Sampling and Convolution on Motion Groups Using Generalized Gaussian Functions

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Abstract

Most engineers are familiar with the concepts of convolution, Gaussian functions and the heat equation on the line. These concepts all extend when dealing with functions on the Lie groups of interest in kinematics (i.e., the rotation and rigid-body-motion groups). In this paper we collect some results concerning differential operators and partial differential equations on the rotation group and three-sphere. We present these results (which have long been known in the mathematics literature) in a manner that is palatable to kinematicians. That is, we do not introduce excessive new notation or definitions. We then build a new sampling theory for the group of rigid-body motions on these results.

1 Introduction

In this paper we develop a sampling theory for functions on the group of rigid-body motions[1], and illustrate why this is a useful computational tool in kinematics.

In a number of recent papers, it has been shown how the workspace boundaries of manipulators in general, and workspace density function for binary manipulators[2] in particular, can be generated in a computationally efficient manner using the concept of group-theoretical convolution [1, 2]. Inverse problems associated with manipulator design can be solved using techniques of noncommutative harmonic analysis [3]. Images of planar and spatial binary manipulators in action can be found at the website: *http://caesar.me.jhu.edu*.

The scope of applications for which techniques from harmonic analysis are useful has been expanded to include the generation of configuration-space obstacles for single-body mobile robots, statistical mechanics of chain molecules and the radiotherapy treatment planning problem [4]. Fast Fourier transform techniques for groups reduce even further the time required to compute the generalized convolutions that arise in these various application areas. Such techniques have been derived for a variety of finite groups and compact Lie groups, but have yet to be implemented for noncompact noncommutative groups like $SE(d)$.

In all of those previous works, it is assumed that the density of points (or frames) is large enough so that direct discretization of $\mathbb{R}^d$ or $SE(d)$ leads to histograms that accurately reflect the density. While this is valid for the case when a large number of points (or frames) occupy the space under consideration, it can lead to unacceptable errors and memory overhead when points are too sparsely dispersed. The technique presented here is an alternative to discretization by counting points and forming histograms. Instead, the approach we take here is to allow each point (or frame) to diffuse. Hence, if each point is originally viewed as a Dirac delta function, after it diffuses it will be a Gaussian. Gaussians convolved with Gaussians result in functions of the same form, and the result of a convolution can be resampled and replaced by a sum of a smaller number of Gaussians. This leads to a sampling method for the motion groups.

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[1]Often called the Euclidean Motion group, or Special Euclidean Group, and denoted as $SE(d)$, with $d = 2, 3$ of primary interest.
[2]Manipulators composed of a serial cascade of units containing two-state actuators

The contribution of this paper is to formulate exactly what is meant by a Gaussian function on the Euclidean groups of the plane and three space, to present the corresponding sampling technique, and to illustrate the technique with numerical examples.

The remainder of this paper is formulated as follows. Section 2 discusses diffusion equations on the circle and quaternion sphere, and defines Gaussians as the solution of these equations. Section 3 examines properties of convolution on groups. Section 4 shows how the convolution of functions on the motion groups can be decomposed into sums of generalized Gaussians. Section 5 presents a sampling technique for sums of Gaussians.

2 Rotational Diffusion and Averaging

2.1 Diffusion on the Circle and the Rotation Group $SO(2)$

In this subsection we review diffusion on the circle. The purpose of this subsection is purely to serve as a connection between what is commonly known to engineers, and that which is presented in the next subsection, which is not known to most engineers.

Rotations in the plane are given by matrices of the form

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

All the information contained in the rotation matrix R is also contained in the unit vector

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},$$

and so we write

$$R(\mathbf{r}) = \begin{bmatrix} r_1 & -r_2 \\ r_2 & r_1 \end{bmatrix}.$$

Similarly, the vector $\mathbf{r}$ is extracted from the rotation matrix R as $\mathbf{r}(R) = R\mathbf{e}_1$ where $\mathbf{e}_1 = [1, 0]^T$.

The identity rotation then corresponds to the vector $\mathbf{r} = \mathbf{e}_1$, and an infinitesimal rotation close to the identity is of the form

$$R(\mathbf{e}_1 + \epsilon\mathbf{e}_2) = \mathbb{I}_{2\times 2} + \epsilon X,$$

where

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbb{I}_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In general we will denote the $n \times n$ identity matrix as $\mathbb{I}_{n\times n}$.

We note that $R(\mathbf{r}(\theta)) = \exp(\theta X)$ where $\exp(\cdot)$ is the matrix exponential.

Given a real-valued function, $f(\cdot)$, that takes its argument from the unit circle (we will use both the notation $f(\theta)$ and $f(\mathbf{r}(\theta))$ to mean the same thing) one defines the differential operator

$$Df = \frac{df}{da}(\exp(aX)\mathbf{r}(\theta))|_{a=0}.$$

It is easy to see that

$$Df = \frac{df}{da}(\mathbf{r}(a+\theta))|_{a=0} = \frac{df}{da}(a+\theta)|_{a=0} = \frac{df}{d\theta}.$$

If f is a function of time, t, in addition to θ, then the full derivative above is replaced with a partial one, and the Laplacian is defined as

$$\nabla^2 f = D^2 f = \frac{\partial^2 f}{\partial\theta^2}.$$

The heat equation on the circle is then

$$\frac{\partial f}{\partial t} = K\nabla^2 f.$$

It is well known that the Fourier series solution of this equation under the initial condition $f(\theta,0) = \delta(\theta - 0)$ is of the form

$$f(\theta,t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-n^2Kt}e^{in\theta} = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2Kt}\cos n\theta. \tag{1}$$

Another well-known form of the solution to the heat equation on the circle is what results from "wrapping" the solution of the heat equation on the line [3],

$$F(x,t) = \frac{1}{\sqrt{2\pi Kt}}e^{-x^2/2Kt}$$

for initial conditions $F(x,0) = \delta(x-0)$, around the circle. That is, shifting all intervals on the line of the form $[2\pi n, 2\pi(n+1)]$ for $n \in \mathbb{Z}$ to the interval $[0,2\pi]$, and superposing the values of the function. This is written as

$$f(\theta,t) = \sum_{n=-\infty}^{\infty} F(\theta - 2\pi n). \tag{2}$$

A nice feature of the expansion in Equation (2) is that when Kt is small, only one or at most a few terms in the expansion need to be retained since the Gaussian function decays so rapidly. Another nice feature is that the function $f(\theta,t)$ is always positive (as it should be) when using this expansion, whereas negative values and convergence problems are likely to occur when using truncated Fourier expansions for small values of Kt. To illustrate this, Simulation 1 depicts the series in (1) and (2) truncated at $n = 5$, and animated with values of Kt ranging from 0.01 to 100.

2.2 Diffusion on the Three-Sphere and the Rotation Group $SO(3)$

Spherical coordinates in $\mathbb{R}^4$ can be chosen as

$$\mathbf{r}(\theta_1,\theta_2,\theta_3) = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1\cos\theta_2 \\ \sin\theta_1\sin\theta_2\cos\theta_3 \\ \sin\theta_1\sin\theta_2\sin\theta_3 \end{bmatrix}$$

for $\theta_1 \in [0,\pi]$, $\theta_2 \in [0,\pi]$, $\theta_3 \in [0,2\pi]$.

Using the four-dimensional matrix representation of rotation corresponding to $\mathbf{r}$, i.e., by defining $R_{4\times4}(r)$ such that

$$R_{4\times4}(r)q = rq$$

is consistent with the rules of quaternion multiplication, one has:

$$R_{4\times4}(\mathbf{r}) = \begin{bmatrix} r_1 & -r_2 & -r_3 & -r_4 \\ r_2 & r_1 & -r_4 & r_3 \\ r_3 & r_4 & r_1 & -r_2 \\ r_4 & -r_3 & r_2 & r_1 \end{bmatrix}.$$

Hence, $R_{4\times4}(\mathbf{e}_1)$ is the identity matrix, and corresponds to no rotation. If one considers infinitesimal motions on the sphere S^3 in each of the remaining coordinate directions in the vicinity of the point $\mathbf{r} = \mathbf{e}_1$, the corresponding rotations will be of the form

$$R_{4\times4}(\mathbf{e}_1 + \epsilon\mathbf{e}_i) = \mathbb{I}_{4\times4} + \epsilon X_{i-1}$$

[3] Called a Gaussian or normal distribution

for $i = 2, 3, 4$ and $|\epsilon| << 1$. It is easy to see that $X_{i-1} = R_{4\times 4}(\mathbf{e}_i)$, and explicitly have the form

$$X_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad X_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}; \quad X_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

In analogy with the way the cross product in $I\!R^3$ takes orthonormal basis vectors into orthonormal basis vectors as $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$, $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$, one finds that under the Lie bracket operation (which in this case we take to be half of the matrix commutator),

$$[X_i, X_j] = \frac{1}{2}(X_i X_j - X_j X_i),$$

that

$$[X_1, X_2] = X_3; \quad [X_2, X_3] = X_1; \quad [X_3, X_1] = X_2.$$

This indicates that the matrices X_i are representations of the basis elements of the Lie algebra $so(3)$. These elements are not unique, as can be seen by the fact that

$$R_{4\times 4}(\mathbf{e}_1 + \epsilon R(q)\mathbf{e}_i) = \mathbb{1}_{4\times 4} + \epsilon R_{4\times 4}(q) X_{i-1} R^T_{4\times 4}(q).$$

Hence any set of basis elements of the form $\tilde{X}_i = R_{4\times 4}(q) X_i R^T_{4\times 4}(q)$ is equally acceptable.

Differential motions on the surface of the sphere S^3 can be used to define differential operators that act on real valued-functions of the form $f(\mathbf{r}(\theta_1, \theta_2, \theta_3))$ as:

$$D_i f = \frac{d}{dt} f(\exp(tX_i)\mathbf{r}(\theta_1, \theta_2, \theta_3))|_{t=0}$$

for $i = 1, 2, 3$. Here

$$\exp(A) = \mathbb{1}_{4\times 4} + \sum_{n=1}^{\infty} \frac{A^n}{n!}$$

is the matrix exponential for any $A \in I\!R^{4\times 4}$.

After a little work, it can be shown that the explicit form of these operators (in a basis, $\{\tilde{X}_i\}$ different than ours) is [7]

$$\tilde{D}_1 = -\cos\theta_2 \frac{\partial}{\partial\theta_1} + \sin\theta_2 \cot\theta_1 \frac{\partial}{\partial\theta_2} - \frac{\partial}{\partial\theta_3};$$

$$\tilde{D}_2 = -\sin\theta_2 \cos\theta_3 \frac{\partial}{\partial\theta_1} + (\sin\theta_3 - \cot\theta_1 \cos\theta_2 \cos\theta_3) \frac{\partial}{\partial\theta_2} + \left(\cot\theta_1 \frac{\sin\theta_3}{\sin\theta_2} + \cot\theta_2 \cos\theta_3\right) \frac{\partial}{\partial\theta_3};$$

$$\tilde{D}_3 = -\sin\theta_2 \sin\theta_3 \frac{\partial}{\partial\theta_1} - (\cos\theta_3 + \cot\theta_1 \cos\theta_2 \sin\theta_3) \frac{\partial}{\partial\theta_2} + \left(-\cot\theta_1 \frac{\cos\theta_3}{\sin\theta_2} + \cot\theta_2 \sin\theta_3\right) \frac{\partial}{\partial\theta_3}.$$

The Laplacian, which is invariant under the choice of orthonormal basis in $so(3)$, is defined as

$$\nabla^2 = (D_1)^2 + (D_2)^2 + (D_3)^2 = (\tilde{D}_1)^2 + (\tilde{D}_2)^2 + (\tilde{D}_3)^2 =$$

$$\frac{\partial^2}{\partial\theta_1^2} + 2\cot\theta_1 \frac{\partial}{\partial\theta_1} + \frac{\cot\theta_2}{\sin^2\theta_1} \frac{\partial}{\partial\theta_2} + \frac{1}{\sin^2\theta_1} \frac{\partial^2}{\partial\theta_2^2} + \frac{1}{\sin^2\theta_1 \sin^2\theta_2} \frac{\partial^2}{\partial\theta_3^2}.$$

The heat equation is

$$\frac{\partial f}{\partial t} = K\nabla^2 f. \tag{4}$$

The solution of this equation for $f(\theta_1, \theta_2, \theta_3, t)$ under the initial condition that $f(\theta_1, \theta_2, \theta_3, 0)$ is a Dirac delta function on S^3 has been known for many years. It was first derived by Perrin [8] in the late 1920s, and has been elaborated on by many mathematicians since then. Perrin's solution is of the form

$$f(\theta_1, \theta_2, \theta_3, t) = \frac{1}{\pi^2 \sin\theta_1} \sum_{n=0}^{\infty} (2n+1) e^{-[(2n+1)^2 - 1](Kt)} \sin(2n+1)\theta_1. \tag{5}$$

In a subsequent section we shall present (we believe for the first time in the literature) an alternative form of the solution for small values of Kt which is analogous to the wrapped (folded) Gaussian for the case of the circle. It is this alternate form that will be central to our formulation of a sampling method for efficient generation of convolutions on $SE(3)$. The main issues regarding these convolutions are discussed in the following section.

3 Convolution on Rotation and Motion Groups

Given a group $(G, \circ)$ for which a left and right invariant integration measure ($d\mu(g) = d\mu(g \circ h) = d\mu(h \circ g)$ for all $h, g \in G$) exists[4], the convolution of functions square-integrable with respect to this measure is defined as

$$(f_1 \star f_2)(g) = \int_G f_1(h) f_2(h^{-1} \circ g) d\mu(h). \tag{6}$$

Here $\circ$ is the group operation, which is matrix multiplication in the case of rotations and rigid-body motions represented as homogeneous transforms. The geometric meaning and applications of this kind of integral when $G = SE(d)$ has been studied extensively in [2, 3, 4]. In all the discussions that follow, G is assumed to be this kind of a group, which is called a *unimodular* group. In general, the convolution of arbitrary square-integrable functions on such a noncommutative group is not commutative. A notable exception to this is when both functions are class functions, i.e., functions satisfying $f(g \circ h) = f(h \circ g)$ for all $g, h \in G$. This is equivalent to the statement $f(h^{-1} \circ g \circ h) = f(g)$, which clearly shows that a class function is invariant under conjugation, and hence is constant on *conjugacy classes* of the group (hence the name *class* function).

One example of a class function on a unimodular group is the Dirac delta. In complete analogy with the usual Dirac delta function on the real line, we have the properties

$$\int_G \delta(g) d\mu(g) = 1 \qquad \delta(h^{-1} \circ g) = \delta(g^{-1} \circ h)$$

and

$$\int_G f(h)\delta(h^{-1} \circ g) d\mu(h) = f(g).$$

Using these properties, it is easy to see that the convolution of a function with a shifted delta function, $\delta(g_i^{-1} \circ g)$, results in a shifted version of the original function.

4 Decomposition into Basic Convolutions

In what follows, rigid-body motions are denoted as the translation-rotation pair $g = (\mathbf{p}, q)$ where q is the quaternion describing rotation.

4.1 Decomposition

4.1.1 Distributivity

Suppose that in the convolution $\varphi'(f') = \psi(g) \star \varphi(f)$ the density functions consist of sums of more *basic functions*

$$\psi(g) = \sum_{j=1}^{n} \psi_j(g_j^{-1} g) \tag{7}$$

$$\varphi(f) = \sum_{i=1}^{m} \varphi_i(f_i^{-1} f). \tag{8}$$

[4]This includes the rotation and motion groups of all $I\!R^d$ for $d = 1, 2, 3, \ldots$

Then, the sums can be taken out of the convolution since integrals and sums commute

$$\left[\sum_{j=1}^{n}\psi_j(g_j^{-1}g)\right]\star\left[\sum_{i=1}^{m}\varphi_i(f_i^{-1}f)\right]=\sum_{i=1}^{m}\sum_{j=1}^{n}\psi_j(g_j^{-1}g)\star\varphi_i(f_i^{-1}f) \tag{9}$$

4.1.2 Normalization

The next possible simplification is to "center" the density function $\psi_j(g_j^{-1}g)$

$$\psi_j(g_j^{-1}g)=\delta(g_j^{-1}g)\star\psi_j(g) \tag{10}$$

The same can be applied to the density function $\varphi_j(f_j^{-1}f)$.

4.1.3 Swapping of Rotations

The expression $\delta(f_i^{-1}f)$ for $f_i=(\mathbf{p}_i,q_i)$ means that a rotation $\delta(q_i^{-1}f)$ is followed by a translation $\delta(\mathbf{p}_i^{-1}f)$. In other words

$$\varphi_i(f_i^{-1}f)=\delta(\mathbf{p}_i^{-1}f)\star\delta(q_i^{-1}f)\star\varphi_i(f). \tag{11}$$

Now, assume that $\varphi_i(f)$ for $f=(\mathbf{p},q)$ may be written as

$$\varphi_i(f)=\varphi_i(\mathbf{p})\varphi_i(\sigma) \tag{12}$$

where $p=\sqrt{\mathbf{p}^T\mathbf{p}}$ and $\sigma=\arccos(q_1)$. In this case, the rotation part commutes and one obtains

$$\varphi_i(f_i^{-1}f)=\delta(\mathbf{p}_i^{-1}f)\star\varphi_i(f)\star\delta(q_i^{-1}f). \tag{13}$$

4.1.4 Reduction of Translations

The translation component $\delta(\mathbf{p}_i^{-1}f)$ is determined by the vector $\mathbf{p}_i$ which has a direction $R(r_{p_i})\mathbf{e}_1$ and a magnitude p_i, where r_{p_i} is the quaternion that describes a rotation that converts $(p_i,0,0)$ to $\mathbf{p}_i$. Thus, a translation may be understood as [5]

$$\delta(\mathbf{p}_i^{-1}f)=\delta(r_{p_i}^{-1}f)\star\delta(p_i^{-1}f)\star\delta(r_{p_i}f) \tag{14}$$

Taking into account this property, one obtains

$$\varphi_i(f_i^{-1}f)=\delta(r_{p_i}^{-1}f)\star\delta(p_i^{-1}f)\star\delta(r_{p_i}f)\star\varphi_i(f)\star\delta(q_i^{-1}f) \tag{15}$$

which is also

$$\varphi_i(f_i^{-1}f)=\delta(r_{p_i}^{-1}f)\star\delta(p_i^{-1}f)\star\varphi_i(f)\star\delta(r_{p_i}f)\star\delta(q_i^{-1}f) \tag{16}$$

or, finally

$$\varphi_i(f_i^{-1}f)=\delta(r_{p_i}^{-1}f)\star\delta(p_i^{-1}f)\star\varphi_i(f)\star\delta(q_i^{-1}r_{p_i}f). \tag{17}$$

4.1.5 First Basic Convolution

Putting things together, the decomposition process leads to

$$\varphi'(f')=\sum_{i=1}^{m}\sum_{j=1}^{n}\delta(g_j^{-1}g)\star\psi_j(g)\star\delta(r_{p_i}^{-1}f)\star\delta(p_i^{-1}f)\star\varphi_i(f)\star\delta(q_i^{-1}r_{p_i}f). \tag{18}$$

[5] ${p_i}^{-1}f$ is shorthand for $(p_i\mathbf{e}_1)^{-1}f$. That is, a frame f is translated, here along the x-axis, by a distance p_i. In other words, the product is still that of a translation of arbitrary element of $SE(3)$. Do not get confused and think $p_i^{-1}f=f/p_i$. Division of a motion by a scalar is not defined.

Suppose that the same assumption as for $\varphi_i(f)$ may be made for $\psi_j(g)$; that is,

$$\psi_j(g) = \psi_j(\mathbf{d})\psi_j(r) \tag{19}$$

where $d = \sqrt{\mathbf{d}^T\mathbf{d}}$, the rotation $\delta(r_{p_i}^{-1} f)$ can be moved in front and combined with the rotation part r_j of g_j

$$\varphi'(f') = \sum_{i=1}^{m}\sum_{j=1}^{n} \delta(\mathbf{d}_j^{-1} g) \star \delta(r_{p_i}^{-1} r_j^{-1} g) \star \psi_j(g) \star \delta(p_i^{-1} f) \star \varphi_i(f) \star \delta(q_i^{-1} r_{p_i} f). \tag{20}$$

Suppose that the convolution $\psi_j(g) \star \delta(p_i^{-1} f)$ is now simple enough to allow a direct computation. This operation is called the *first basic convolution*. Note that the resulting density function cannot be split, because the two variables $\mathbf{p}$ and q are not independent. We define

$$\psi'_{ij}(g') = \psi_j(g) \star \delta(p_i^{-1} f). \tag{21}$$

4.1.6 Second Basic Convolution

Section 5 will describe how one can transform the density function $\psi'_{ij}(g')$ into a sum of basic functions like

$$\psi'_{ij}(g') = \sum_{k=1}^{l} \psi'_{ijk}((g'_k)^{-1} g') \tag{22}$$

Then the sum can be moved again out of the convolution:

$$\varphi'(f') = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} \delta(\mathbf{d}_j^{-1} g) \star \delta(r_{p_i}^{-1} r_j^{-1} g) \star \psi'_{ijk}((g'_k)^{-1} g') \star \varphi_i(f) \star \delta(q_i^{-1} r_{p_i} f) \tag{23}$$

and the function $\psi'_{ijk}((g'_k)^{-1} g')$ can be "centered"

$$\varphi'(f') = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} \delta(\mathbf{d}_j^{-1} g) \star \delta(r_{p_i}^{-1} r_j^{-1} g) \star \delta((g'_k)^{-1} g') \star \psi'_{ijk}(g') \star \varphi_i(f) \star \delta(q_i^{-1} r_{p_i} f). \tag{24}$$

The convolution $\psi'_{ijk}(g') \star \varphi_i(f)$ is now assumed to be simple enough to be carried out by direct computation. This is the *second basic convolution*

$$\varphi'_{ijk}(f') = \psi'_{ijk}(g') \star \varphi_i(f). \tag{25}$$

4.1.7 Decentering

The last step is to "move" this function according to the original uncentered functions:

$$\varphi'(f') = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} \varphi'_{ijk}(g_k^{-1} r_{p_i}^{-1} r_j^{-1} \mathbf{d}_j^{-1} f' q_i^{-1} r_{p_i}). \tag{26}$$

4.2 Basic Functions

4.2.1 The One-Dimensional Case

In order to thoroughly exploit the power of decomposition, the basic functions should belong to a single family generated by one or two parameters only. Moreover, the convolution of two basic functions should still result in a basic function. For the one-dimensional case the chosen basic function is a Gaussian function

$$\psi(d) = b\mathrm{e}^{-\pi \mathrm{b}^2 \mathrm{d}^2} \tag{27}$$

$$\varphi(p) = a\mathrm{e}^{-\pi \mathrm{a}^2 \mathrm{p}^2} \tag{28}$$

because it fulfills the above requirements and also because the Gaussian function is rapidly decreasing in both the spatial and Fourier domains. It is therefore a good tradeoff between finitely supported functions and spectrums, which is a necessary property for numerical computation, as will be discussed through Section 5. Note that the spectrum of these functions is simply a Gaussian function. The convolution for the one-dimensional case degenerates into a usual convolution, and the usual convolution of two Gaussian functions still results in a Gaussian function

$$\begin{aligned}\varphi'(p') &= \int_{-\infty}^{\infty} \psi(x)\varphi(-x+p')\mathrm{dx} &(29)\\ &= \int_{-\infty}^{\infty} \left[b\mathrm{e}^{-\pi \mathrm{b}^2\mathrm{x}^2}\right]\left[a\mathrm{e}^{-\pi \mathrm{a}^2(-\mathrm{x}+\mathrm{p}')^2}\right]\mathrm{dx} &(30)\\ &= a'\mathrm{e}^{-\pi(\mathrm{a}')^2(\mathrm{p}')^2} &(31)\end{aligned}$$

where

$$\frac{1}{a'^2} = \frac{1}{a^2} + \frac{1}{b^2}. \tag{32}$$

We remark that the convolution is simply a product in the spectrum (frequency) domain and the product of two Gaussian functions is still a Gaussian function.

4.2.2 The Two-Dimensional Case

The two-dimensional case also involves rotations and orientations. One can first factor the density function as

$$\begin{aligned}\psi(\mathbf{d}, r) &= \psi(\mathbf{d})\psi(r) &(33)\\ \varphi(\mathbf{p}, q) &= \varphi(\mathbf{p})\varphi(q). &(34)\end{aligned}$$

The translation and position part are still defined as Gaussian functions, that is

$$\begin{aligned}\psi(\mathbf{d}) &= b^2\mathrm{e}^{-\pi \mathrm{b}^2\mathbf{d}^\mathrm{T}\mathbf{d}} &(35)\\ \varphi(\mathbf{p}) &= a^2\mathrm{e}^{-\pi \mathrm{a}^2\mathbf{p}^\mathrm{T}\mathbf{p}}. &(36)\end{aligned}$$

For the rotation and orientation part, r and q are used and parameterized in the following way

$$r = (\cos\theta, \sin\theta) \quad \text{and} \quad q = (\cos\sigma, \sin\sigma). \tag{37}$$

The basic function is defined as a Gaussian function "spread on the circle"

$$\begin{aligned}\psi(r) &= \beta \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\pi\beta^2(\theta-2\mathrm{k}\pi)^2} &(38)\\ \varphi(q) &= \alpha \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\pi\alpha^2(\sigma-2\mathrm{k}\pi)^2}. &(39)\end{aligned}$$

Note that the spectrum of these functions is simply a sampled Gaussian function (as a result of the periodicity). For these basic functions, rotations and translations are decoupled, since the rotation/orientation part is independent of the translation/position part and since $\varphi(\mathbf{p})$ is invariant under rotation. Thus, the result of the convolution is

$$\varphi'(\mathbf{p}', q') = \varphi'(\mathbf{p}')\varphi'(q') \tag{40}$$

where

$$\begin{aligned}\varphi'(\mathbf{p}') &= (a')^2\mathrm{e}^{-\pi(\mathrm{a}')^2(\mathbf{p}')^\mathrm{T}\mathbf{p}'} &(41)\\ \varphi'(q') &= \alpha' \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\pi(\alpha')^2(\sigma'-2\mathrm{k}\pi)^2} &(42)\end{aligned}$$

with

$$\frac{1}{(a')^2} = \frac{1}{a^2} + \frac{1}{b^2} \tag{43}$$

$$\frac{1}{\alpha'^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2}. \tag{44}$$

4.2.3 The Three-Dimensional Case

The extension of the translation/position part to the three-dimensional case is straightforward:

$$\psi(\mathbf{d}) = b^3 \mathrm{e}^{-\pi \mathrm{b}^2 \mathbf{d}^\mathrm{T} \mathbf{d}} \tag{45}$$

$$\varphi(\mathbf{p}) = a^3 \mathrm{e}^{-\pi \mathrm{a}^2 \mathbf{p}^\mathrm{T} \mathbf{p}}. \tag{46}$$

For the rotation/orientation part the following parameterization is proposed

$$r = (\cos\theta, \mathbf{v}\sin\theta) \quad \text{and} \quad q = (\cos\sigma, \mathbf{u}\sin\sigma) \tag{47}$$

where $\mathbf{v}$ and $\mathbf{u}$ are unit vectors, that is

$$\mathbf{v} = \begin{bmatrix} \cos\nu_1 \\ \cos\nu_2 \sin\nu_1 \\ \sin\nu_2 \sin\nu_1 \end{bmatrix} \tag{48}$$

$$\mathbf{u} = \begin{bmatrix} \cos\mu_1 \\ \cos\mu_2 \sin\mu_1 \\ \sin\mu_2 \sin\mu_1 \end{bmatrix}. \tag{49}$$

Then, the basic functions are defined as

$$\psi(\theta) = \frac{\beta^3}{\sin\theta} \sum_{k=-\infty}^{\infty} (\theta - 2k\pi) \mathrm{e}^{-\pi\beta^2(\theta - 2\mathrm{k}\pi)^2} \tag{50}$$

$$\varphi(\sigma) = \frac{\alpha^3}{\sin\sigma} \sum_{k=-\infty}^{\infty} (\sigma - 2k\pi) \mathrm{e}^{-\pi\alpha^2(\sigma - 2\mathrm{k}\pi)^2}. \tag{51}$$

Hence we use r and θ (q and σ) interchangably as the indepedent variables.

The result of the convolution of the translation/position part is again a basic function

$$\varphi'(\mathbf{p}') = (a')^3 \mathrm{e}^{-\pi(\mathrm{a}')^2 (\mathbf{p}')^\mathrm{T} \mathbf{p}'} \tag{52}$$

where

$$\frac{1}{a'^2} = \frac{1}{a^2} + \frac{1}{b^2}. \tag{53}$$

The convolution of the rotation/orientation part is more difficult to perform. First, note that the basic functions do not depend on $\mathbf{u}$ or $\mathbf{v}$ and neither does the density function $\varphi'(q')$ resulting from the convolution. Thus, without any loss of generality, one can assume that $\mu_1' = 0$ and $\mu_2' = 0$. The convolution to perform is then

$$\varphi'(q') = \psi(r) \star \varphi(q) \tag{54}$$

$$= \int_0^\pi \int_0^\pi \int_{-\pi}^\pi \psi(\theta)\varphi(\sigma) \sin^2\theta \sin\nu_1 \mathrm{d}\nu_2 \mathrm{d}\nu_1 \mathrm{d}\theta \tag{55}$$

where

$$\cos\sigma = \cos\theta\cos\sigma' - \cos\nu_1 \sin\theta \sin\sigma'. \tag{56}$$

Since neither $\psi(\theta)$ nor $\varphi(\sigma)$ depends on ν_2, one obtains

$$\varphi'(\sigma') \quad = \quad 2\pi \int_0^\pi \psi(\theta) \sin^2\theta \int_0^\pi \varphi(\sigma) \sin\nu_1 \mathrm{d}\nu_1 \mathrm{d}\theta. \tag{57}$$

As σ is a function of ν_1, the variable of the integral can be changed:

$$-\sin\sigma \mathrm{d}\sigma = \sin\nu_1 \sin\theta \sin\sigma' \mathrm{d}\nu_1 \tag{58}$$

$$\varphi'(\sigma') \quad = \quad -\frac{2\pi}{\sin\sigma'} \int_0^\pi \tilde{\psi}(\theta) \int_{\sigma'+\theta}^{\sigma'-\theta} \tilde{\varphi}(\sigma) \mathrm{d}\sigma \mathrm{d}\theta \tag{59}$$

where

$$\tilde{\psi}(\theta) \quad = \quad \psi(\theta) \sin\theta \tag{60}$$

$$\tilde{\varphi}(\sigma) \quad = \quad \varphi(\sigma) \sin\sigma. \tag{61}$$

Assuming that $\tilde{\Phi}$ is the indefinite integral of $\tilde{\varphi}$, the convolution can be rewritten as

$$\begin{aligned}
\varphi'(\sigma') &= \frac{2\pi}{\sin\sigma'} \int_0^\pi \tilde{\psi}(\theta)\tilde{\Phi}(\sigma'-\theta)\mathrm{d}\theta - \frac{2\pi}{\sin\sigma'} \int_0^\pi \tilde{\psi}(\theta)\tilde{\Phi}(\sigma'+\theta)\mathrm{d}\theta & (62)\\
&= \frac{2\pi}{\sin\sigma'} \int_0^\pi \tilde{\psi}(\theta)\tilde{\Phi}(\sigma'-\theta)\mathrm{d}\theta - \frac{2\pi}{\sin\sigma'} \int_{-\pi}^0 \tilde{\psi}(-\theta)\tilde{\Phi}(\sigma'-\theta)\mathrm{d}\theta & (63)\\
&= \frac{2\pi}{\sin\sigma'} \int_0^\pi \tilde{\psi}(\theta)\tilde{\Phi}(\sigma'-\theta)\mathrm{d}\theta + \frac{2\pi}{\sin\sigma'} \int_{-\pi}^0 \tilde{\psi}(\theta)\tilde{\Phi}(\sigma'-\theta)\mathrm{d}\theta & (64)\\
&= \frac{2\pi}{\sin\sigma'} \int_{-\pi}^\pi \tilde{\psi}(\theta)\tilde{\Phi}(\sigma'-\theta)\mathrm{d}\theta & (65)\\
&= \frac{(\alpha')^3}{\sin\sigma'} \sum_{k=-\infty}^{\infty} (\sigma'-2k\pi)\mathrm{e}^{-\pi(\alpha')^2(\sigma'-2\mathrm{k}\pi)^2} & (66)
\end{aligned}$$

where

$$\frac{1}{\alpha'^2} \quad = \quad \frac{1}{\alpha^2} + \frac{1}{\beta^2}. \tag{67}$$

Observe that the convolution of two basic functions is commutative, which is not usual in $SO(3)$.

5 Sampling and Gaussian Reconstruction

Methods of harmonic analysis of functions on the circle, line, and $\mathbb{R}^d$ are well known to many engineers. The Shannon sampling theorem is of particular importance in regard to the reconstruction of band-limited functions on the line. We present an alternative sampling technique for functions on the line based on Gaussians, and show how this formulation is applicable to the convolution of functions on $SE(d)$.

5.1 Functions of Finite Support

A numerical computation can be carried out only if i) the involved functions $h(\mathbf{x})$ can be sampled without any loss of information and ii) the *support of the functions* $h(\mathbf{x})$ is finite, for example

$$h(x) = \begin{cases} \text{arbitrary} & \text{if} \quad |x| < \xi \\ 0 & \text{elsewhere} \end{cases}. \tag{68}$$

Unfortunately these two conditions cannot be fulfilled simultaneously. A reasonable tradeoff is again the Gaussian function since for $w = \xi a$ sufficiently large (typically $w > 1.32$), one can assume in practice that the support of the function is finite. At any rate, if the result of a computation based on this assumption is $r(w)$, owing to the rapidly decreasing nature of Gaussian functions, $r(w)$ will converge rapidly to its asymptotic value.

5.2 Reconstruction

A function $h(\mathbf{x})$ can be reconstructed from its sampled version $\hat{h}(\mathbf{x})$ by a sum of sine cardinal functions. However, such an operation can be costly to perform since the sine-cardinal function does not have a finite support, which means that to interpolate between two samples, all the other samples have to be considered It would be more convenient if instead of the sine cardinal, one had a function with finite support, since only the nearest neighbors would have to be considered.

What is the price to pay if a Gaussian function is used instead of the sine cardinal ? The rejection of the aliases by a Gaussian alters the original spectrum. Consequently, the reconstructed function is somewhat different from the original one. The following equation holds, which means that the reconstructed function is low-pass filtered in comparison with the original function

$$h(\mathbf{x}) * (\hat{a}\Delta)^n e^{-\pi \hat{a}^2 \mathbf{x}^T \mathbf{x}} \approx \hat{h}(\mathbf{x}) * (\hat{a}\Delta)^n e^{-\pi \hat{a}^2 \mathbf{x}^T \mathbf{x}} \tag{69}$$

where $\hat{a} = \frac{1}{2w\Delta}$.

5.3 High-pass Filtering

In the previous subsection it was shown that the reconstructed function by Gaussian filtering is low-pass filtered in comparison with the original function $h(\mathbf{x})$. If one samples a high-pass filtered version $h'(\mathbf{x})$ instead of $h(\mathbf{x})$, one can compensate for the effect of the subsequent low-pass process. For example, assume that

$$h(\mathbf{x}) = \sum_i c_i e^{-\pi a^2 (\mathbf{x}-\mathbf{x}_i)^T (\mathbf{x}-\mathbf{x}_i)}. \tag{70}$$

Then, the high-pass filtered version is

$$h'(\mathbf{x}) = \left[\frac{a'}{a\Delta}\right]^n \sum_i c_i e^{-\pi a'^2 (\mathbf{x}-\mathbf{x}_i)^T (\mathbf{x}-\mathbf{x}_i)} \tag{71}$$

where

$$\frac{1}{a'^2} = \frac{1}{\hat{a}^2} - \frac{1}{a^2}. \tag{72}$$

The reconstructed version of the sampled function $\hat{h}'(\mathbf{x})$ by a Gaussian filtering is then (almost, depending on w) equal to the original function $h(\mathbf{x})$

$$h(\mathbf{x}) \approx \hat{h}'(\mathbf{x}) * (\hat{a}\Delta)^n e^{-\pi \hat{a}^2 \mathbf{x}^T \mathbf{x}}. \tag{73}$$

However, the maximum sampling step Δ must be smaller for $h'(\mathbf{x})$ than for $h(\mathbf{x})$.

5.4 Application to Motion-Group Convolutions

The Gaussian reconstruction technique outlined previously in this section is directly applicable to the numerical computation of motion-group convolutions calculated in the order

$$\psi_n \star \ldots \star \psi_1 \star \varphi_0 = \psi_n \star (\psi_{n-1} \star \ldots \star (\psi_1 \star \varphi_0)\ldots)$$

when ψ_i are general functions on the motion group and φ_0 is a function only of position (i.e., it is constant over all orientations).

Geometrically, this can be viewed as φ_0 being swept by ψ_i; then, the result $\psi_1 \star \varphi_0$ being swept by ψ_2, etc. At each stage the quantity being swept is a function of position only, and hence can be expanded as a sum of basic (Gaussian) functions. The formulation of Section 4 is then used to perform the convolution of basic functions.

6 Conclusion

We have formulated and implemented a general numerical proceedure for decomposing functions on motion groups into sums of shifted generalized Gaussians. These generalized Gaussians are the product of Gaussians in $\mathbb{R}^d$ and "folded normal" distributions on the circle and quaternion sphere. The relationship between these Gaussians and solutions of the heat equation on the circle and quaternion sphere were explored. It was concluded that in the case of the circle the solutions are identical, while for the quaternion sphere they are not identical, but become indistinguishable in the range of parameter values of greatest interest in applications. It was shown that the convolution of these Gaussians is closed under the operation of convolution on $SE(d)$. By extending Shannon's sampling theorem, a numerical tool for future use in generating manipulator workspaces and analyzing error propogation in serial and hybrid manipulators resulted.

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