \\ 1\le i+j\le 4;i+j+k\le 8}} h''_{5\text{-}ijk}\, c_1^i c_2^j c_3^k = \sum_{\substack{i,j=0,\ldots,4 \\ 1\le i+j\le 4}} H'_{5\text{-}ij}\, c_1^i c_2^j = 0 \tag{32}$$

where $h''_{n\text{-}ijk}$ (n=1,...,5) are real constants depending on input data only, and the coefficients $H'_{n\text{-}ij}$ of the rewritten equations are comprised of $h''_{n\text{-}ijk}$ and c_3.

In order to derive a univariate polynomial in c_3, we are required to construct a Sylvester's matrix containing c_3 only. The following 15 intermediate polynomials are the best set out of the tested ones to constitute the matrix leading to the univariate polynomial of exact degree 40.

$$\Theta_n = 0, \quad (n = 1,2,3) \tag{33}$$

$$\Theta_3 \times c_1 = 0, \quad \Theta_3 \times c_2 = 0 \tag{34}$$

$$\Phi'_n = 0, \quad (n = 1,...,6) \tag{35}$$

$$\Phi''_n = 0, \quad (n = 1,2,5) \tag{36}$$

$$g_{3-301}\Phi''_3 + g_{3-211}\Phi''_4 = 0 \tag{37}$$

where g_{3-301} and g_{3-211} are the real coefficients of $c_1^3 c_3$ and $c_1^2 c_2 c_3$ of Eq. (17). Considering the power product of $c_1^i c_2^j$ $(i, j = 0,...,4, i + j \le 4)$ as linear unknowns, we can arrange Eqs. (33)-(37) in the following matrix form

$$\mathbf{Qw} = \mathbf{0} \tag{38}$$

where **Q** is the 15×15 coefficient matrix of one unknown c_3, and its elements are $G_{n\text{-}ij}$ (n=1,2,3), $H_{n\text{-}ij}$ (n=1,...,6), $H'_{n\text{-}ij}$ (n=1,2,5), and $g_{3-301}H'_{3-ij} + g_{3-211}H'_{4-ij}$ shown in Eqs. (15)-(17) and (30)-(32). **w** is the 15 element vector containing all the power products $c_1^i c_2^j$ $(i, j = 0,...,4, i + j \le 4)$.

If we check the number of equations and linear unknowns (power products) of Eq. (38), there should be linear dependency for the system to have a solution. Hence, the following condition must be satisfied

$$\det(\mathbf{Q}) = 0 \tag{39}$$

The condition (39) directly leads to a 40th-degree univariate equation. That is

$$\sum_{i=0}^{40} s_i c_3^i = 0 \tag{40}$$

where s_i (i=0,...,40) are real constants depending on input data only. Equation (40) gives 40 roots of c_3 in the complex domain.

3.3 Back substitution

By removing any one row from the matrix **Q** in Eq. (38), we can make the linear system comprised of 14×15 coefficient matrix and 15 power products. Since $c_1^0 c_2^0 (=1)$ is known, the system can be considered as linear equations of 14 unknowns. Therefore, for the 40 roots of c_3 of the univariate equation (40), we can obtain the corresponding values of c_1 and c_2 by solving the respective linear systems. Then, if we substitute the values of c_1, c_2, and c_3 into Eq. (23), the values of translational parameter **p** are computed.

4. Implementation and a Numerical Example

4.1 Implementation of the proposed algorithm

The proposed algorithm has been implemented in C^{++} computer language. However, since the maximum number of digits is within 15 on PC base, an extra class-type data [7] of 30 digits is adopted here to diminish the round-off errors. All the symbolic calculation of determinant is replaced with numerical expansion of the polynomial coefficients [3, 8]. By numerical determinants of matrices formed by substituting 41 arbitrarily selected values for c_3 in the 15×15 Sylvester's matrix in Eq. (38), a set of 41 linear equations is derived with the unknowns of the 41 coefficients of the univariate polynomial. Solving this system determines the univariate polynomial, in which we can save computation time by using a predetermined inverse of the 41×41 coefficient matrix of the linear set. For solving the univariate polynomial, a function in Cephes Math Library [9] is used, which shows fast performance and high accuracy. The total required time to compute all the solutions by the implemented algorithm is within 0.025s on a PC (PentiumIII-600MHz) and almost uniform computation time is maintained over various examples.

4.2 Numerical example

For the set of leg lengths and geometrical parameters of the Stewart-Gough platform given in Table 1, the 40 solutions in the complex domain are calculated. However, the actual number of poses the platform can have is 8, since out of the 40 solutions only 8 ones are real as shown in Table 2.

Table 1. Input data for example

i	$\mathbf{a}_i^T$	$\mathbf{b}_i^T$	L_i
1	[0, 0, 0]	[0, 0, 0]	14
2	[5, 0, 0]	[4, 0, 0]	12
3	[12, -15, 0]	[8, -6, 0]	17
4	[18, -6, 3]	[13, -3, -5]	15
5	[20, 1, -3]	[14, 5, 2]	23
6	[10, 8, 5]	[6, 10, 3]	19

Table 2. All the solutions for example

Sols.	Rotational parameters **c**			Translational parameters **p**		
	c_1	c_2	c_3	p_x	p_y	p_z
1	-0.0580	-0.9158	-0.0201	-2.2081	-1.3658	-13.7571
2	1.4357	-1.7068	-0.6716	6.3779	0.7328	-12.4413
3	-3.7761	2.9783	0.4853	2.1076	3.3472	13.4296
4	-0.3979	0.4307	0.5806	-2.5981	-2.8977	13.4482
5	0.6420	0.1643	0.7277	8.3596	-6.4555	9.1893
6	-0.5600	-0.9822	0.6016	0.7725	-13.7260	2.6457
7	0.1817	0.0454	-1.0664	6.8571	0.2821	12.2025
8	6.0419	-4.6719	2.9816	13.1037	-0.9971	4.8270
9,10	$0.2458\mp0.3252i$	$-0.6699\pm0.0119i$	$0.1009\mp0.0075i$	$-3.4158\pm0.7220i$	$4.1701\mp0.5634i$	$-12.9584\mp0.3716i$
11,12	$0.3270\pm1.0826i$	$0.2694\mp0.2255i$	$0.0325\mp0.1788i$	$29.4259\mp43.7380i$	$16.6072\mp33.0945i$	$56.0151\pm32.7884i$
13,14	$0.6269\pm0.2664i$	$0.2875\pm0.1194i$	$0.0177\pm0.1905i$	$-2.6573\pm27.5144i$	$10.2114\pm2.3910i$	$29.1585\pm1.6701i$
15,16	$0.2496\mp0.8634i$	$0.1412\pm0.3450i$	$-0.1924\pm0.0709i$	$13.5427\mp27.0328i$	$29.7614\pm18.2159i$	$17.1647\mp10.2555i$
17,18	$0.0512\pm0.4917i$	$-0.5026\pm0.1708i$	$0.1428\mp0.2192i$	$10.9433\mp17.1196i$	$14.9244\pm9.6792i$	$-15.7384\mp2.7250i$
19,20	$0.2137\pm0.1105i$	$0.4543\pm0.2486i$	$0.0426\pm0.2984i$	$11.9804\pm28.6754i$	$18.3316\mp5.2268i$	$25.6734\mp9.6492i$
21,22	$-0.2962\pm0.7799i$	$-0.9344\pm1.0915i$	$0.8978\pm0.6664i$	$12.4876\pm1.1385i$	$6.9045\mp1.9010i$	$0.6211\mp1.7571i$
23,24	$-0.0235\pm0.7989i$	$0.1011\pm0.7814i$	$-1.0031\pm0.1117i$	$27.0476\pm0.5335i$	$37.1806\pm3.1714i$	$-3.0229\pm43.7809i$
25,26	$-0.1983\mp1.0204i$	$-0.4870\pm0.8927i$	$0.0590\pm0.3049i$	$94.9024\mp53.9004i$	$8.7475\mp29.1278i$	$-58.1829\mp92.2958i$
27,28	$0.2376\pm0.8210i$	$-0.1102\mp0.5150i$	$-0.6812\pm0.1759i$	$-12.8176\pm4.4635i$	$19.8221\pm2.7490i$	$-0.1490\mp18.2688i$
29,30	$0.3786\pm0.7400i$	$-0.2442\mp0.7984i$	$0.8540\pm0.0175i$	$16.2959\pm4.2539i$	$-11.4286\pm14.8188i$	$-10.9792\mp9.1115i$
31,32	$-0.1959\pm1.0147i$	$-0.0942\pm1.0348i$	$-1.0874\pm0.2039i$	$-0.7630\pm2.1869i$	$-15.2060\pm32.5272i$	$-35.0374\mp14.1642i$
33,34	$0.2514\pm0.6691i$	$-0.2267\pm1.0117i$	$1.0519\pm0.2127i$	$-1.2056\pm2.3653i$	$-26.1047\mp4.3756i$	$-5.0436\pm22.0818i$
35,36	$-0.4422\pm0.1136i$	$-0.2287\pm0.1239i$	$-1.1639\pm0.0136i$	$12.2063\mp0.6634i$	$0.1520\pm0.4124i$	$6.9939\pm1.1488i$
37,38	$-0.5138\mp0.7758i$	$0.5700\pm0.8531i$	$0.8293\mp0.2358i$	$-4.1178\pm0.3837i$	$4.7174\pm15.1518i$	$19.9691\mp3.5002i$
39,40	$0.0476\pm1.7023i$	$0.4890\mp1.4534i$	$-0.5991\mp0.9204i$	$12.9734\mp0.5431i$	$-4.3507\pm2.0945i$	$-4.9218\mp3.2832i$

5. Conclusion

This paper presents an algebraic elimination method for the forward kinematics of the general Stewart-Gough platform, which directly derives a 40th-degree univariate polynomial, having no extraneous factors, from the 15×15 Sylvester's matrix. Since the method requires determinant calculation of smaller square matrices, the computation time is greatly reduced compared to the existing algorithms. In this paper, it has been implemented in C^{++} language using the class-type data of 30 significant digits in order that the solutions have enough accuracy. Further enhancement in computation time is expected if it is implemented in other computer environment that supports 16-byte precision data.

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