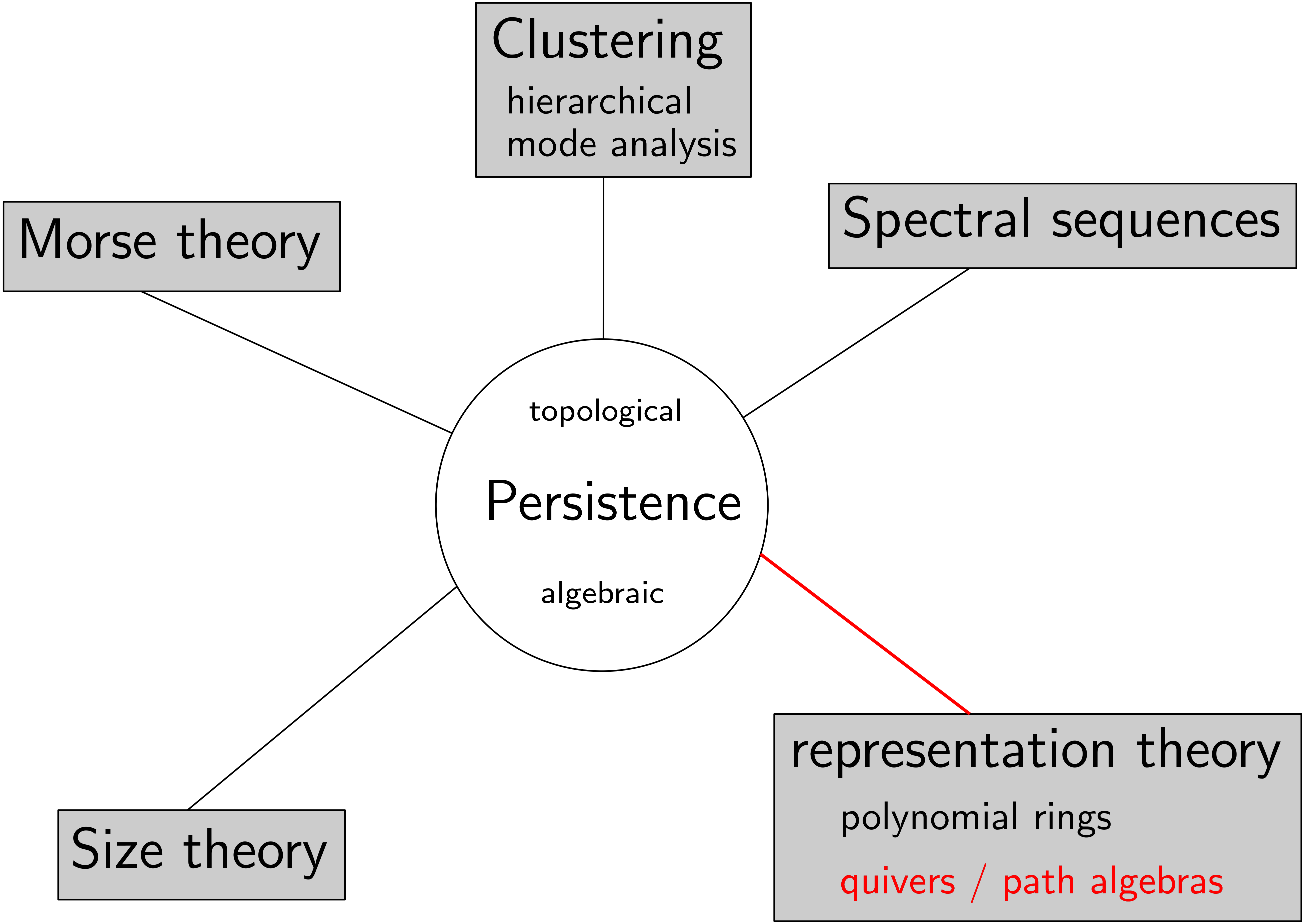
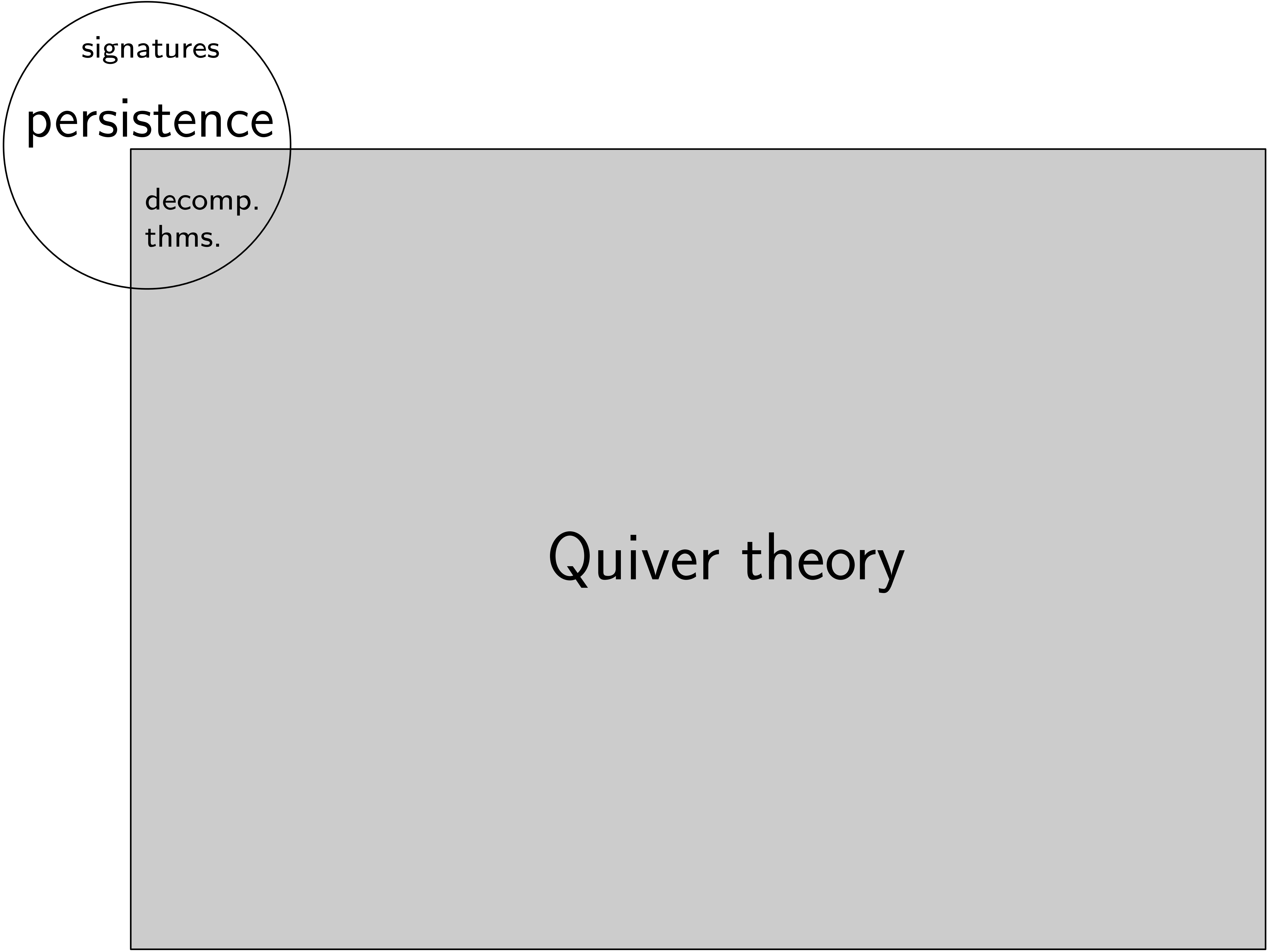
Geometrica Seminar— April 20, 2015

Reflections in Persistence and Quiver Theories

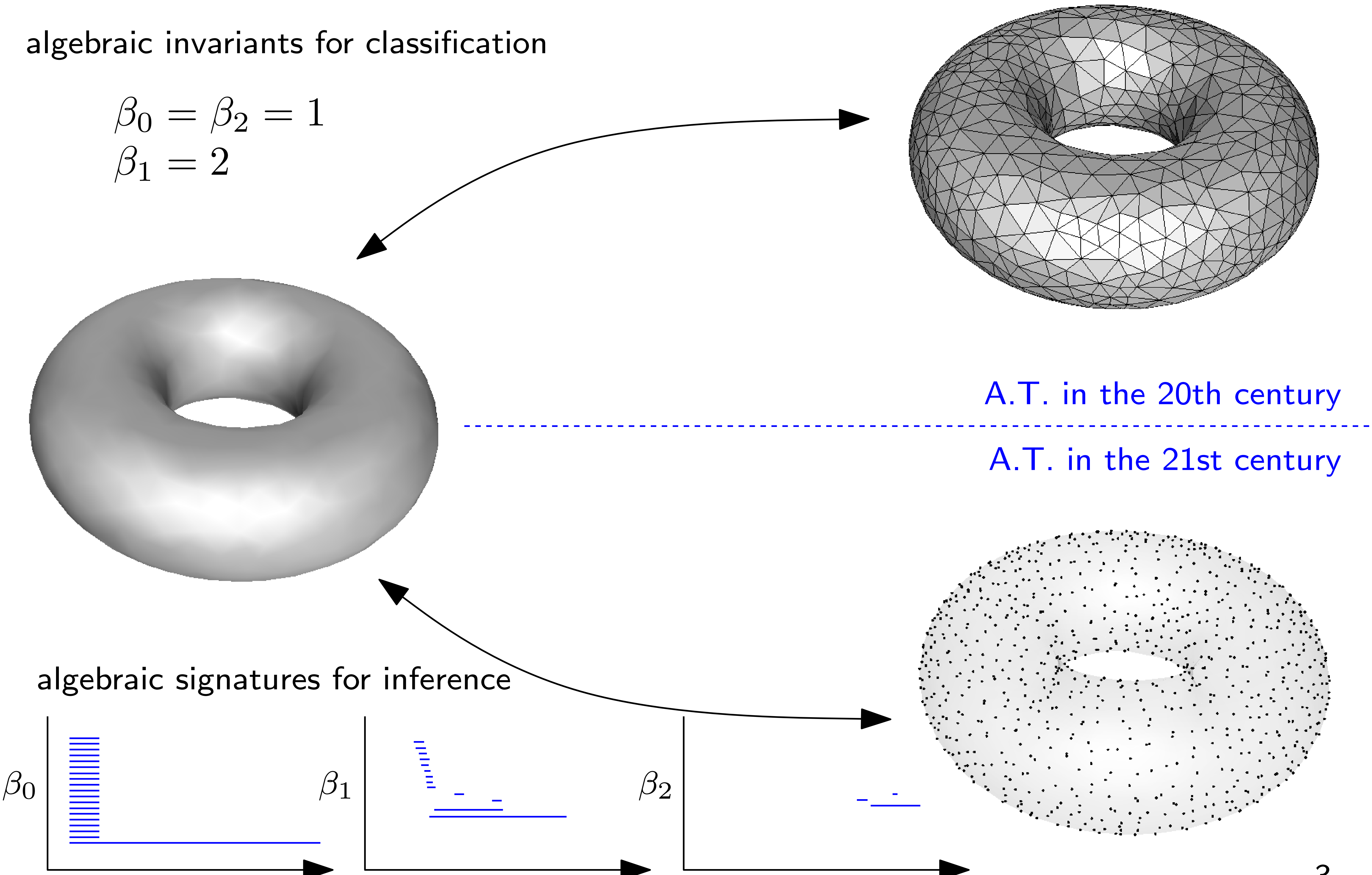
Steve Oudot

- S. O. *Persistence Theory: From Quiver Representations to Data Analysis* (see appendix). AMS Mathematical Surveys and Monographs. To appear.
- C. Maria and S. O. Zigzag Persistence via Reflections and Transpositions. Proc. Symposium on Discrete Algorithms (SODA), 2015, pp. 181–199.

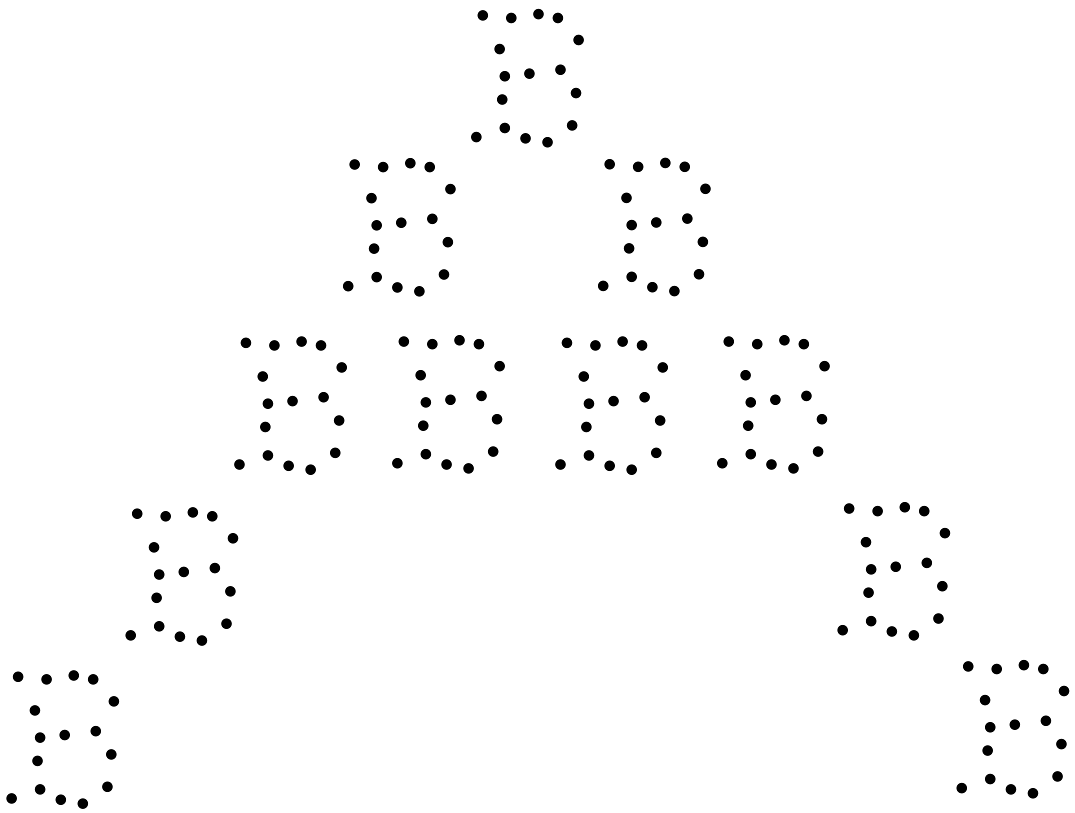

Clustering
hierarchical
mode analysis
Morse theory
Spectral sequences
topological
Persistence
algebraic
Size theory
representation theory
polynomial rings
quivers / path algebras

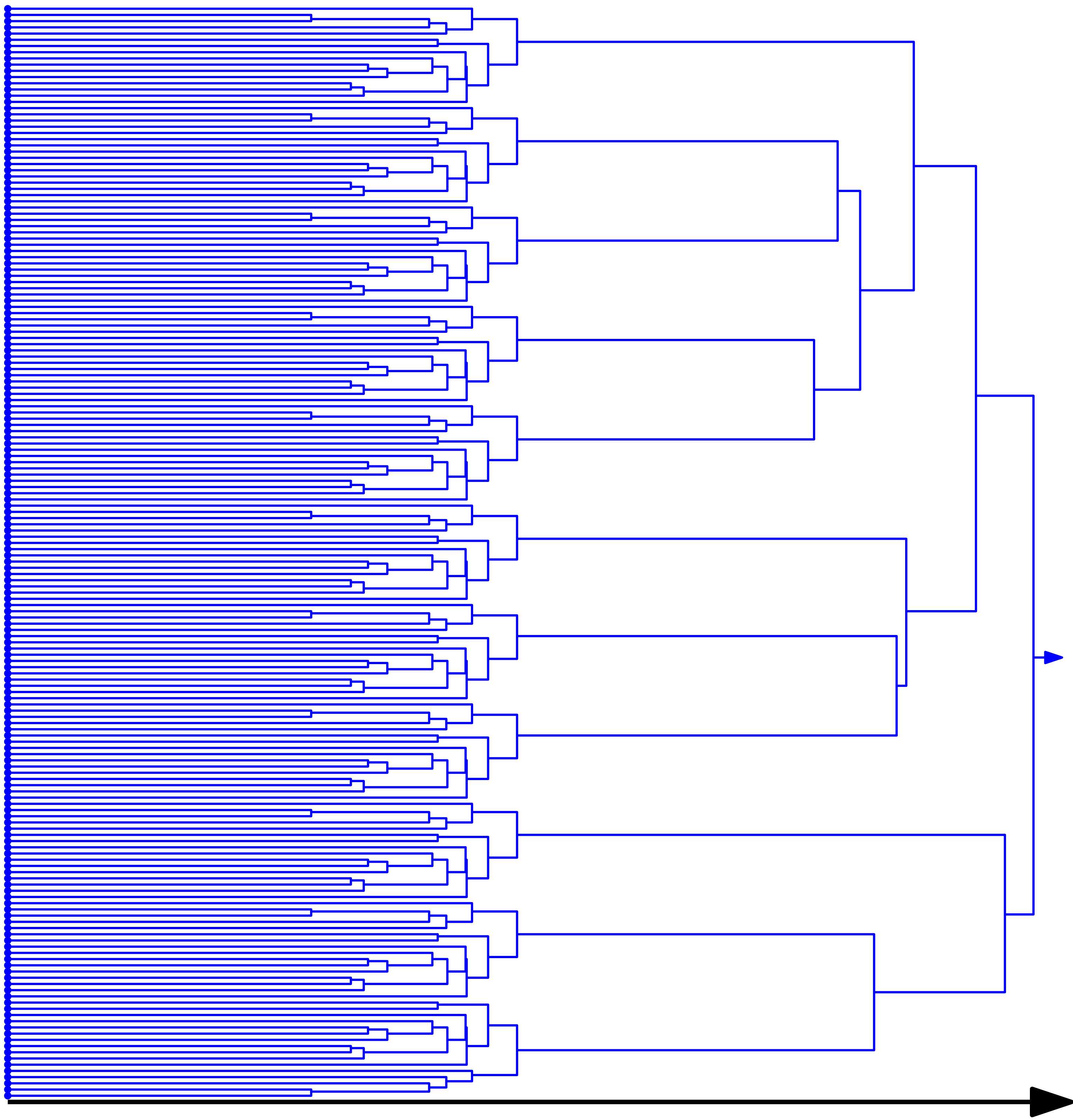

signatures
persistence
decomp.
thms.
Quiver theory

Inferring the topology of data



Intuitive viewpoint: hierarchical clustering





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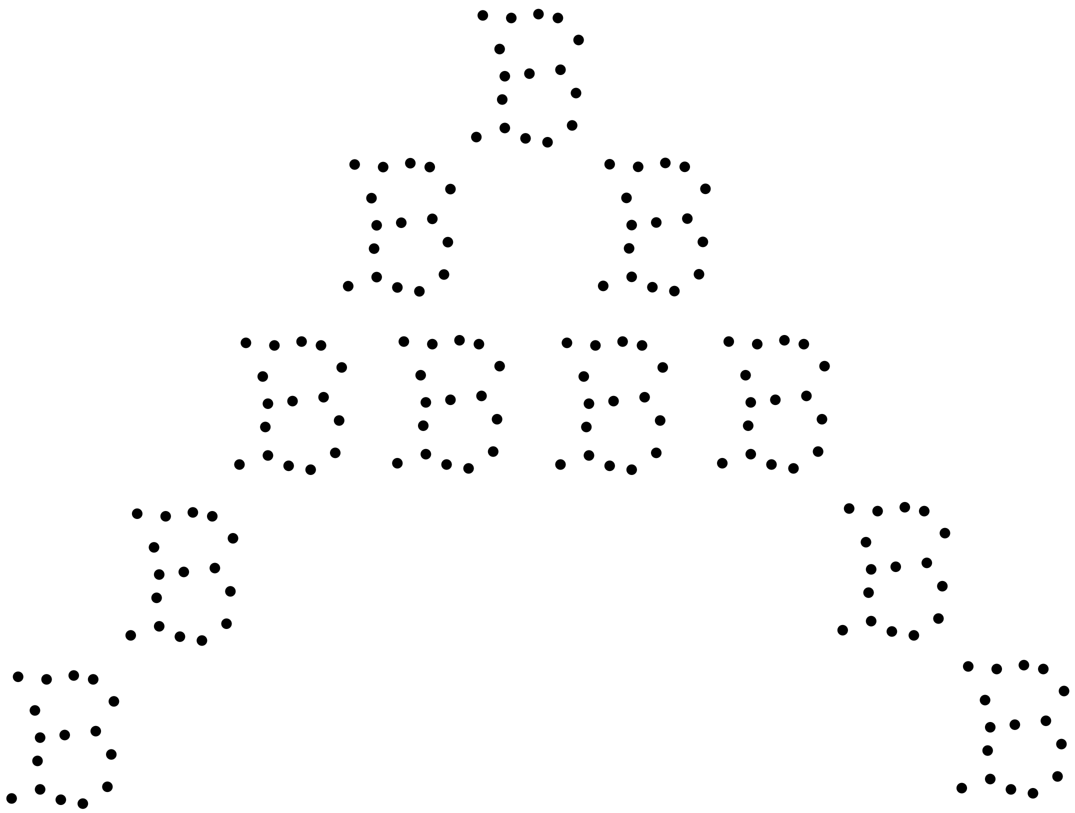

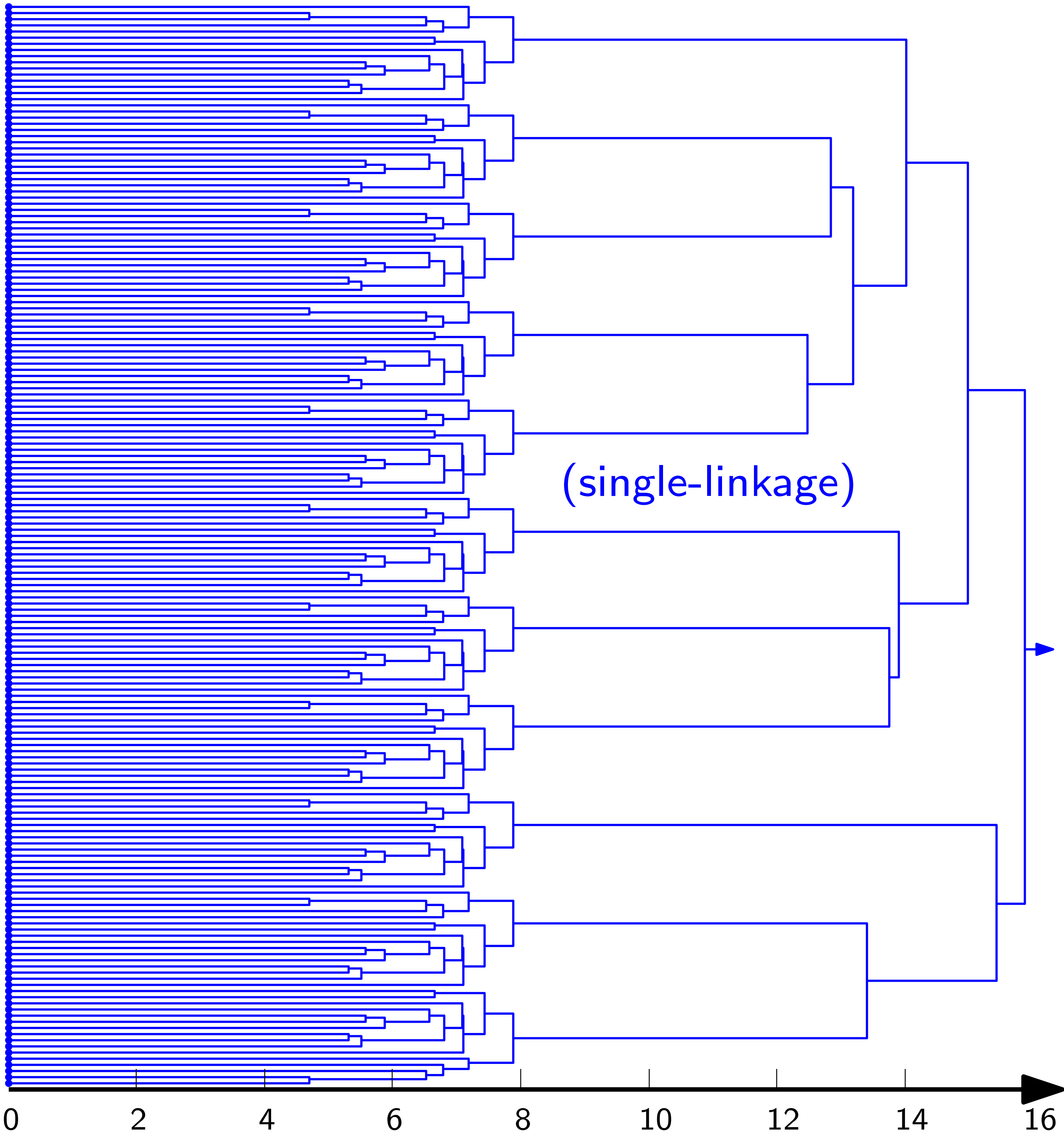

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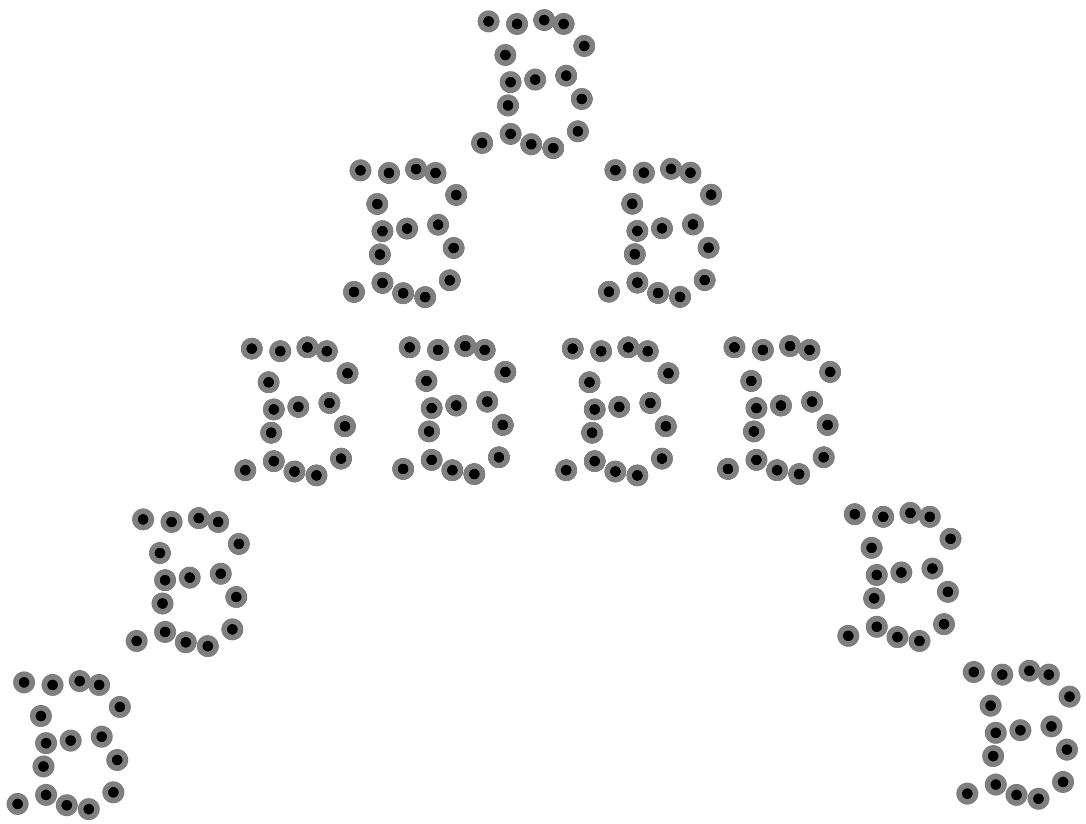

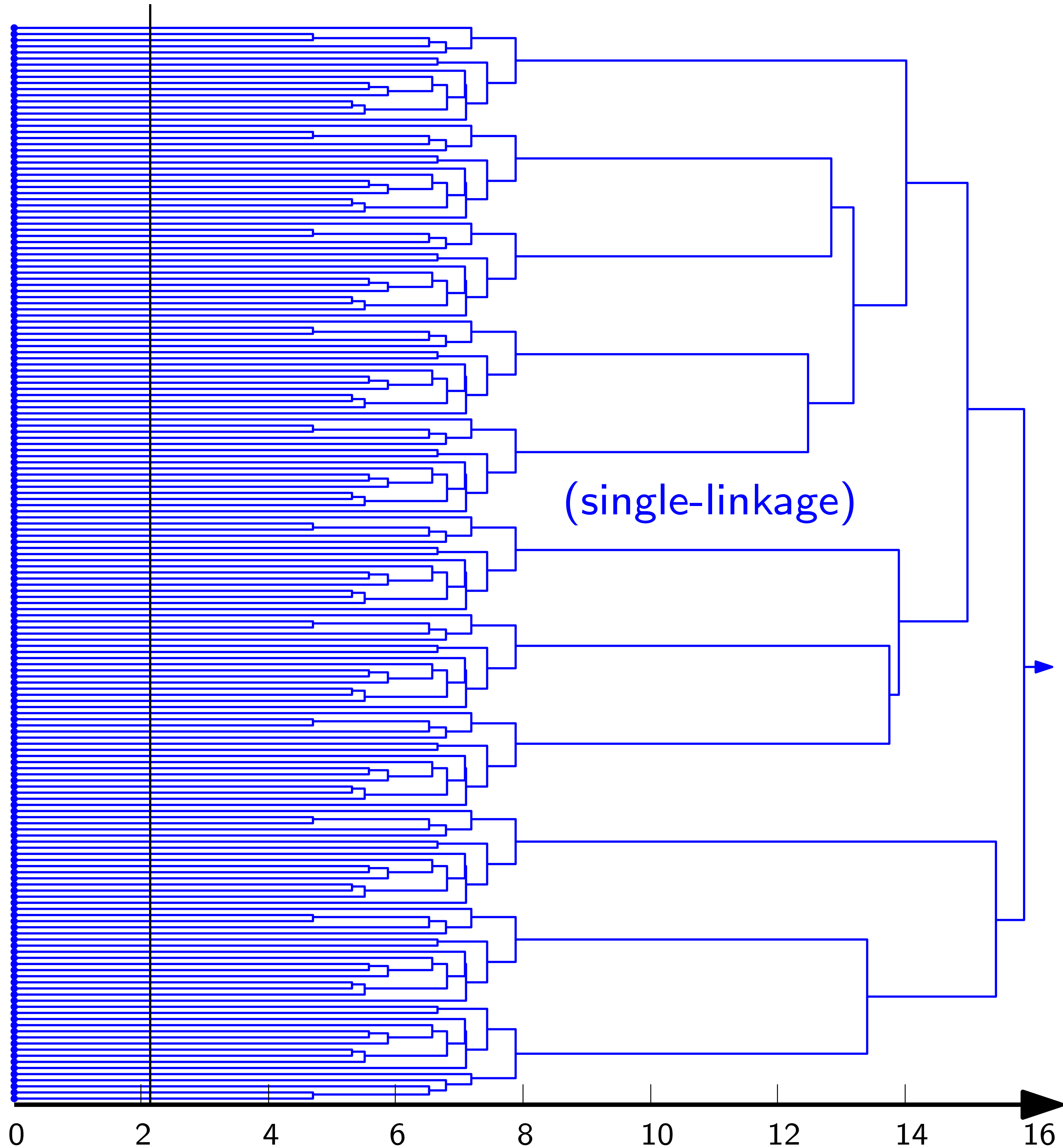

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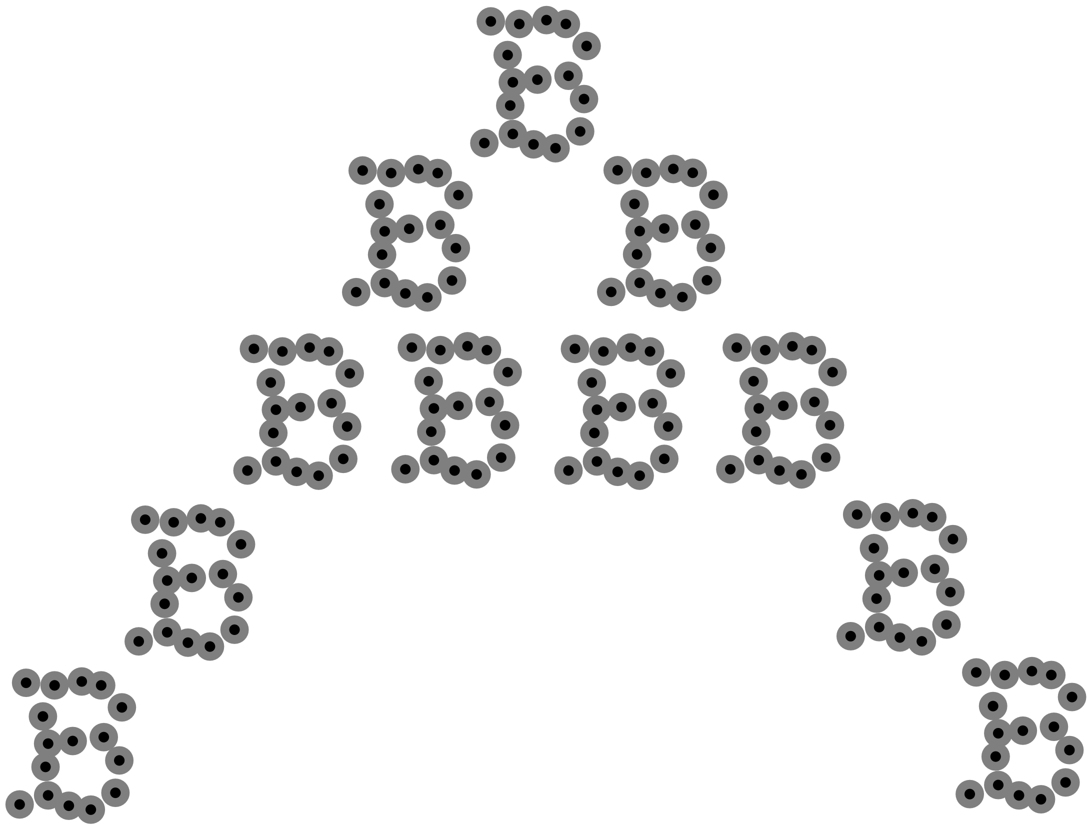

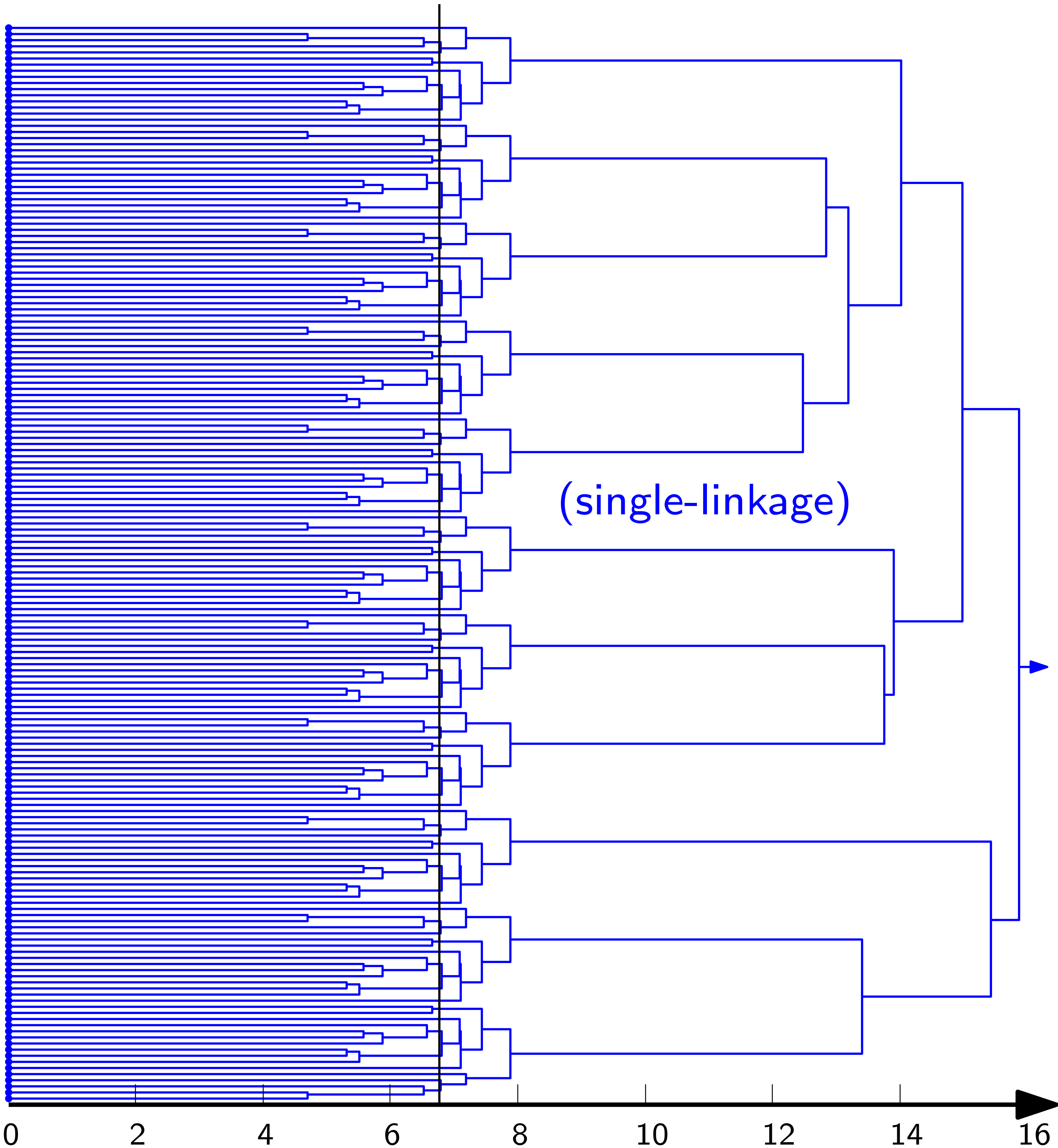

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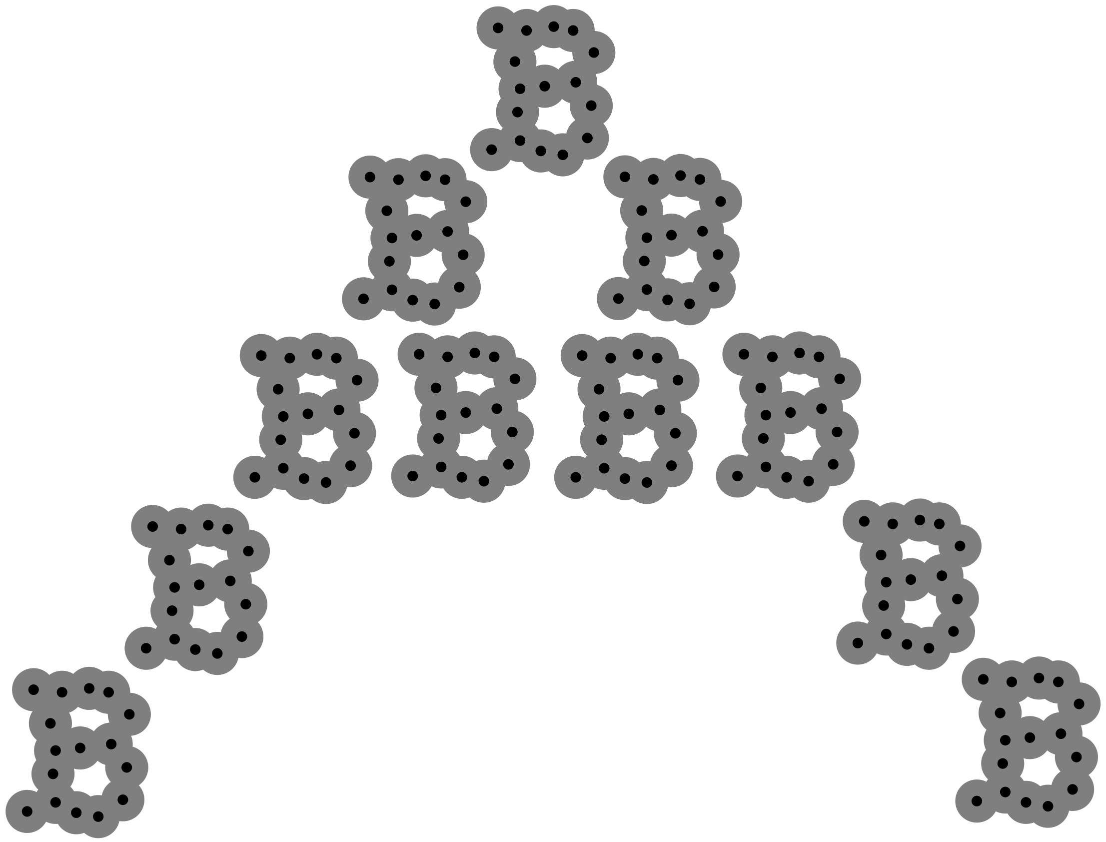

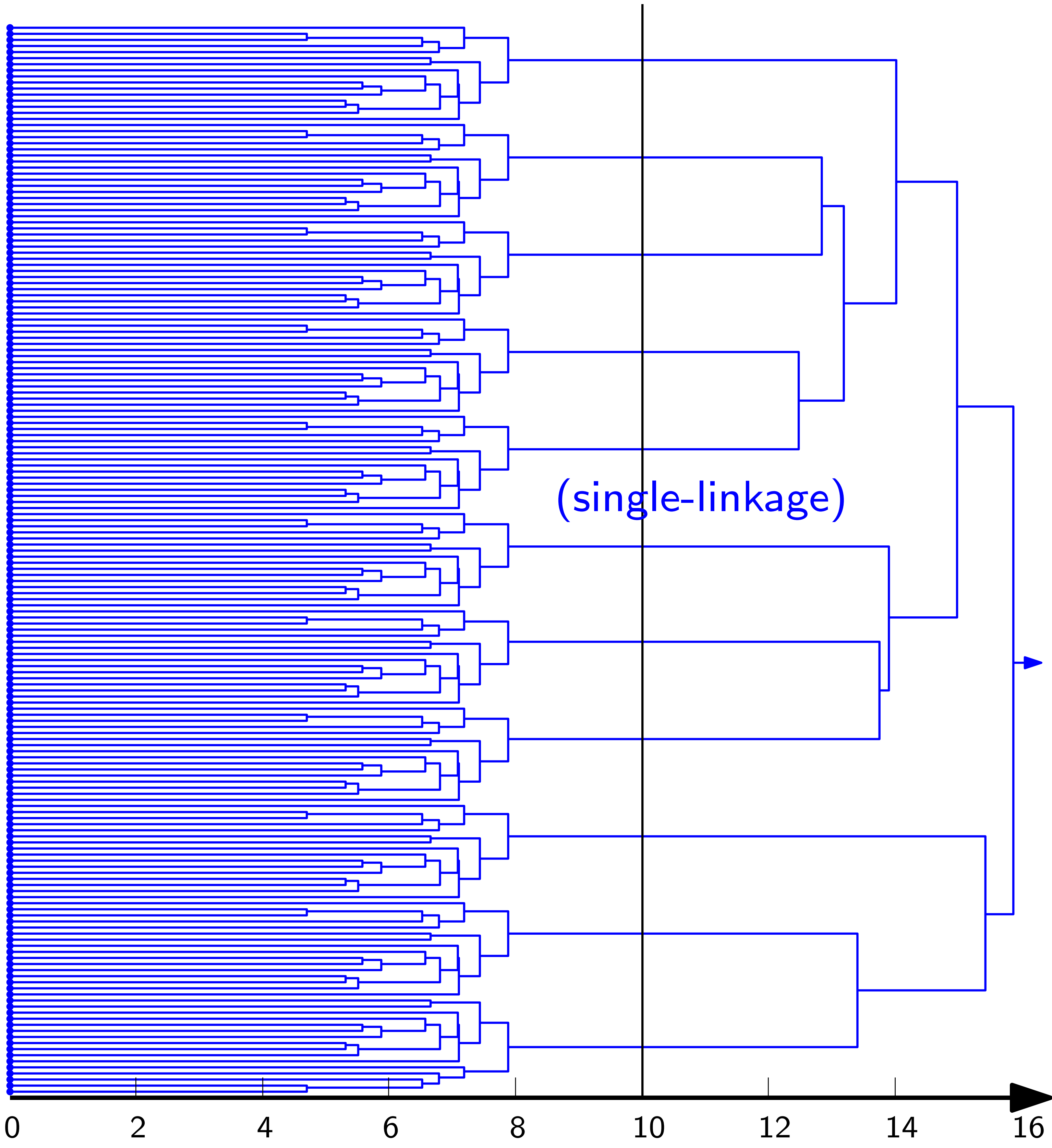

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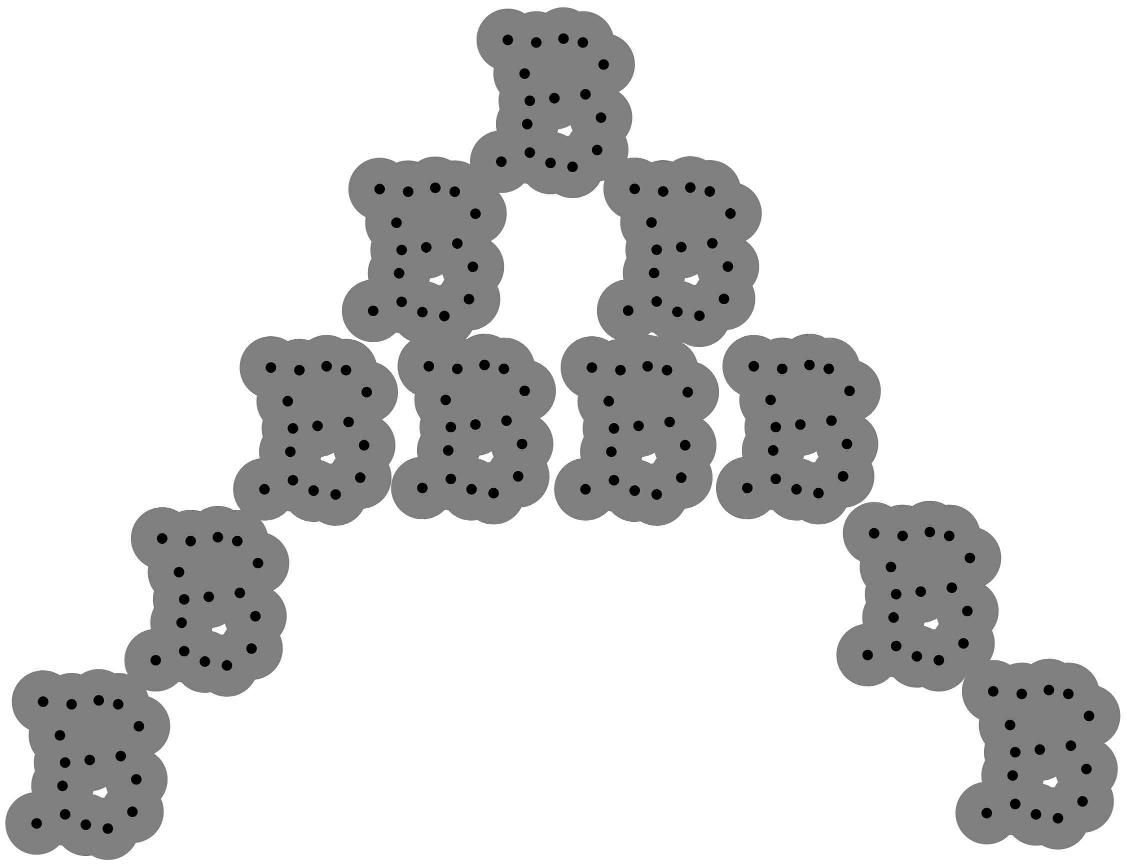

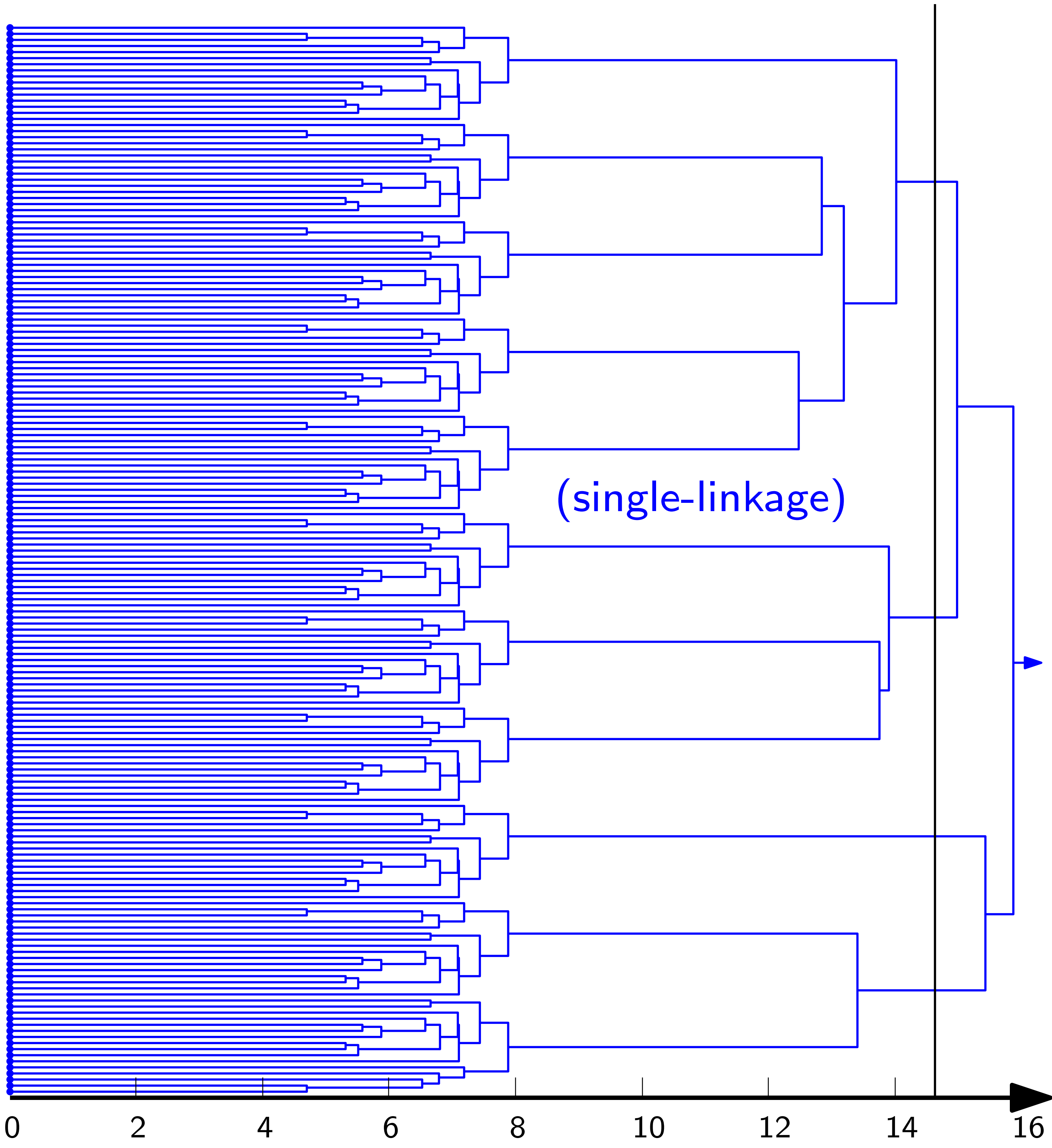

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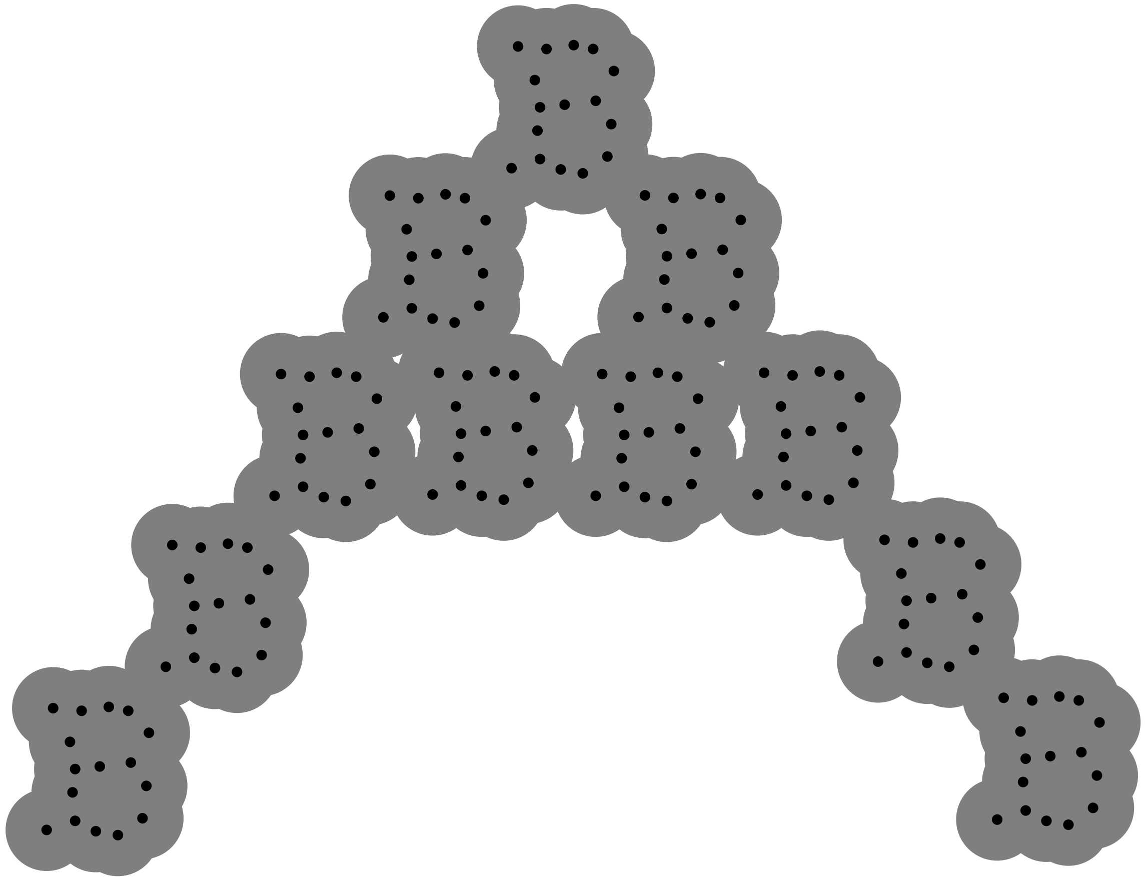

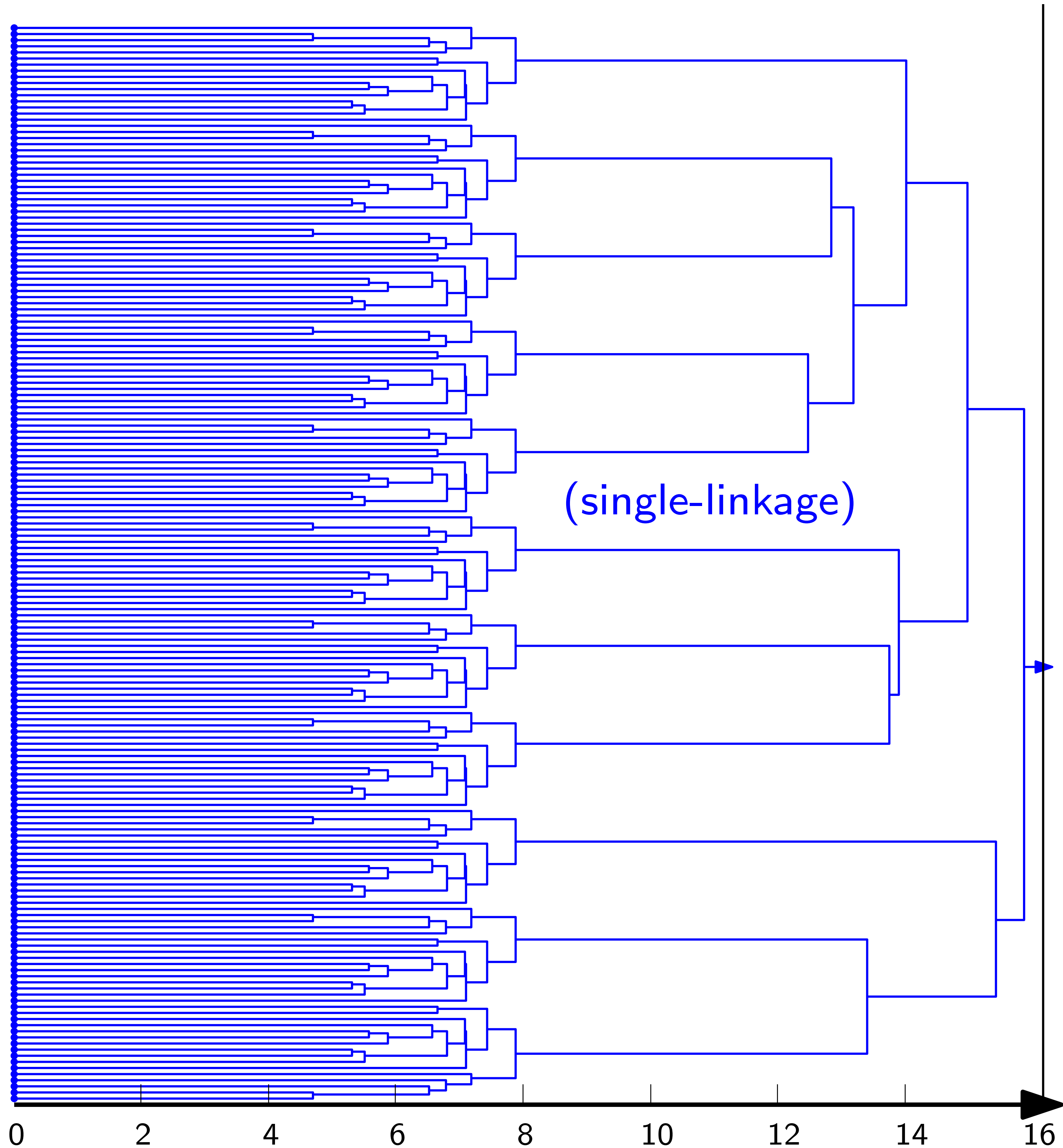

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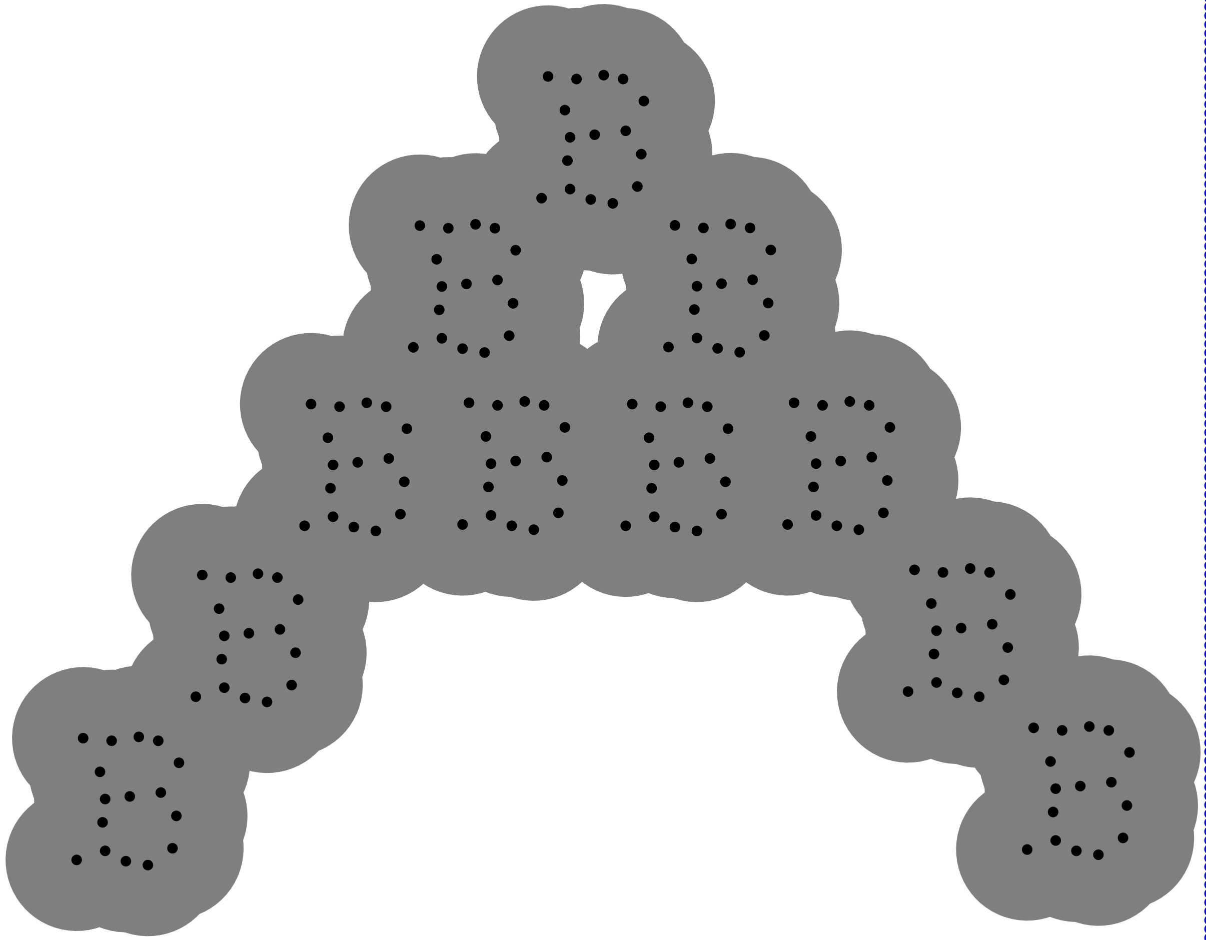

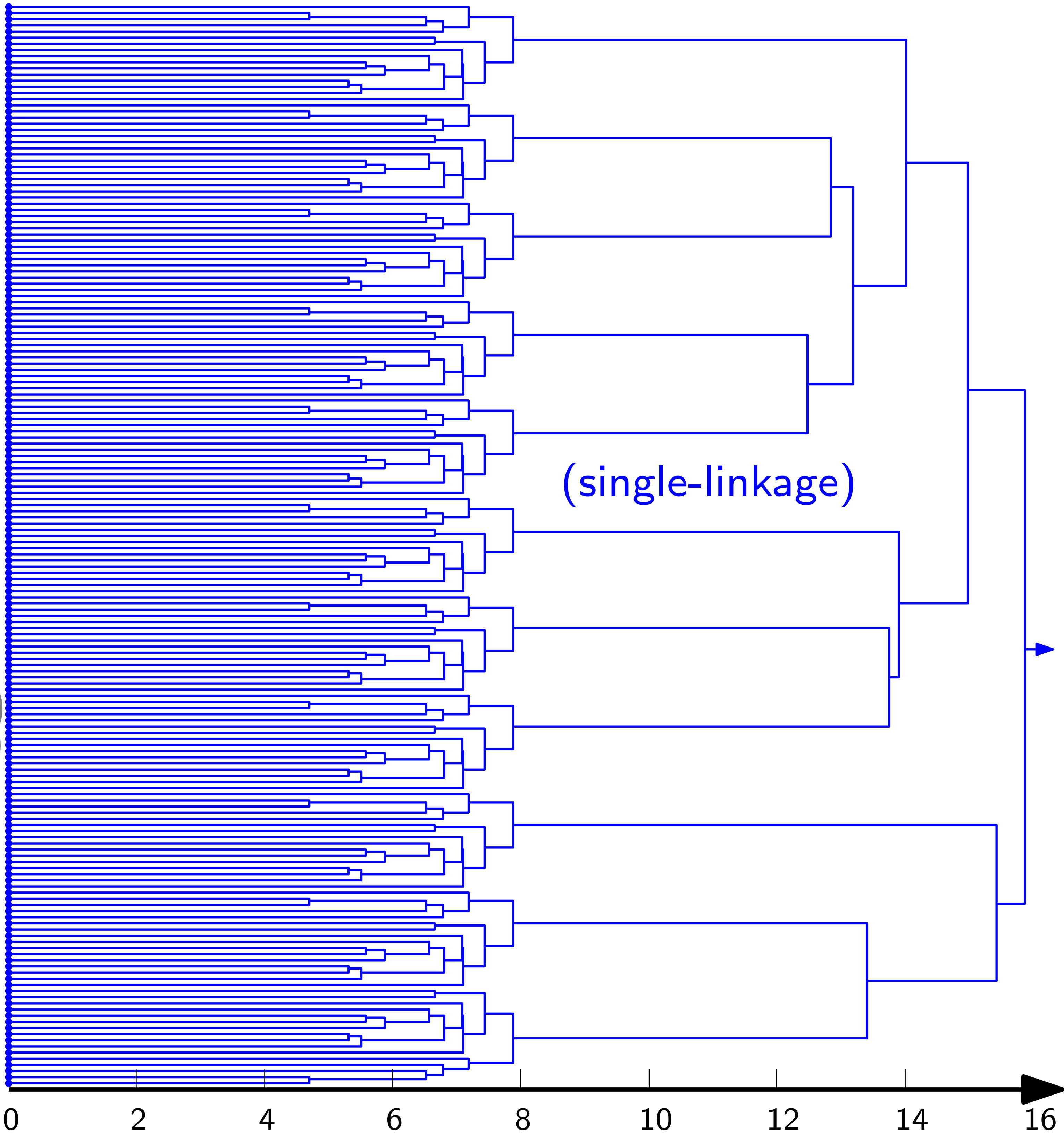

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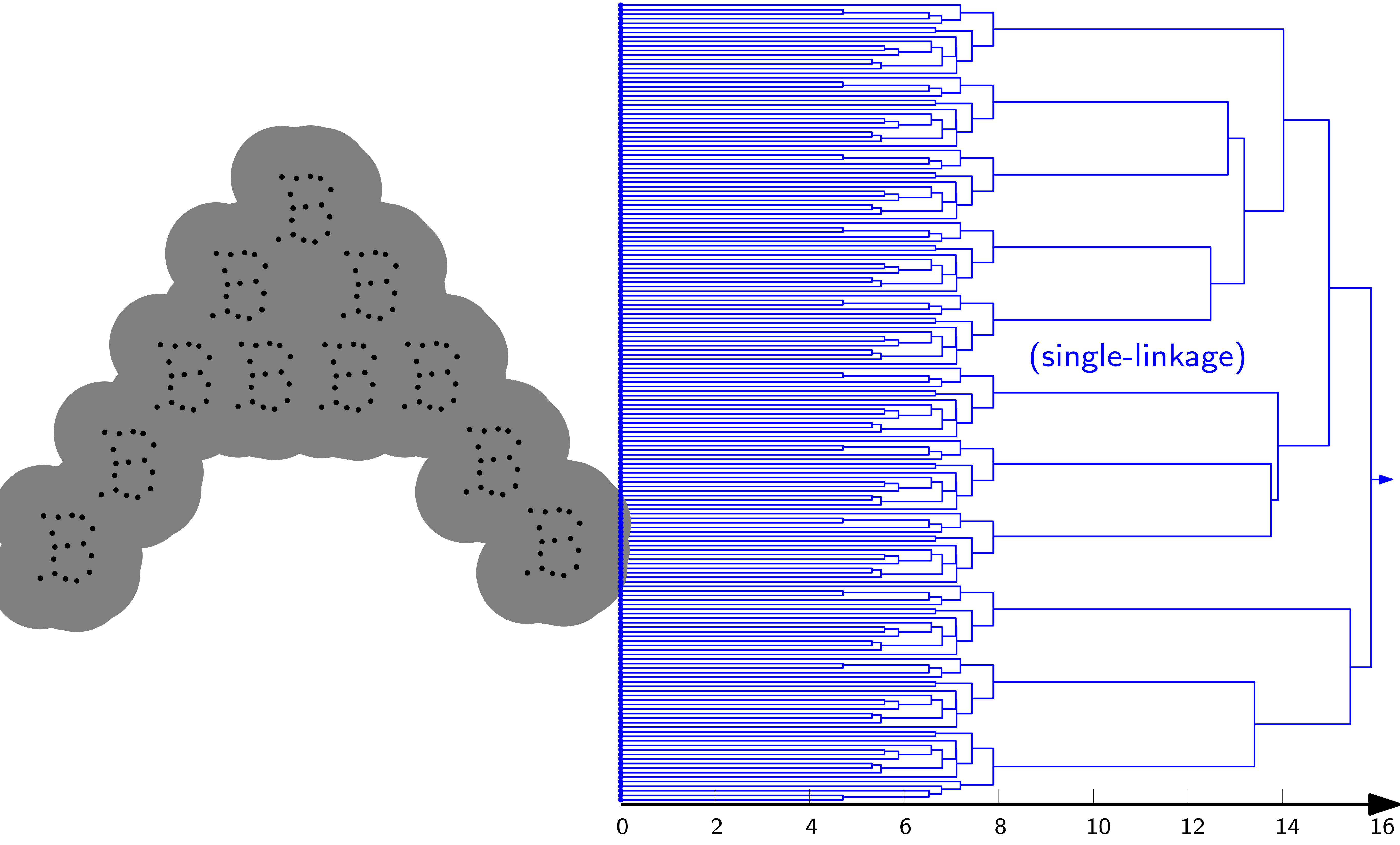

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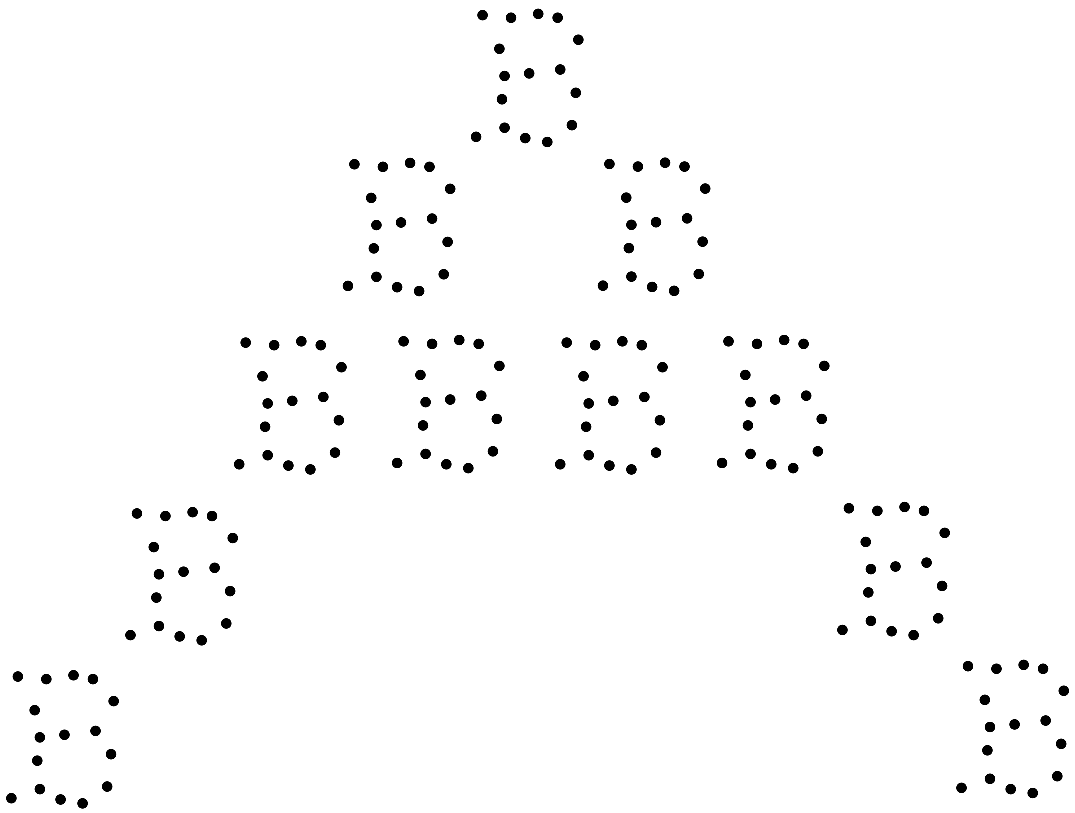
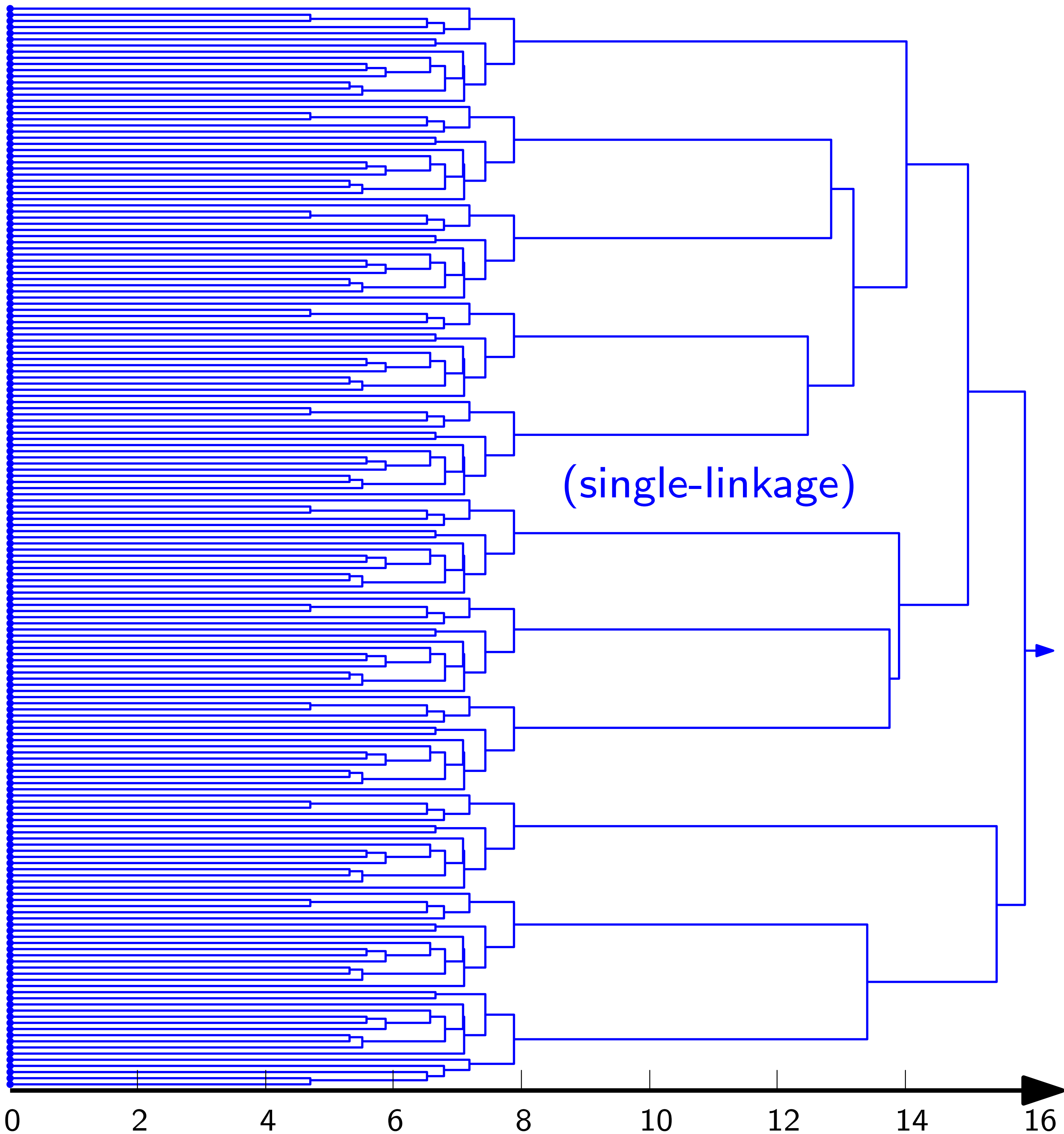


dendrogram is:

- informative
- unstable



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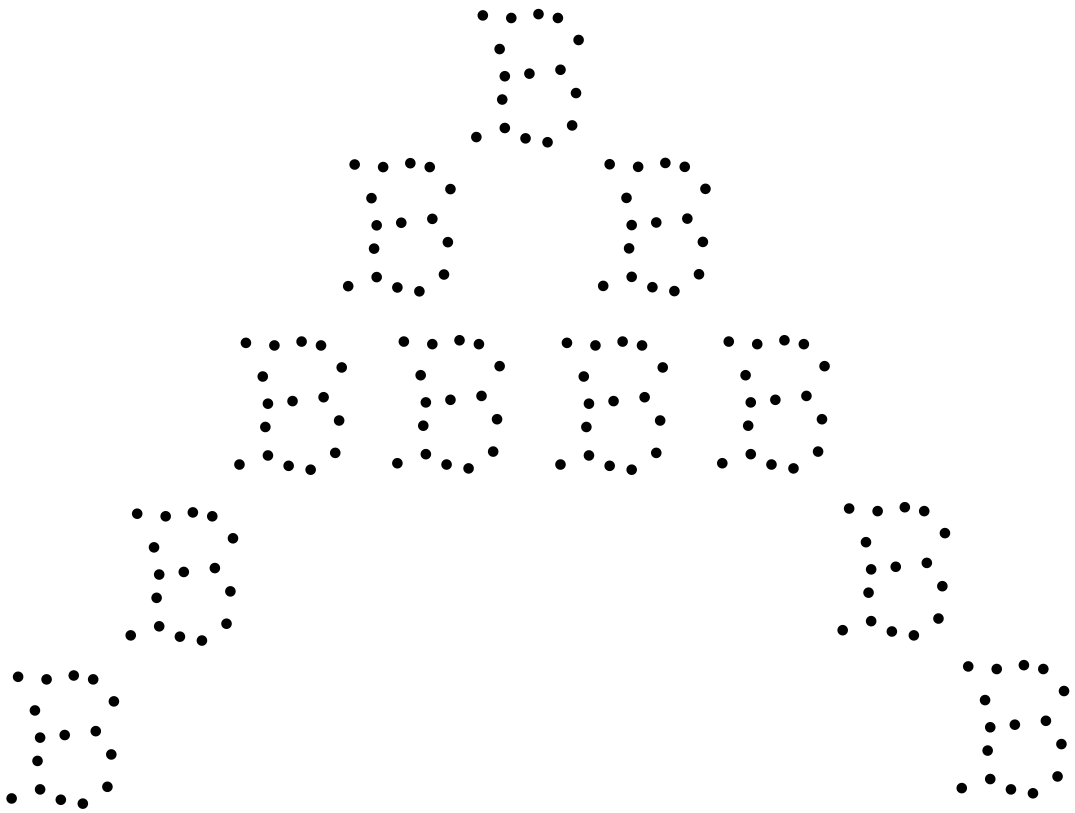
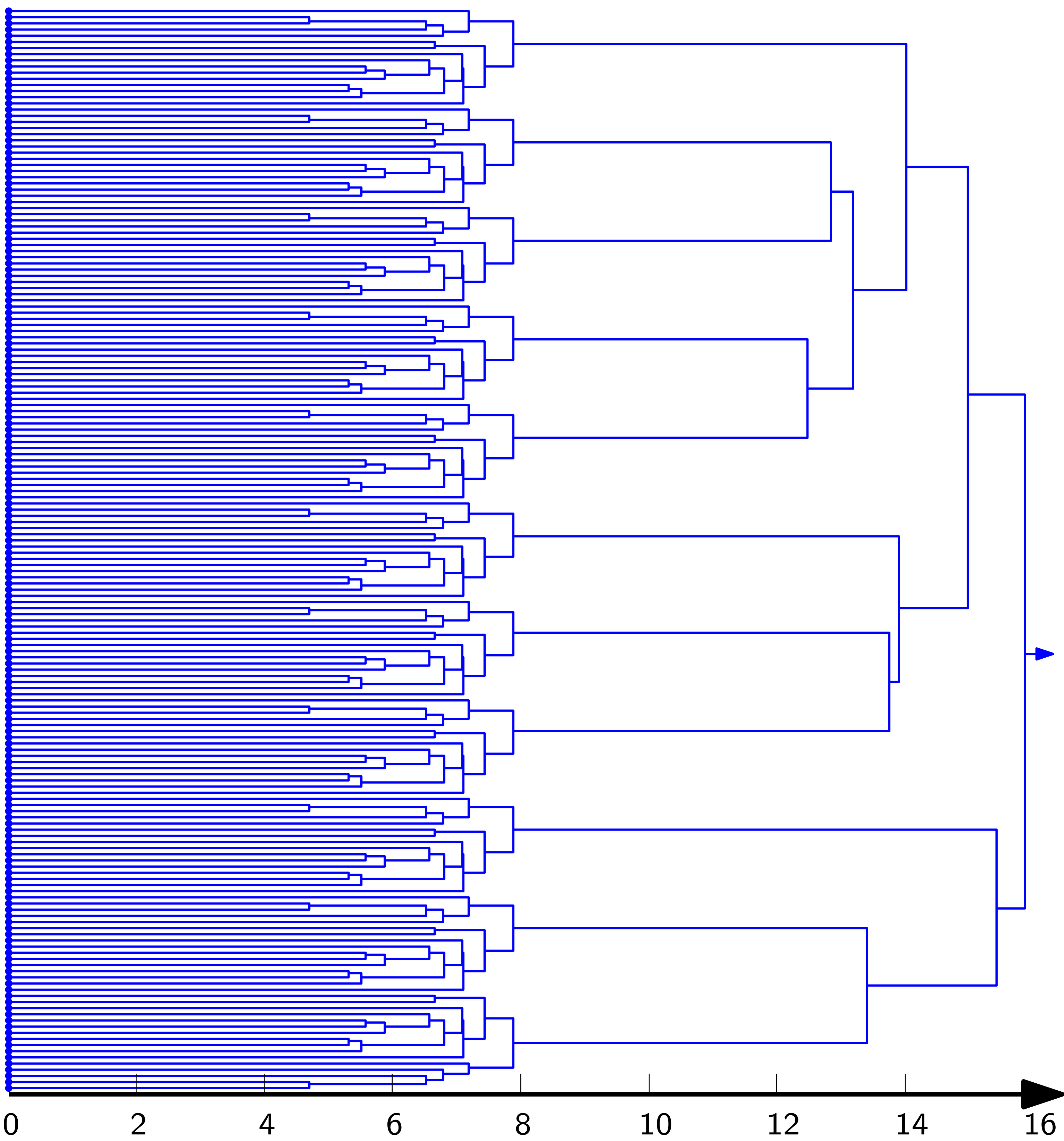


dendrogram → barcode



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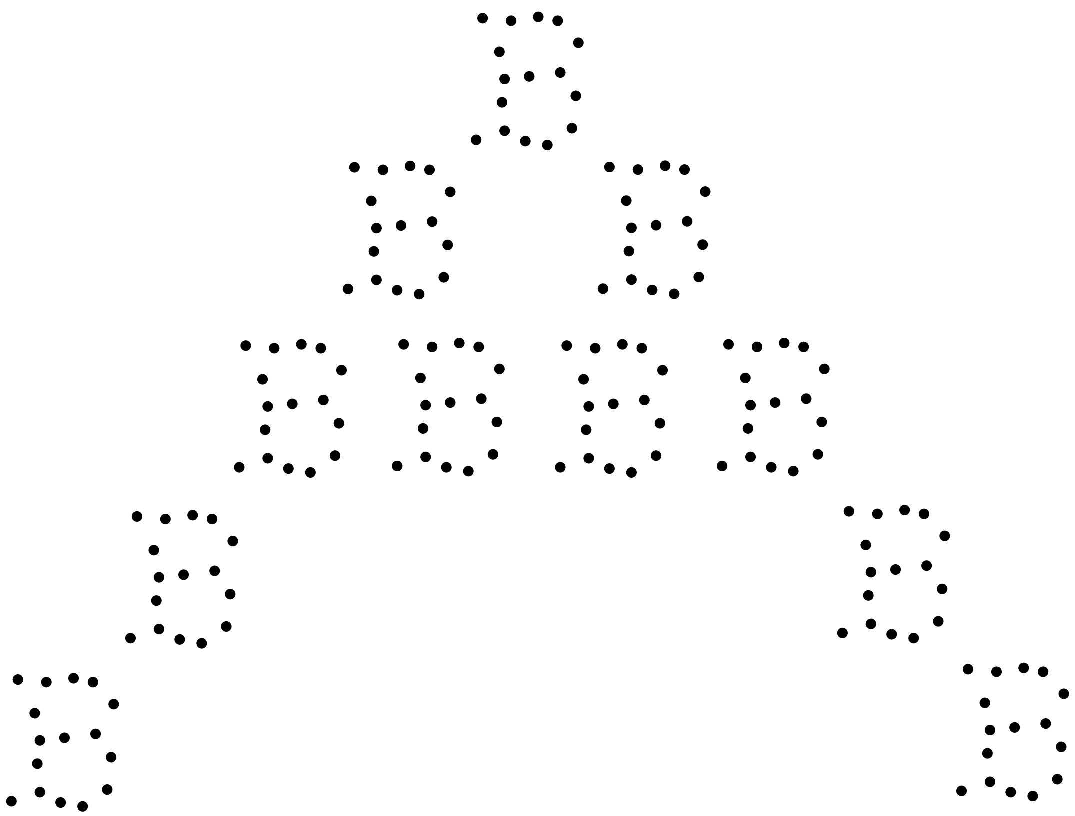
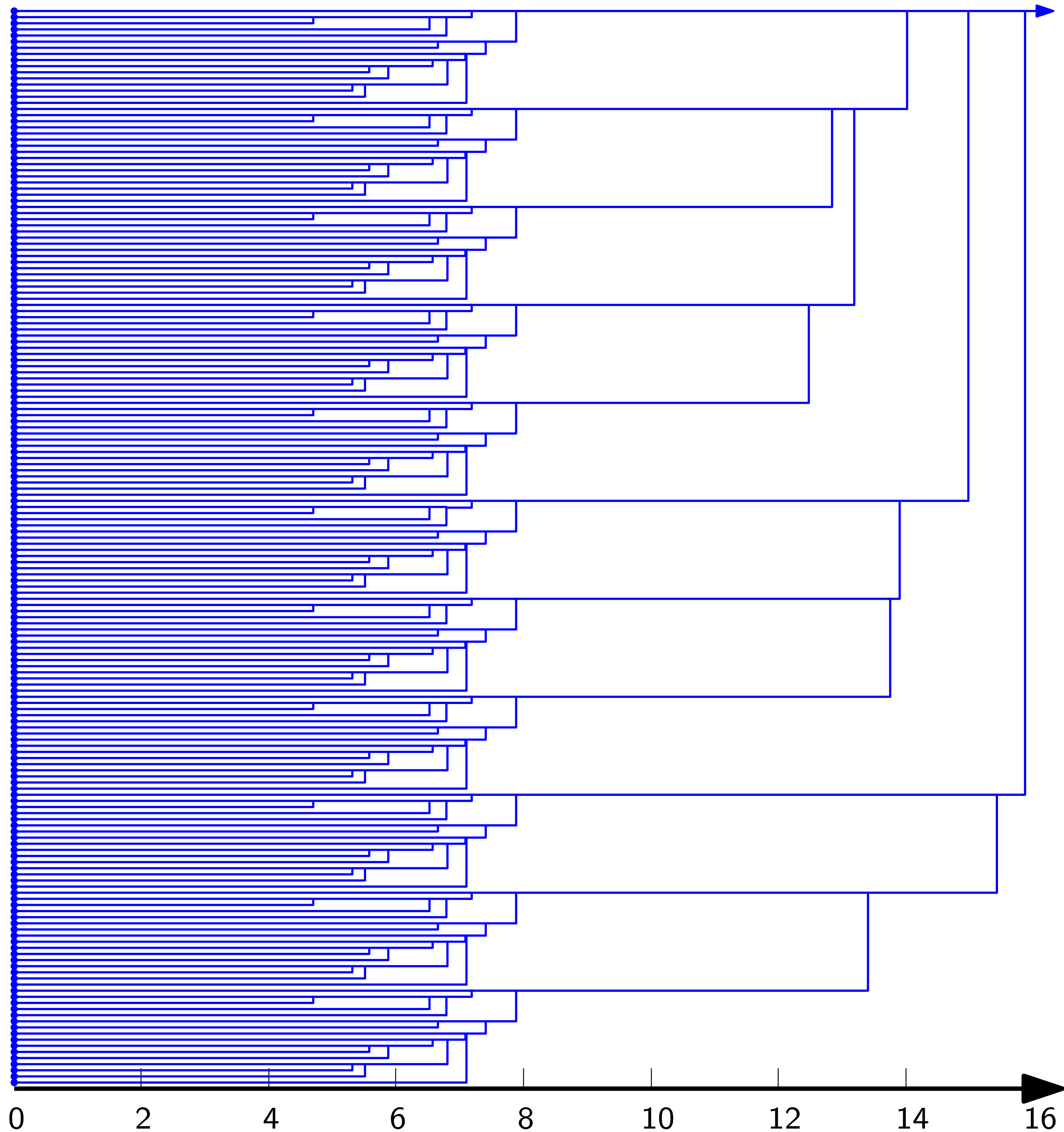


dendrogram $\rightarrow$ barcode



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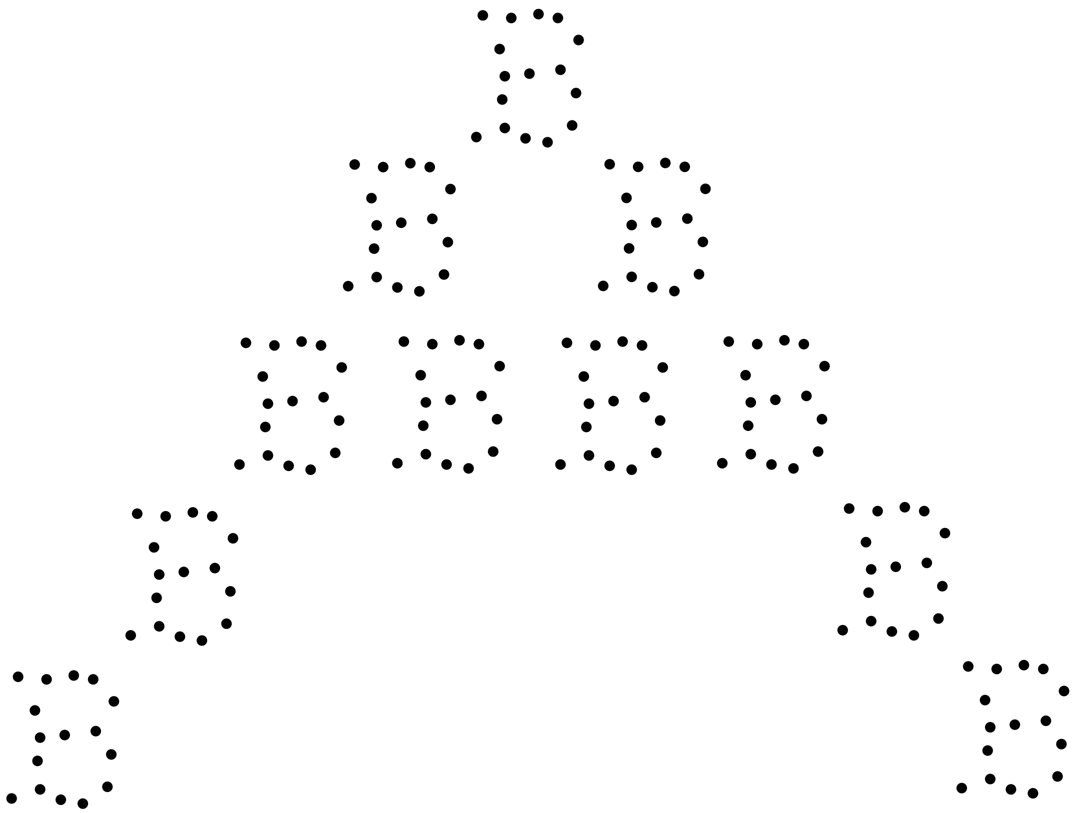

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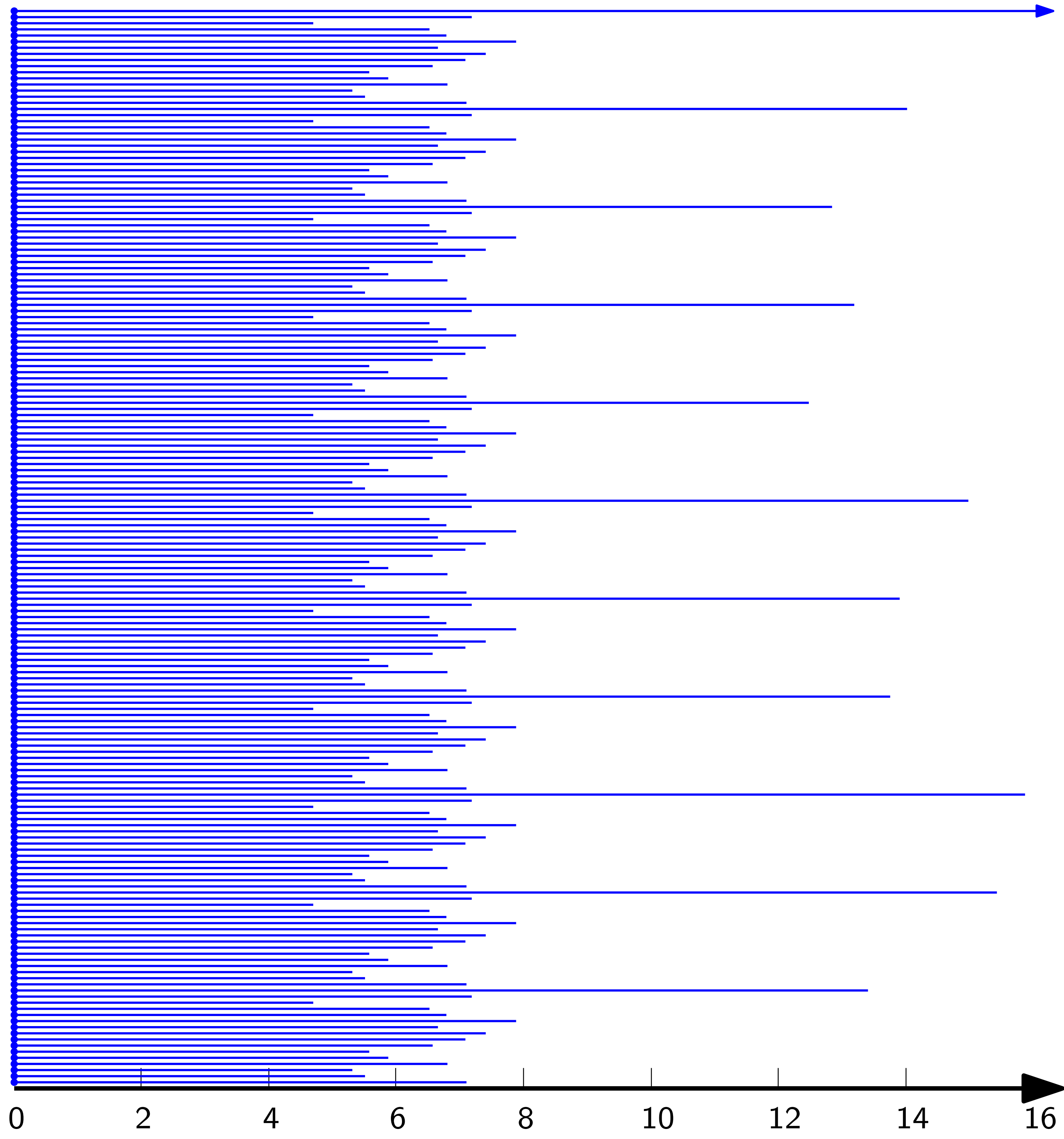

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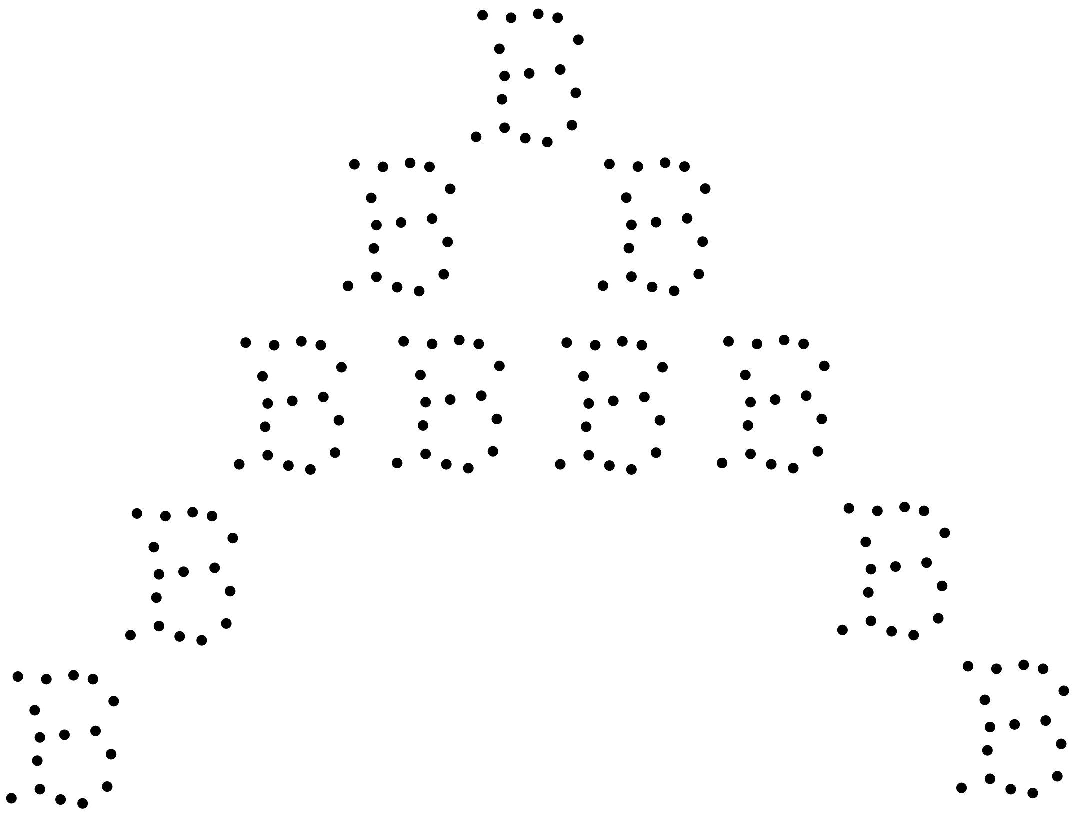
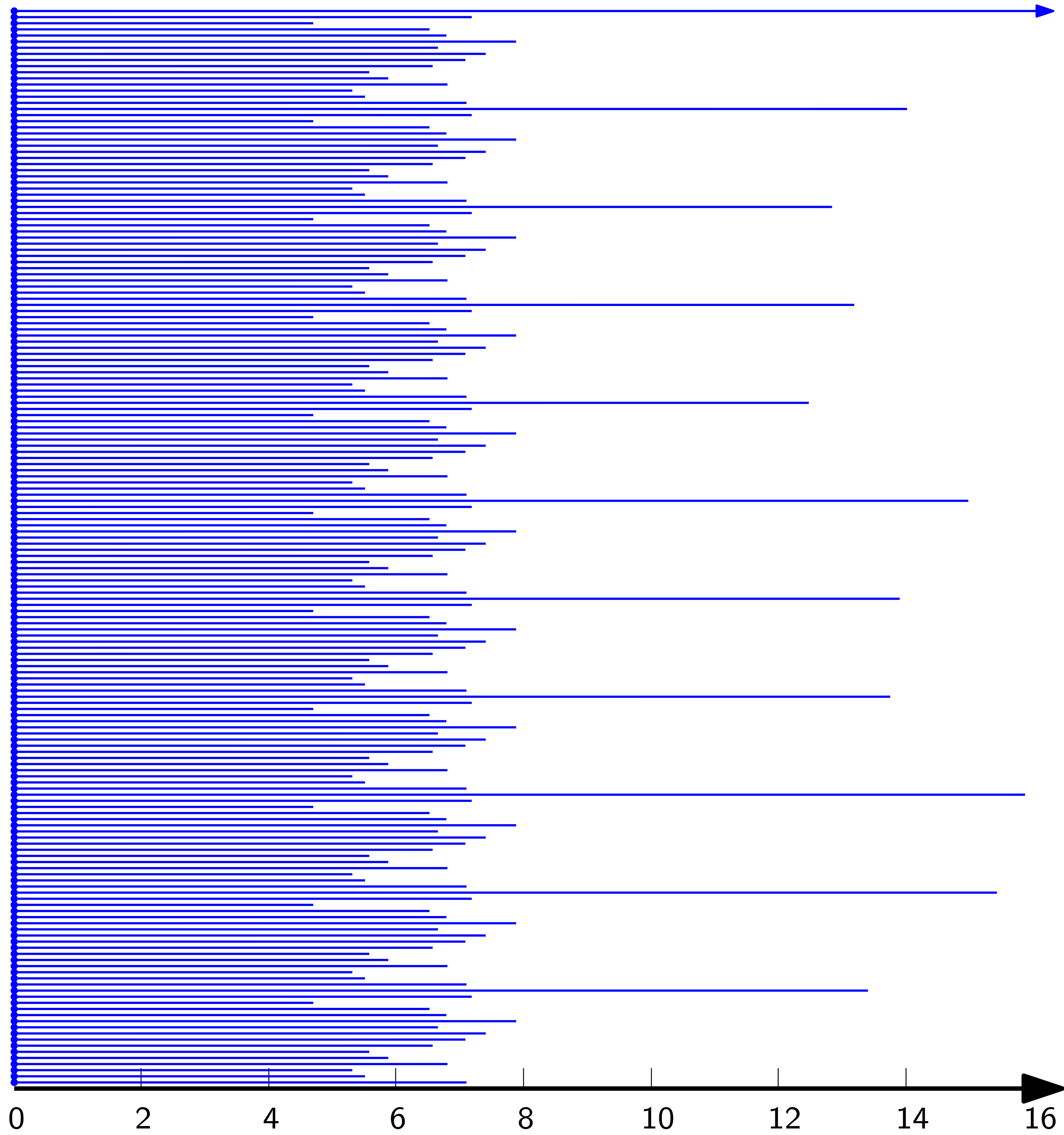
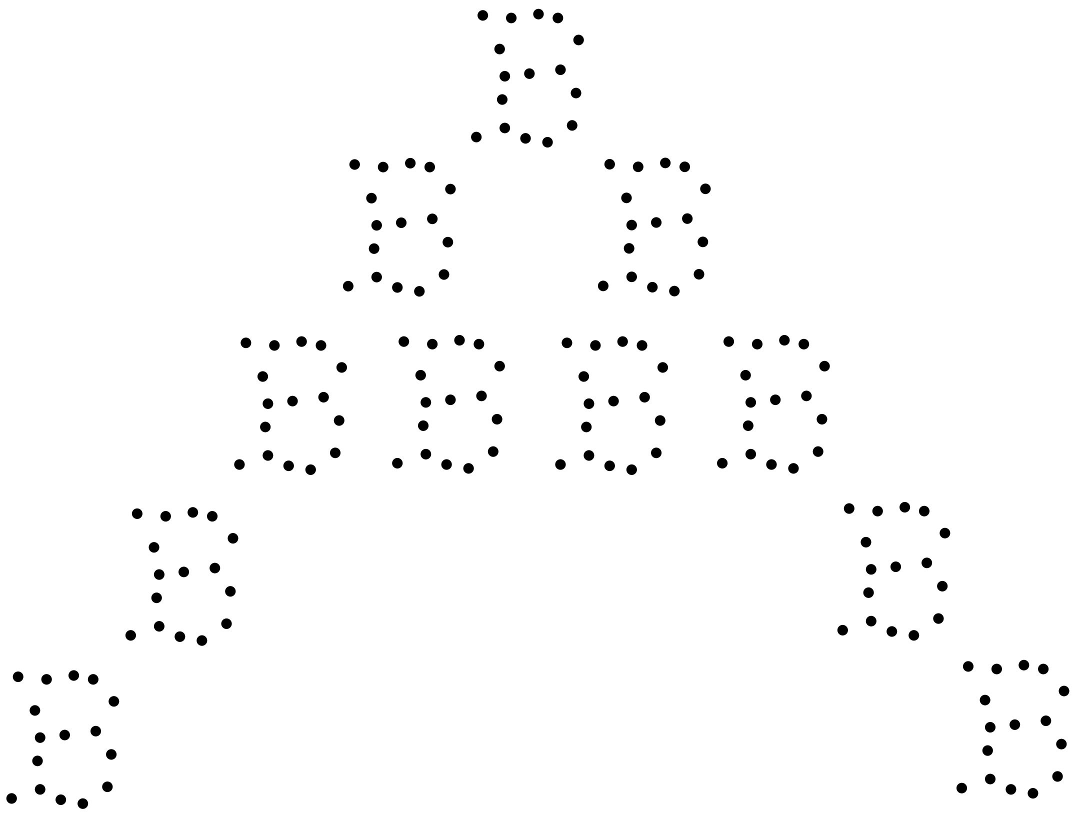




barcode is:

- less (but still) informative
- more stable

Intuitive viewpoint: hierarchical clustering



barcode is:

- less (but still) informative
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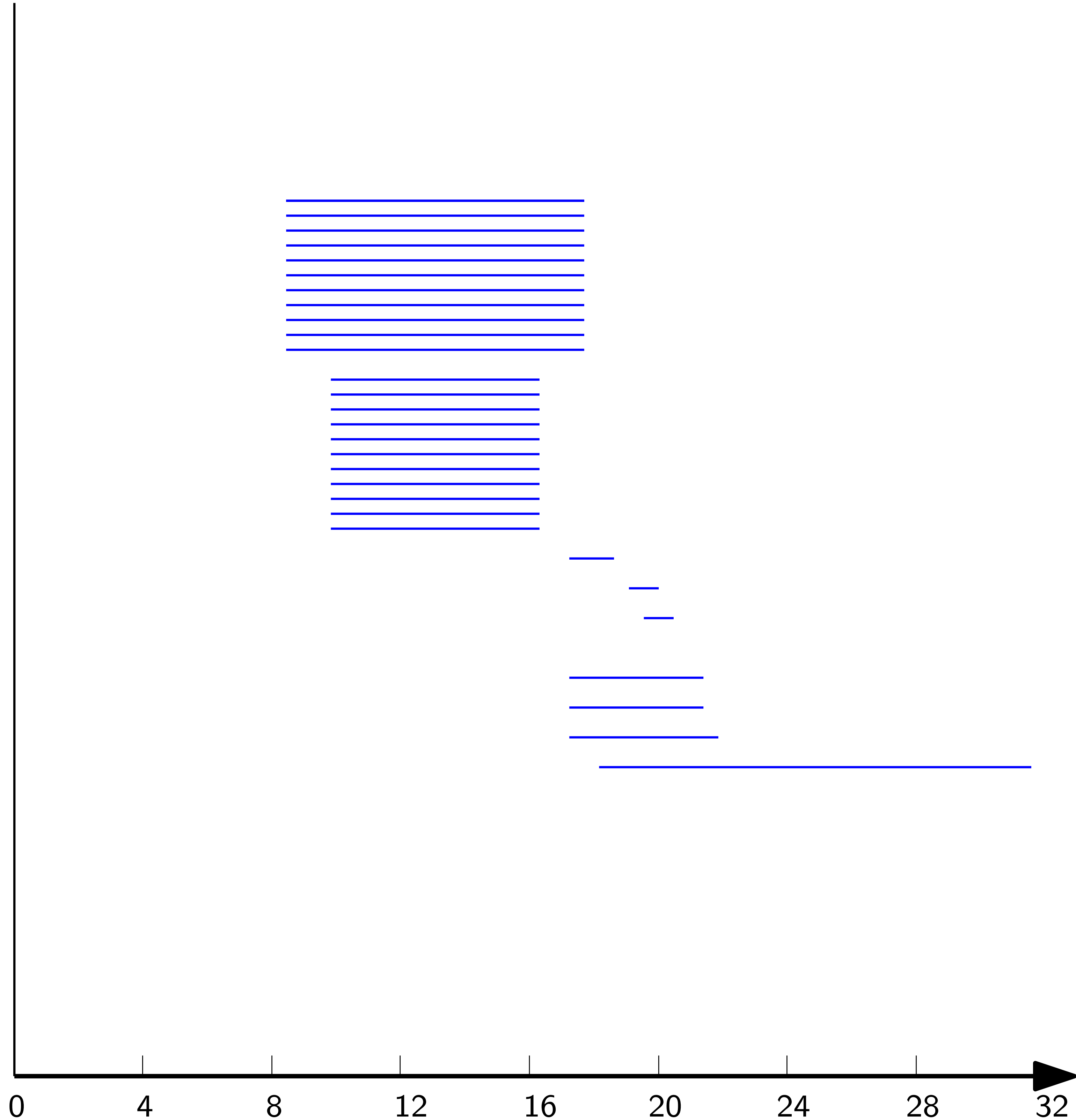
- generalizable



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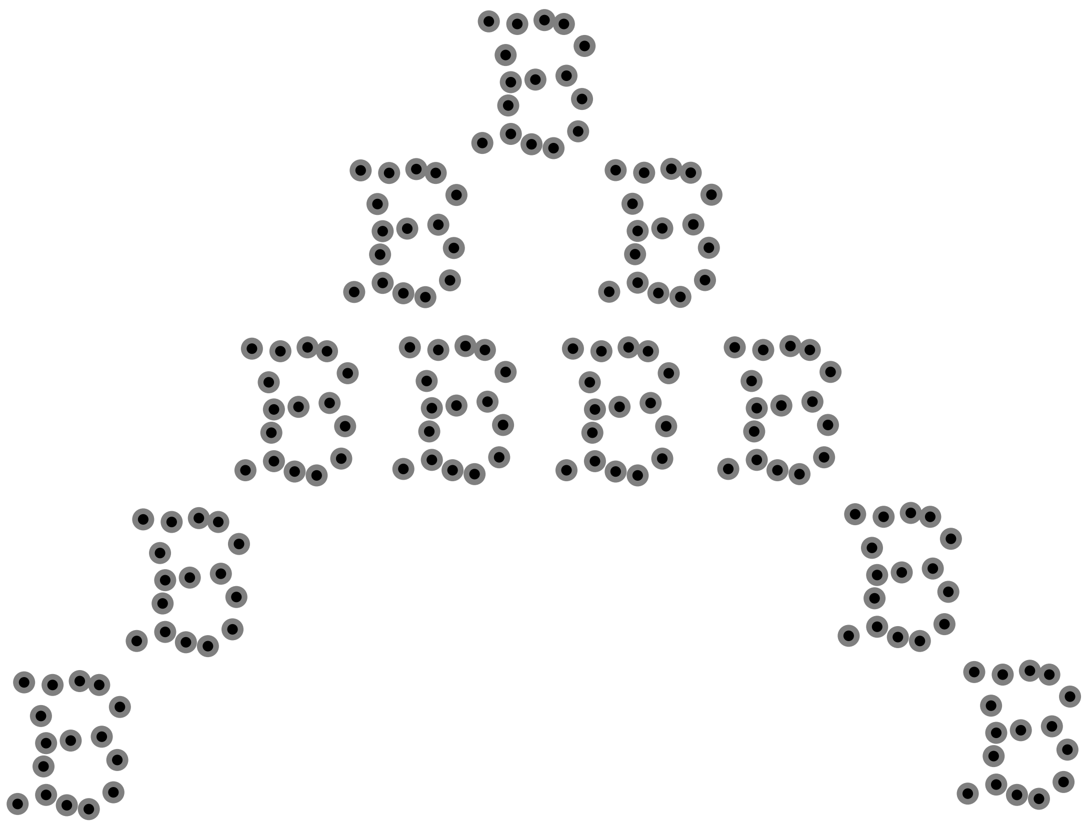

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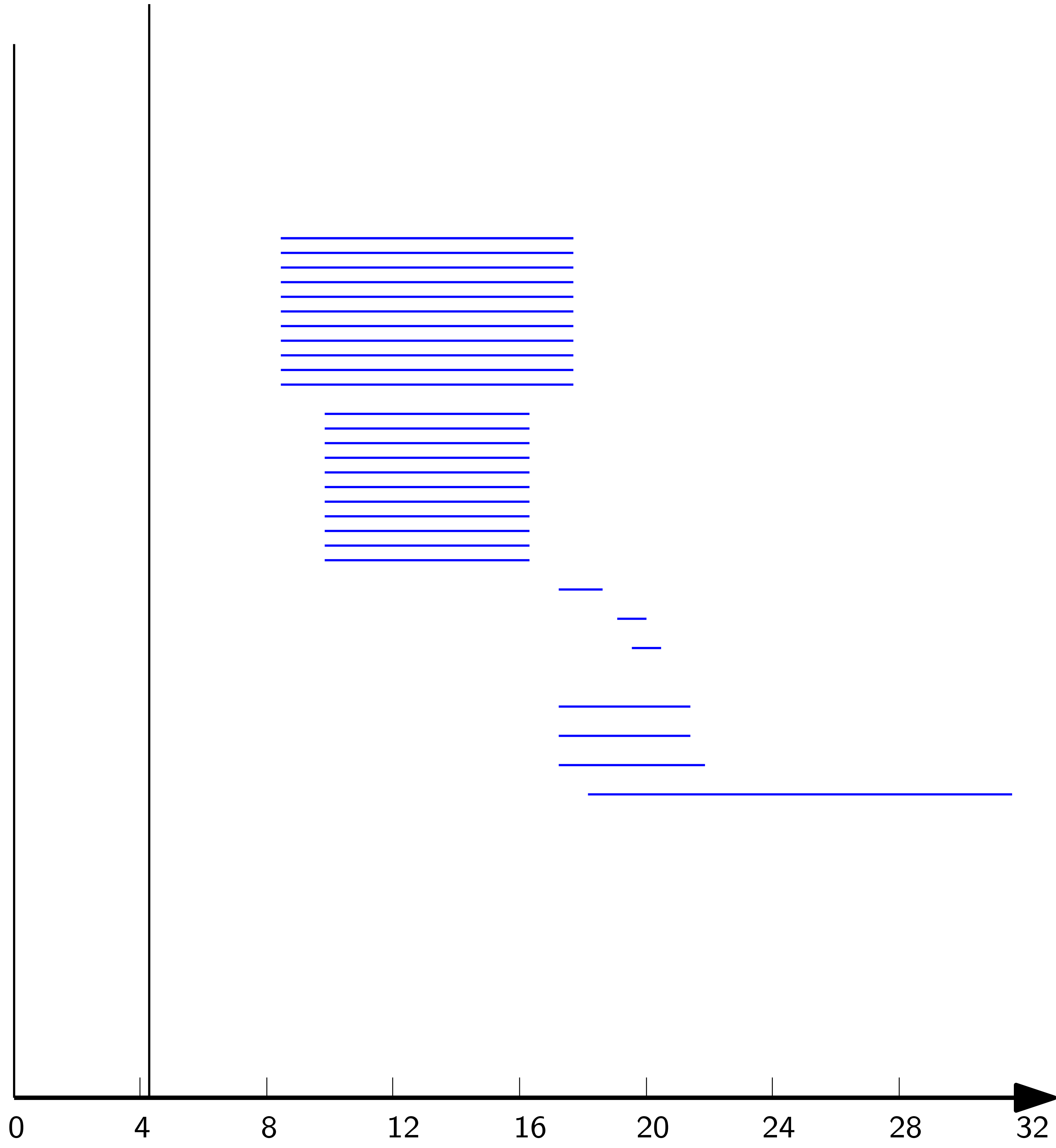

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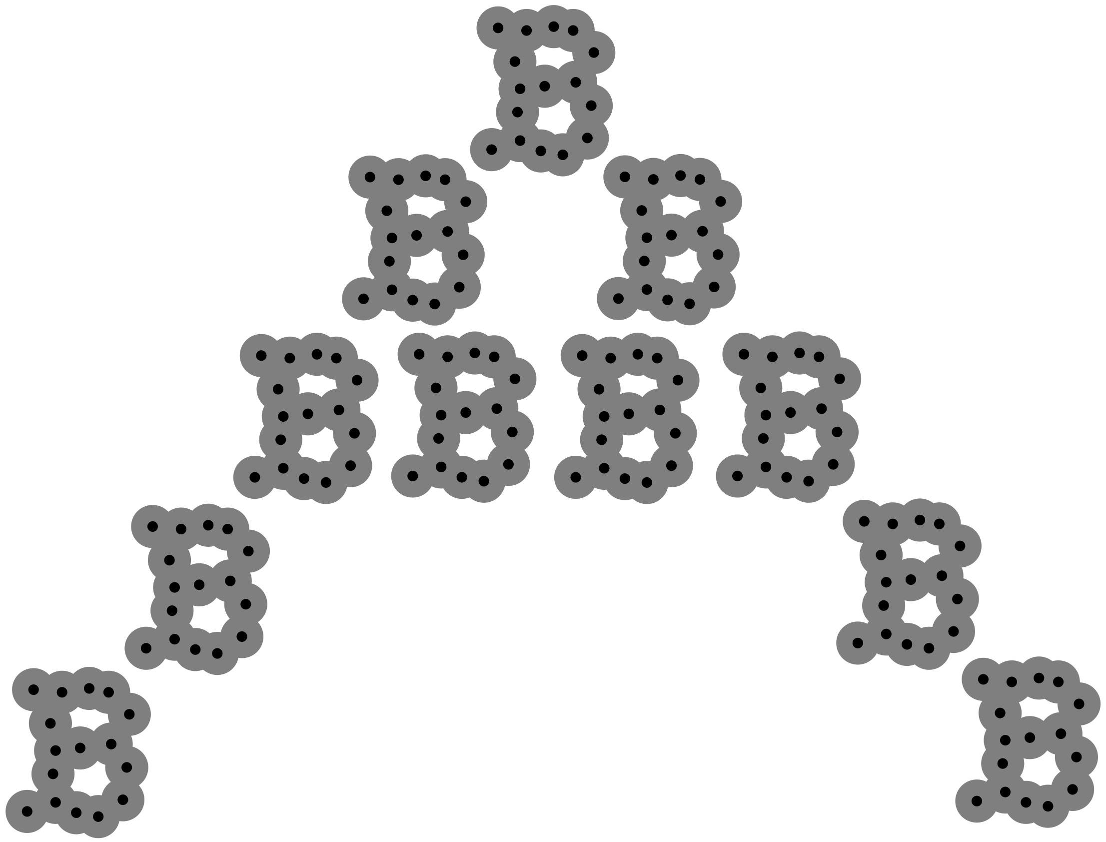

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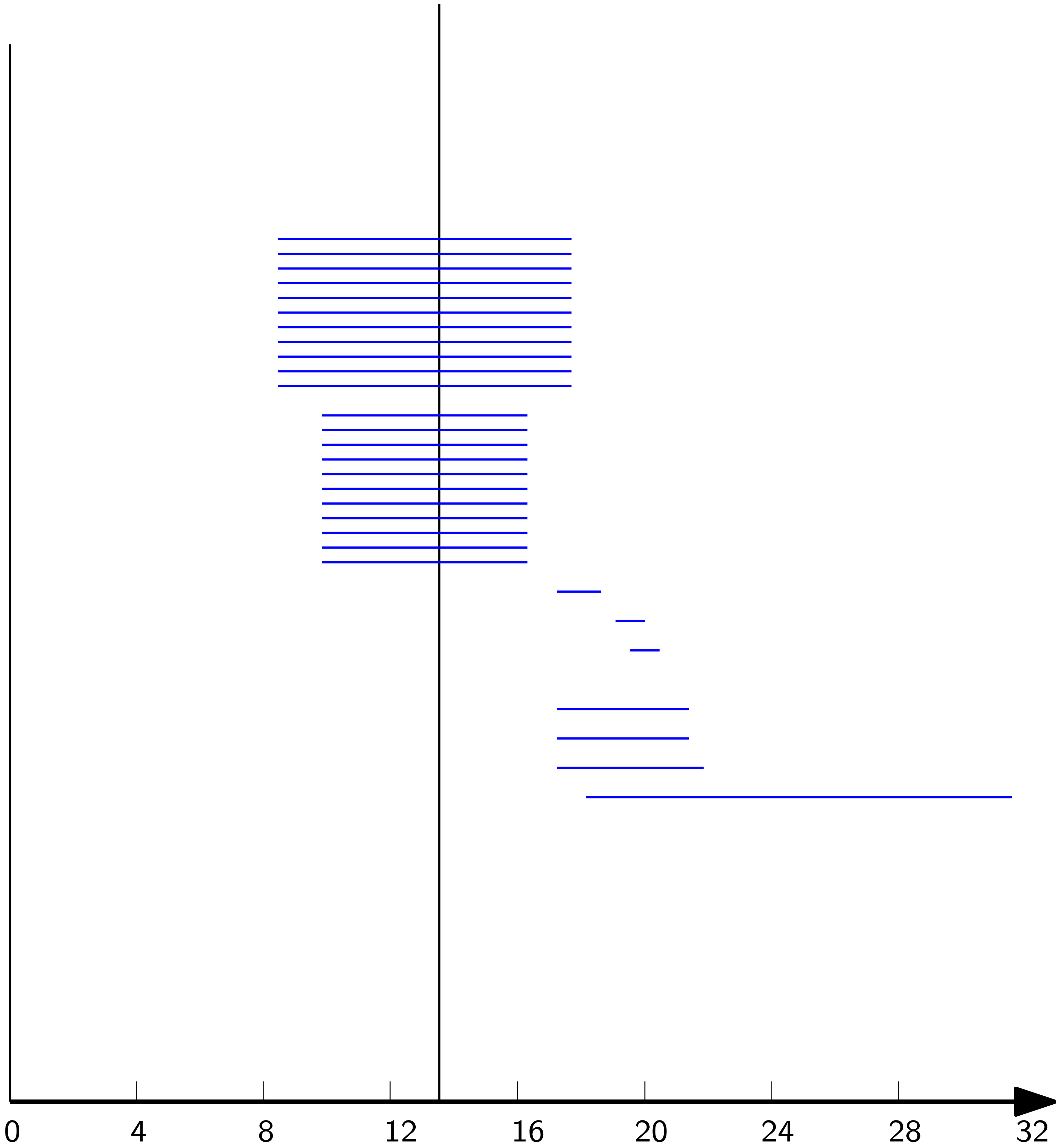

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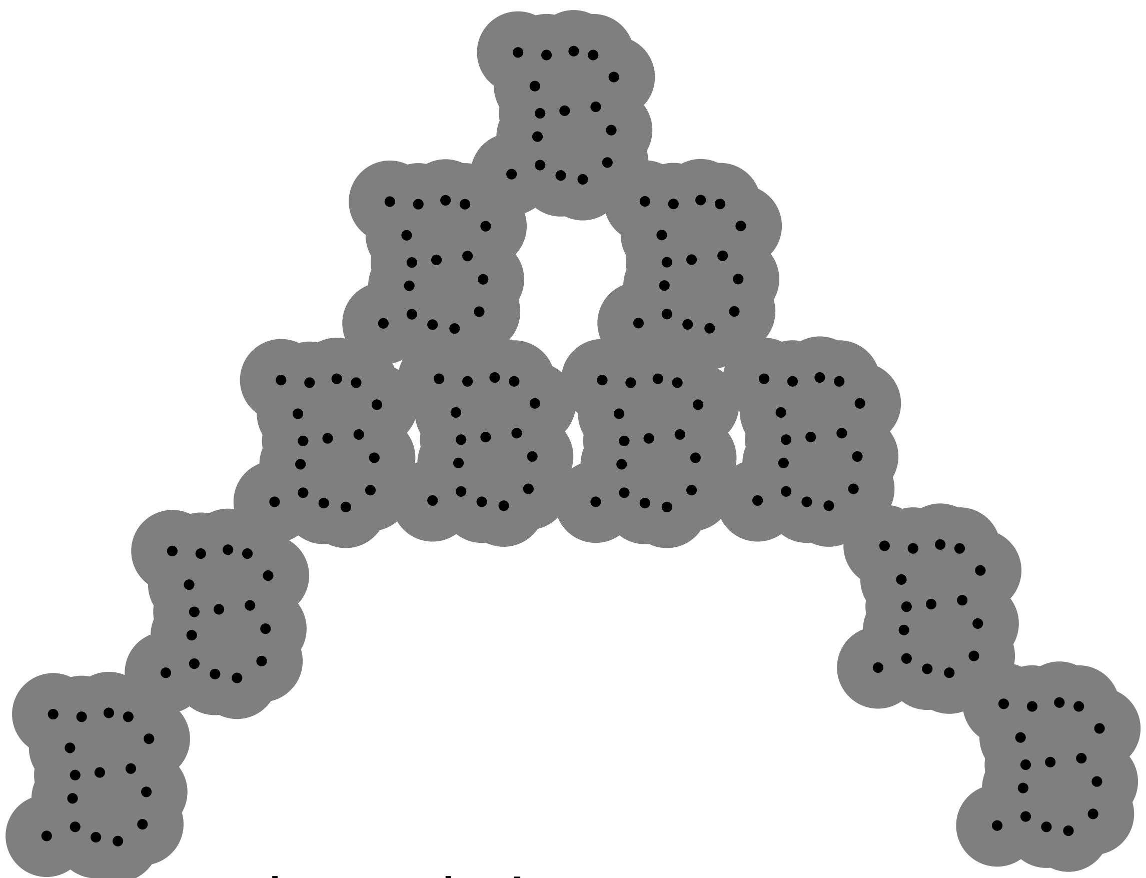

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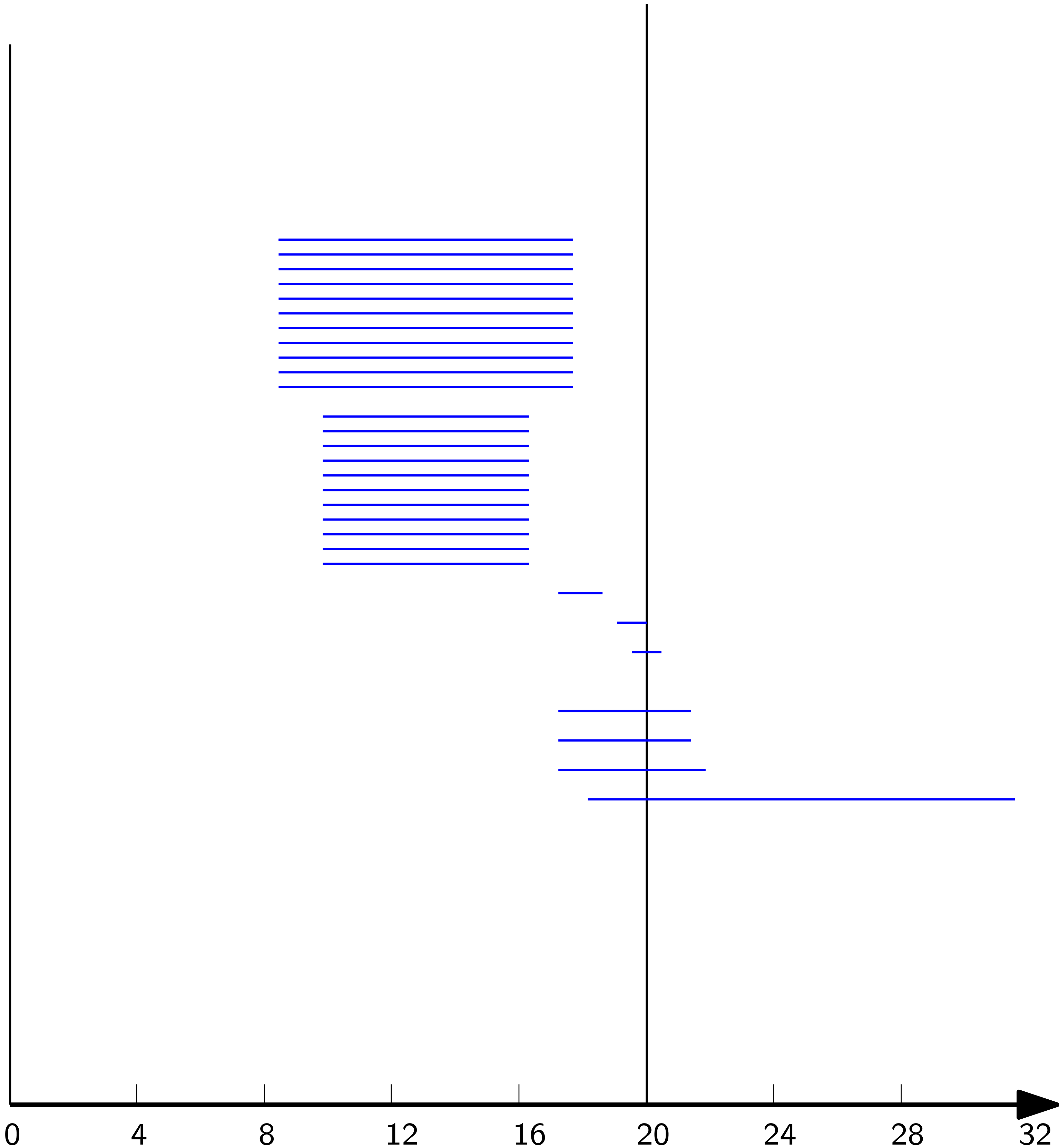

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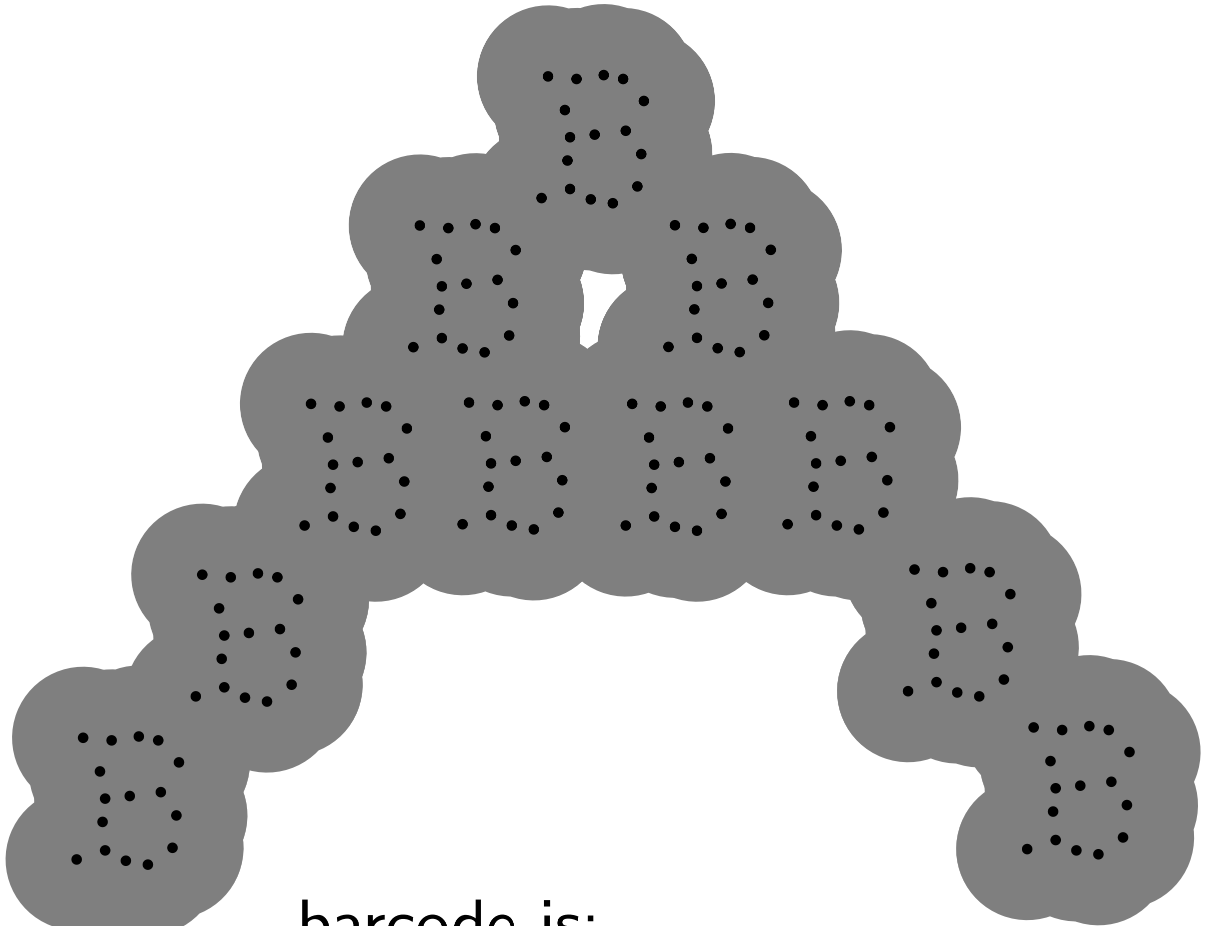

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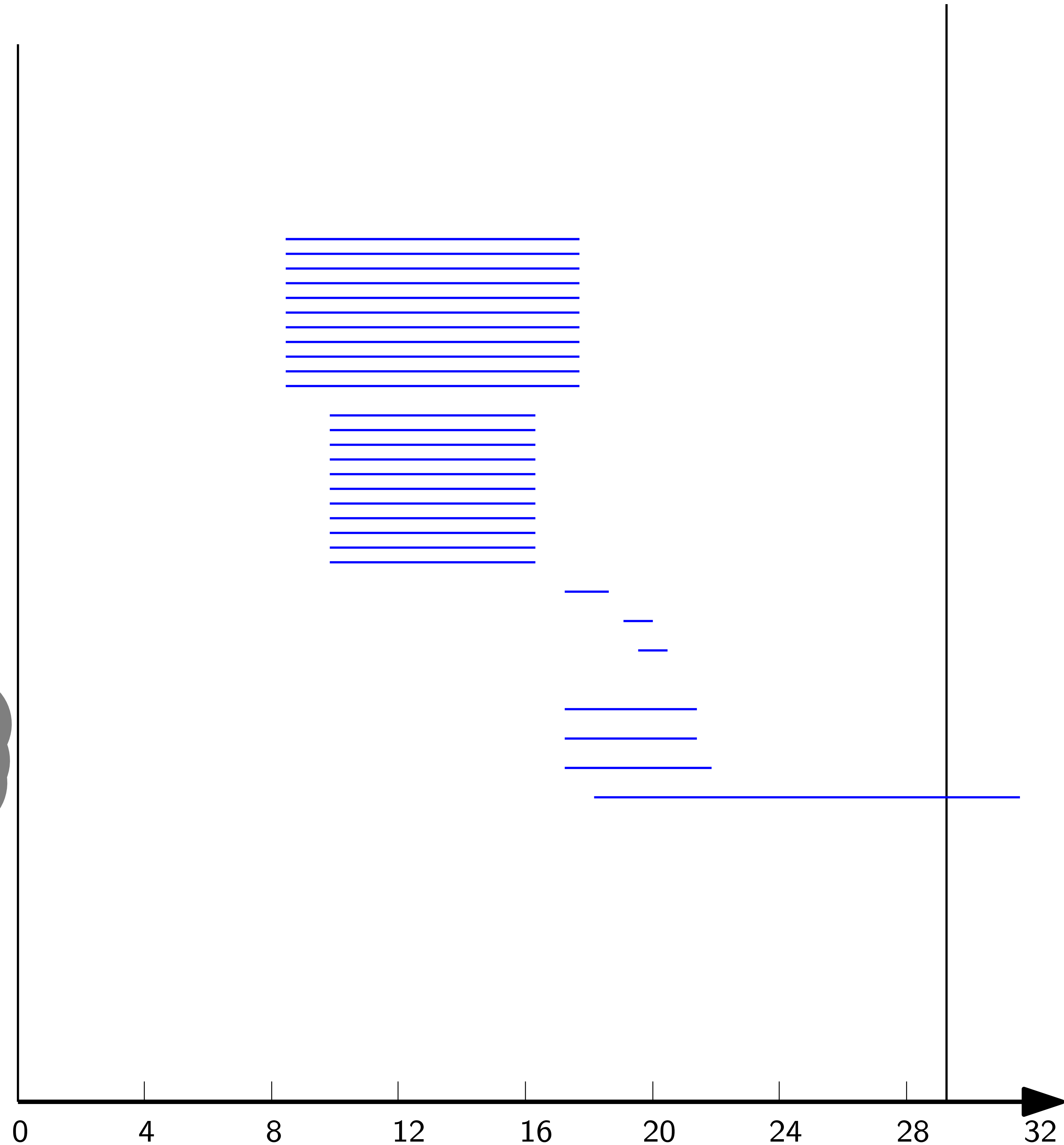

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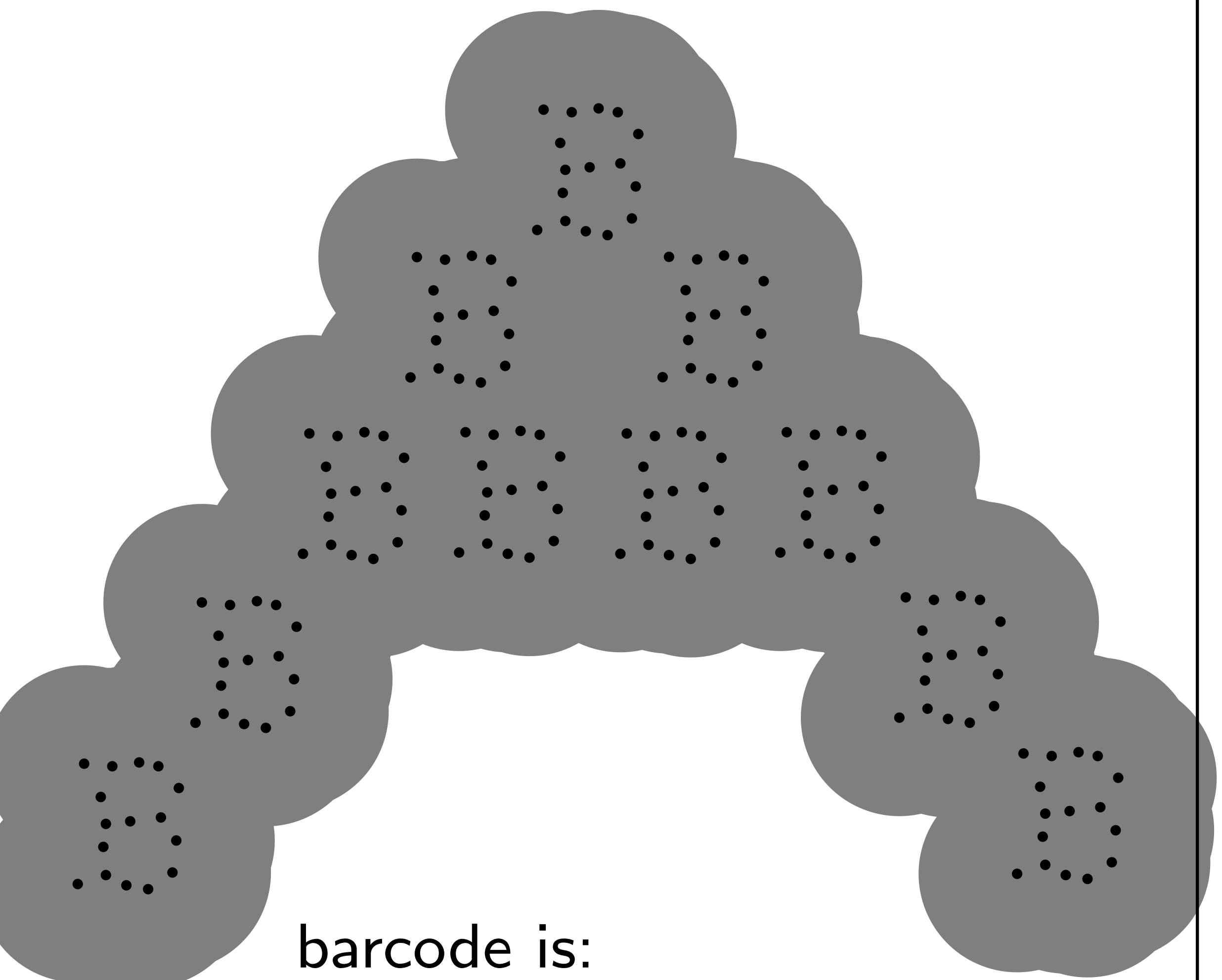

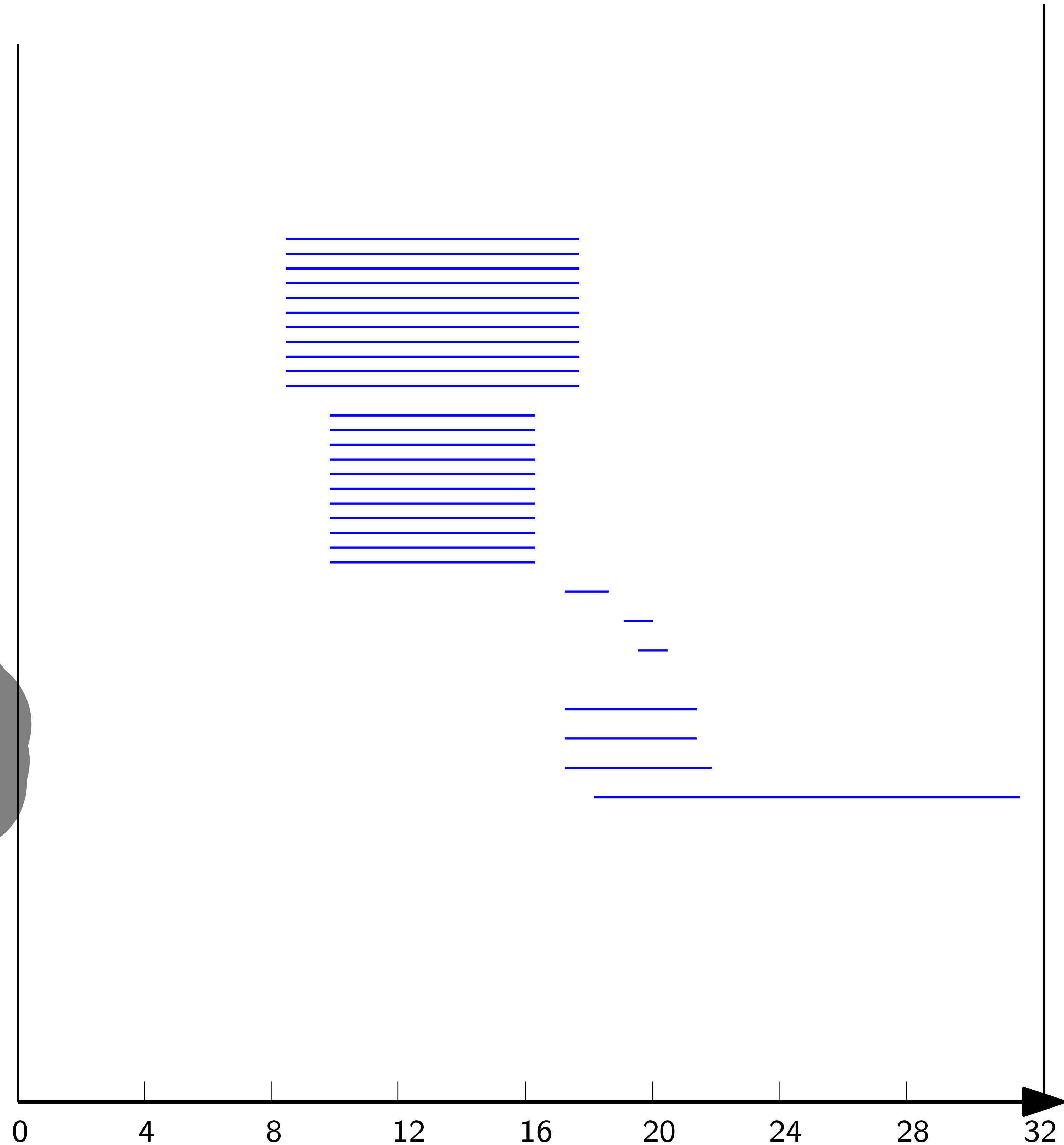

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Mathematical viewpoint

Filtration: $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 \subseteq F_5 \cdots$

Example 1: *offsets filtration* (nested family of unions of balls, cf. previous slide)

Mathematical viewpoint

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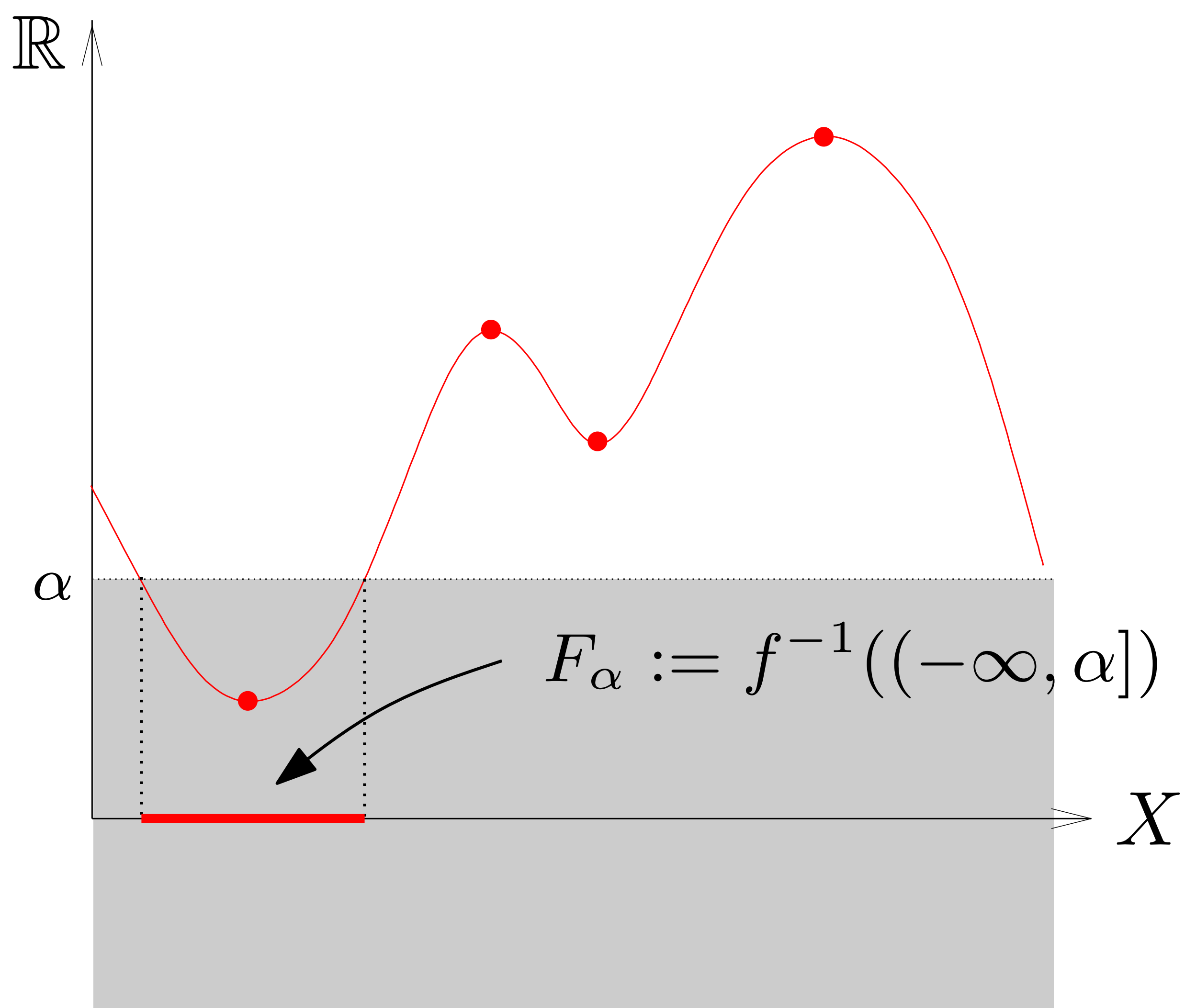
Example 1: *offsets filtration* (nested family of unions of balls, cf. previous slide)

Example 2: *sublevel-sets filtration* (family of sublevel sets of a function $f : X \to \mathbb{R}$)



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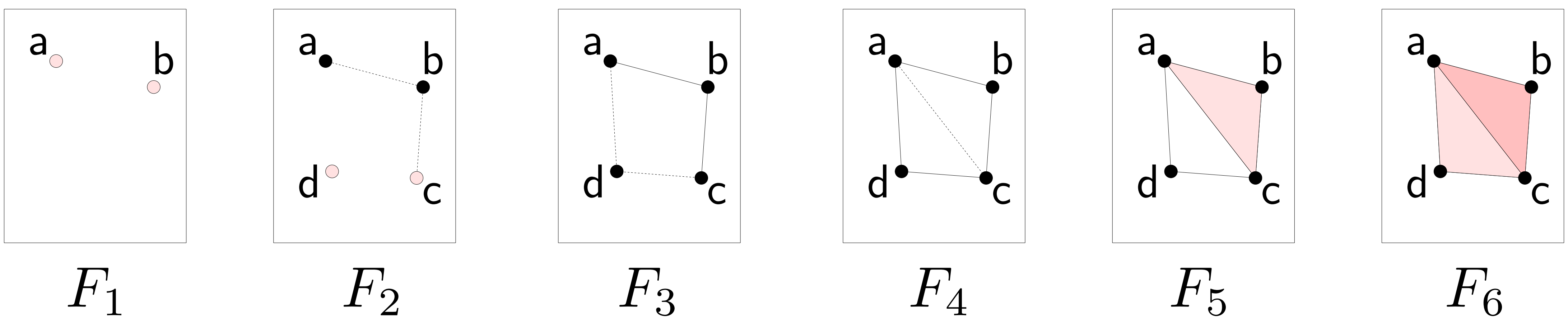
Example 1: *offsets filtration* (nested family of unions of balls, cf. previous slide)

Example 2: *sublevel-sets filtration* (family of sublevel sets of a function $f : X \to \mathbb{R}$)

Example 3: *simplicial filtration* (nested family of simplicial complexes)



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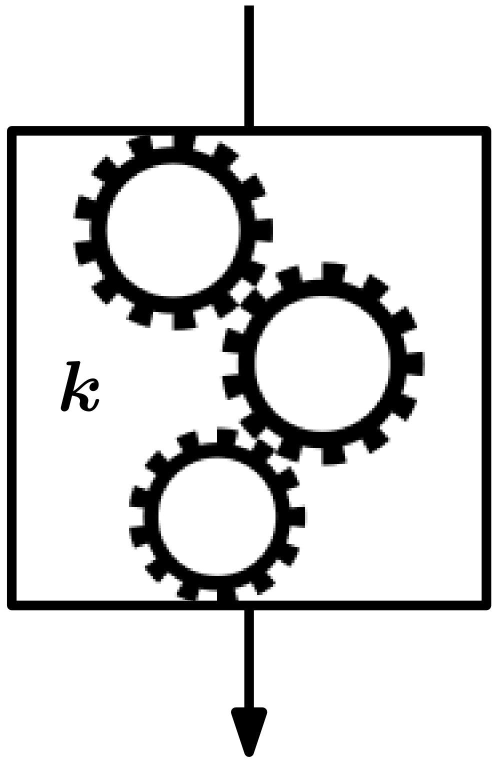


(homology functor)

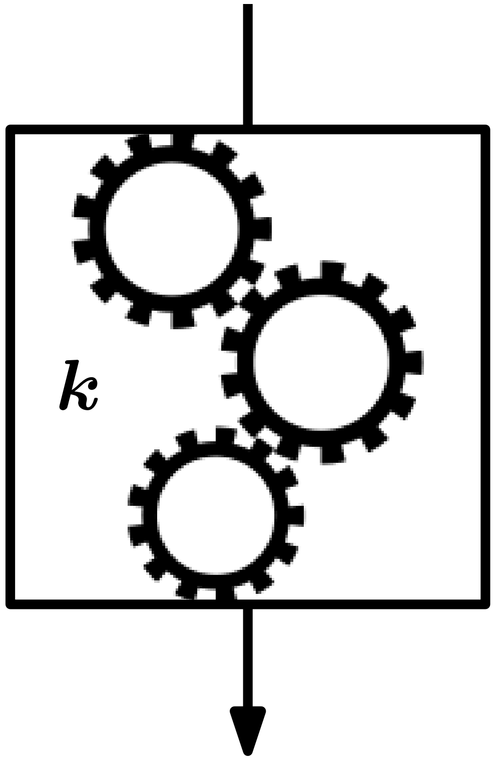
topological level

algebraic level

Persistence module: $H_*(F_1) \to H_*(F_2) \to H_*(F_3) \to H_*(F_4) \to H_*(F_5) \cdots$

Mathematical viewpoint

Zigzag: $F_1 \subseteq F_2 \supseteq F_3 \supseteq F_4 \subseteq F_5 \cdots$



(homology functor)

topological level

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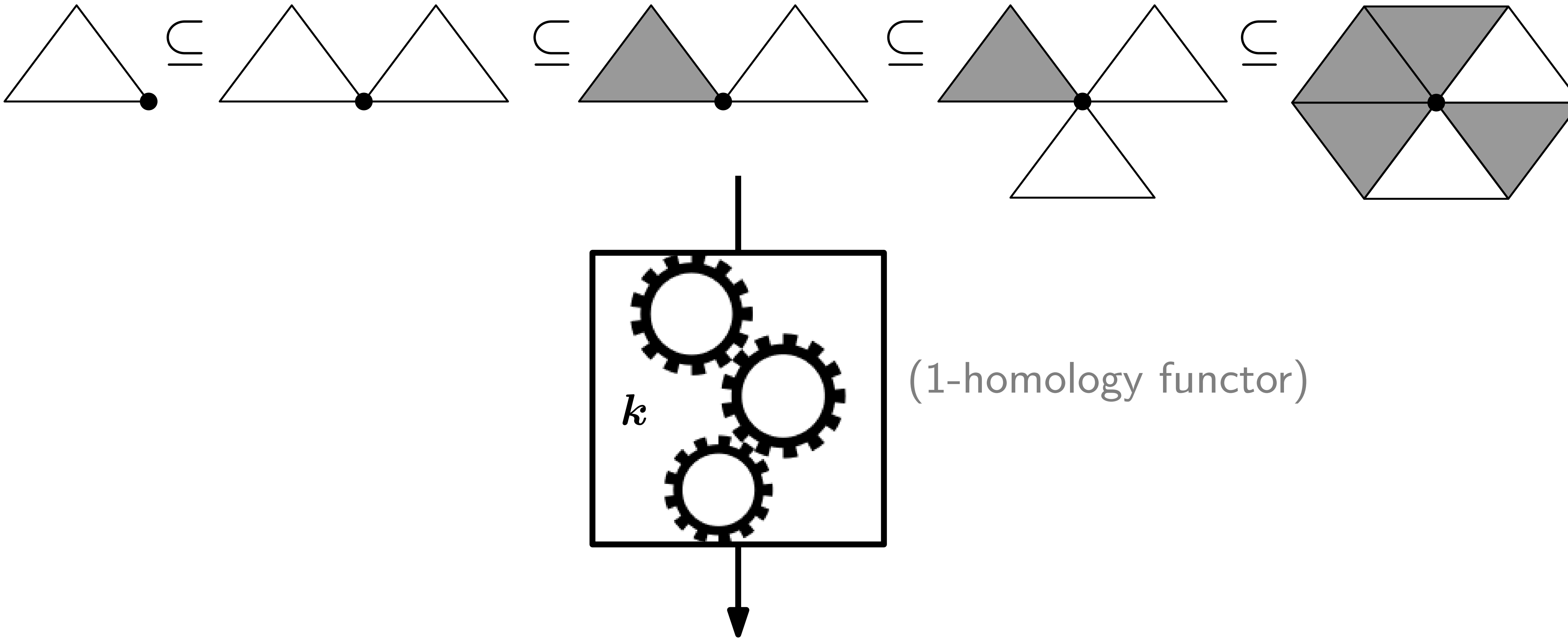
Zigzag module: $H_*(F_1) \to H_*(F_2) \leftarrow H_*(F_3) \leftarrow H_*(F_4) \to H_*(F_5) \cdots$

Mathematical viewpoint

Example:

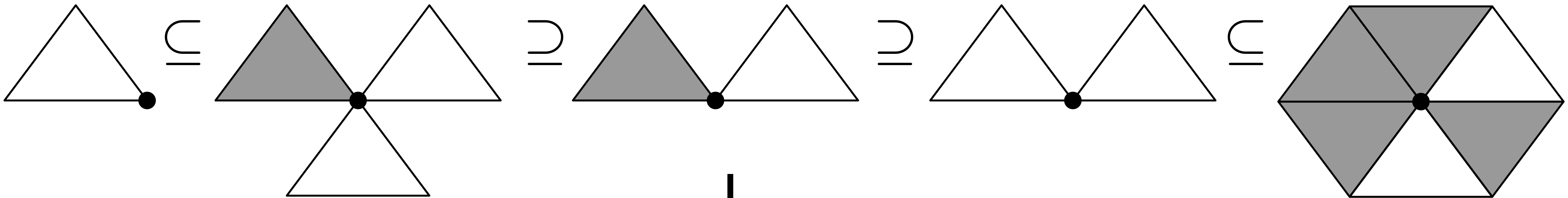


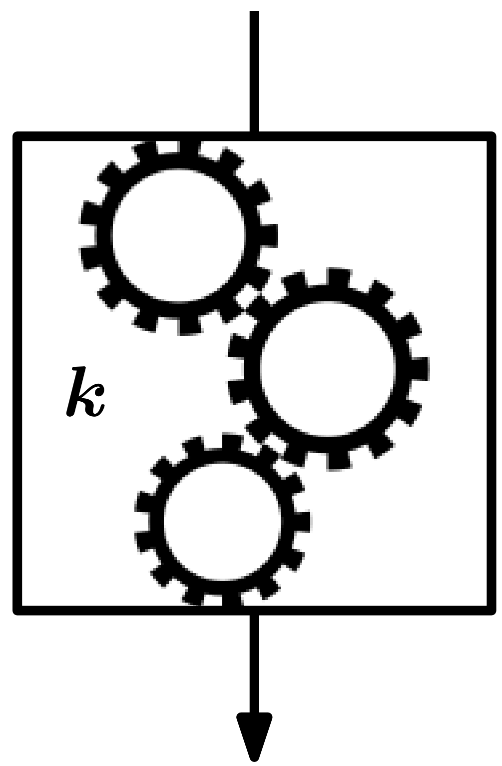
(1-homology functor)

$$\boldsymbol{k} \xrightarrow{\binom{1}{0}} \boldsymbol{k}^2 \xrightarrow{(\,0\ \ 1\,)} \boldsymbol{k} \xrightarrow{\binom{0}{1}} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix}1&0\\0&1\end{pmatrix}} \boldsymbol{k}^2 \ \cdots$$

Mathematical viewpoint

Example:





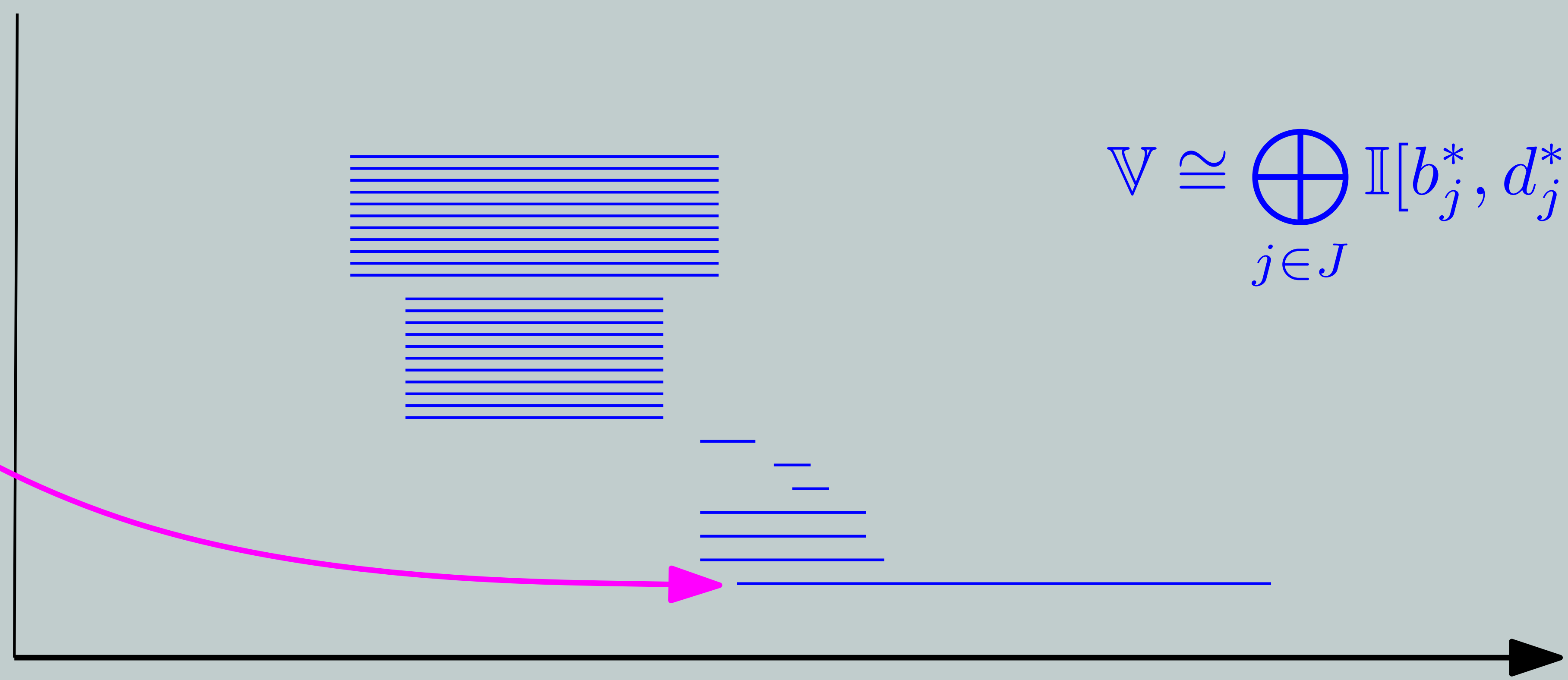
(1-homology functor)

$$\boldsymbol{k} \xrightarrow{\;0\;} \boldsymbol{k}^2 \xleftarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \boldsymbol{k} \xleftarrow{\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)} \boldsymbol{k}^2 \xrightarrow{\left(\begin{smallmatrix}0 & 1\\0 & 0\end{smallmatrix}\right)} \boldsymbol{k}^2 \quad \cdots$$

Mathematical viewpoint

Theorem. Let $\mathbb{V}$ be a persistence/zigzag module over an index set $T \subseteq \mathbb{R}$. Then, $\mathbb{V}$ decomposes as a direct sum of **interval modules** $\mathbb{I}[b^*, d^*]$:

$$\underbrace{0 \overset{0}{-\!\!-\!\!-} \cdots \overset{0}{-\!\!-\!\!-} 0}_{i<b^*} \overset{0}{-\!\!-\!\!-} \underbrace{\boldsymbol{k} \overset{\mathbb{1}}{-\!\!-\!\!-} \cdots \overset{\mathbb{1}}{-\!\!-\!\!-} \boldsymbol{k}}_{[b^*,\, d^*]} \overset{0}{-\!\!-\!\!-} \underbrace{0 \overset{0}{-\!\!-\!\!-} \cdots \overset{0}{-\!\!-\!\!-} 0}_{i>d^*}$$



$$\mathbb{V} \simeq \bigoplus_{j \in J} \mathbb{I}[b_j^*, d_j^*]$$

(the barcode is a complete descriptor of the algebraic structure of $\mathbb{V}$)

Mathematical viewpoint

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in the following cases:

- T is finite [Gabriel 1972] [Auslander 1974],
- all arrows are forward and $\mathbb{V}$ is *pointwise finite-dimensional* (i.e. every space V_t has finite dimension) [Webb 1985] [Crawley-Boevey 2012].

Moreover, when it exists, the decomposition is **unique** up to isomorphism and permutation of the terms [Azumaya 1950].

(Note: this is independent of the choice of field $\boldsymbol{k}$.)

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- T is finite [Gabriel 1972] [Auslander 1974],
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Persistence Modules vs. Quiver Representations

$\boldsymbol{k}$: field of coefficients

persistence/zigzag module: $\boldsymbol{k} \xrightarrow{0} \boldsymbol{k}^2 \xleftarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \boldsymbol{k} \xleftarrow{(0\ 1)} \boldsymbol{k}^2 \xrightarrow{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)} \boldsymbol{k}^2$

module type: $\underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \xleftarrow{b} \underset{3}{\bullet} \xleftarrow{c} \underset{4}{\bullet} \xrightarrow{d} \underset{5}{\bullet}$

Persistence Modules vs. Quiver Representations

$\boldsymbol{k}$: field of coefficients

quiver representation: $\boldsymbol{k} \xrightarrow{0} \boldsymbol{k}^2 \xleftarrow{\binom{1}{0}} \boldsymbol{k} \xleftarrow{(0\ 1)} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \boldsymbol{k}^2$

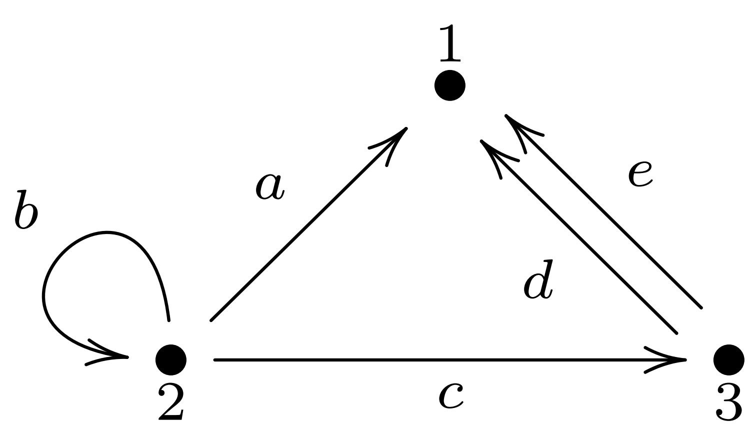
(A_n) quiver: $\underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet} \xleftarrow{b} \underset{3}{\bullet} \xleftarrow{c} \underset{4}{\bullet} \xrightarrow{d} \underset{5}{\bullet}$

Outline

- quivers and representations, classification, Gabriel's theorem
- reflection functors
- algorithm to decompose representations of A_n-type quivers
- application: computing persistence using reflections and transpositions

Quivers and Representations

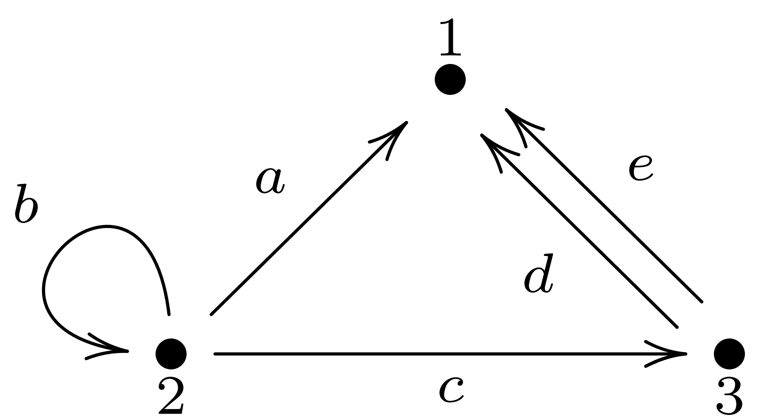
Definition: A *quiver* $\mathtt{Q}$ consists of two sets Q_0, Q_1 and two maps $s, t : Q_1 \to Q_0$. The elements in Q_0 are called the *vertices* of $\mathtt{Q}$, while those of Q_1 are called the *arrows*. The *source map* s assigns a source s_a to every arrow $a \in Q_1$, while the *target map* t assigns a target t_a.



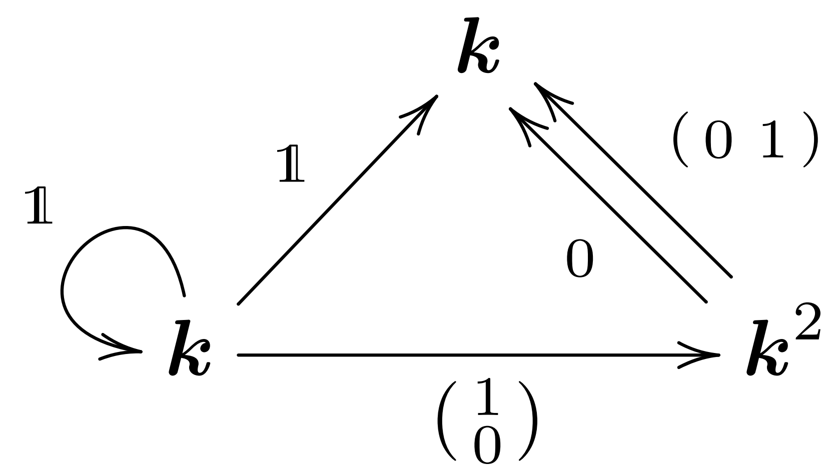
$\mathtt{Q}$

Quivers and Representations

Definition: A *representation* of $\mathtt{Q}$ over a field $\boldsymbol{k}$ is a pair $\mathbb{V} = (V_i, v_a)$ consisting of a set of $\boldsymbol{k}$-vector spaces $\{V_i \mid i \in Q_0\}$ together with a set of $\boldsymbol{k}$-linear maps $\{v_a : V_{s_a} \to V_{t_a} \mid a \in Q_1\}$.



$\mathtt{Q}$



$\in \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$

Quivers and Representations

Definition: A *subrepresentation* $\mathbb{W}$ of $\mathbb{V}$ is defined *pointwise*: $W_i \subseteq V_i$ for all $i \in Q_0$, and $w_a = v_a|_{W_{s_a}}$ for all $a \in Q_1$.

$$
\begin{tikzcd}
 & \boldsymbol{k} & \\
\boldsymbol{k} \arrow[ur, "\mathbb{1}"] \arrow[rr, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] \arrow[loop left, "\mathbb{1}"] & & \boldsymbol{k}^2 \arrow[ul, shift right, "(0\ 1)"'] \arrow[ul, shift left, "0"]
\end{tikzcd}
$$

$$\rotatebox{90}{$\subseteq$}$$

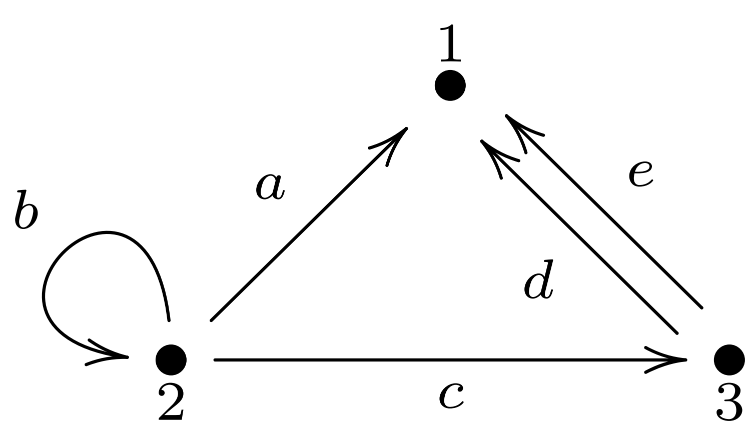
$$
\begin{tikzcd}
 & \boldsymbol{k} & \\
\boldsymbol{k} \arrow[ur, "\mathbb{1}"] \arrow[rr, "\mathbb{1}"'] \arrow[loop left, "\mathbb{1}"] & & \boldsymbol{k} \arrow[ul, shift right, "0"'] \arrow[ul, shift left, "0"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 1 & \\
2 \arrow[ur, "a"] \arrow[rr, "c"'] \arrow[loop left, "b"] & & 3 \arrow[ul, shift right, "e"'] \arrow[ul, shift left, "d"]
\end{tikzcd}
$$

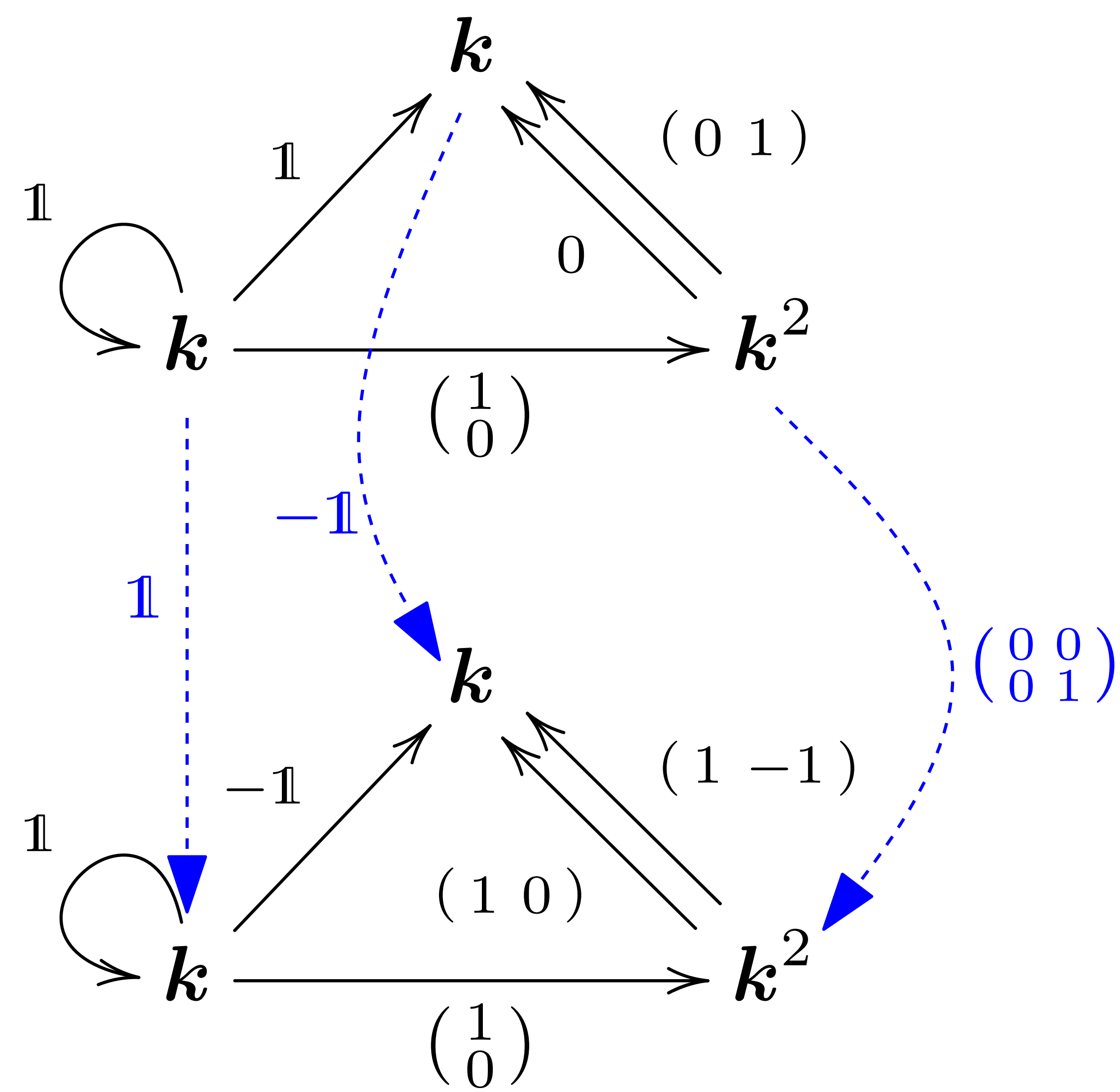
Q

Quivers and Representations

Definition: A *morphism* ϕ between two $\boldsymbol{k}$-representations $\mathbb{V}, \mathbb{W}$ of $\mathtt{Q}$ is a set of $\boldsymbol{k}$-linear maps $\phi_i : V_i \to W_i$ such that $w_a \circ \phi_{s_a} = \phi_{t_a} \circ v_a$ for every arrow $a \in Q_1$.



$\mathtt{Q}$



$\in \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$

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\boldsymbol{k} \arrow[r, "0"] & 0 & \boldsymbol{k} \arrow[l, "0"'] & \boldsymbol{k} \arrow[l, "\mathbb{1}"'] \arrow[r, "0"] & 0
\end{tikzcd}
$$

every quadrangle commutes

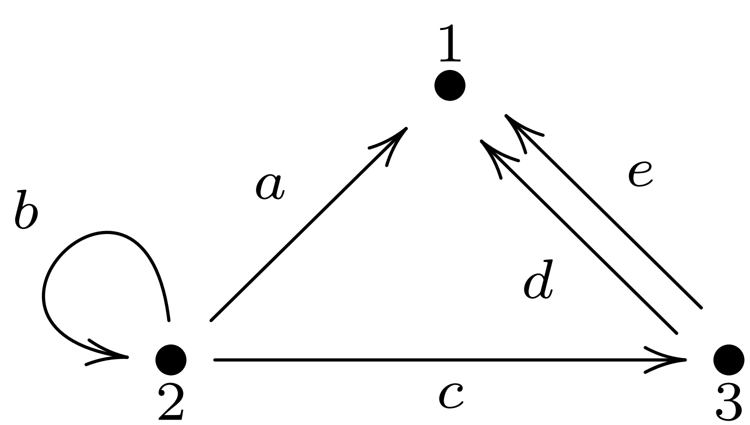
Quivers and Representations

In categorical terms:

quiver $\equiv$ category

representation $\equiv$ functor

morphism $\equiv$ natural transformation



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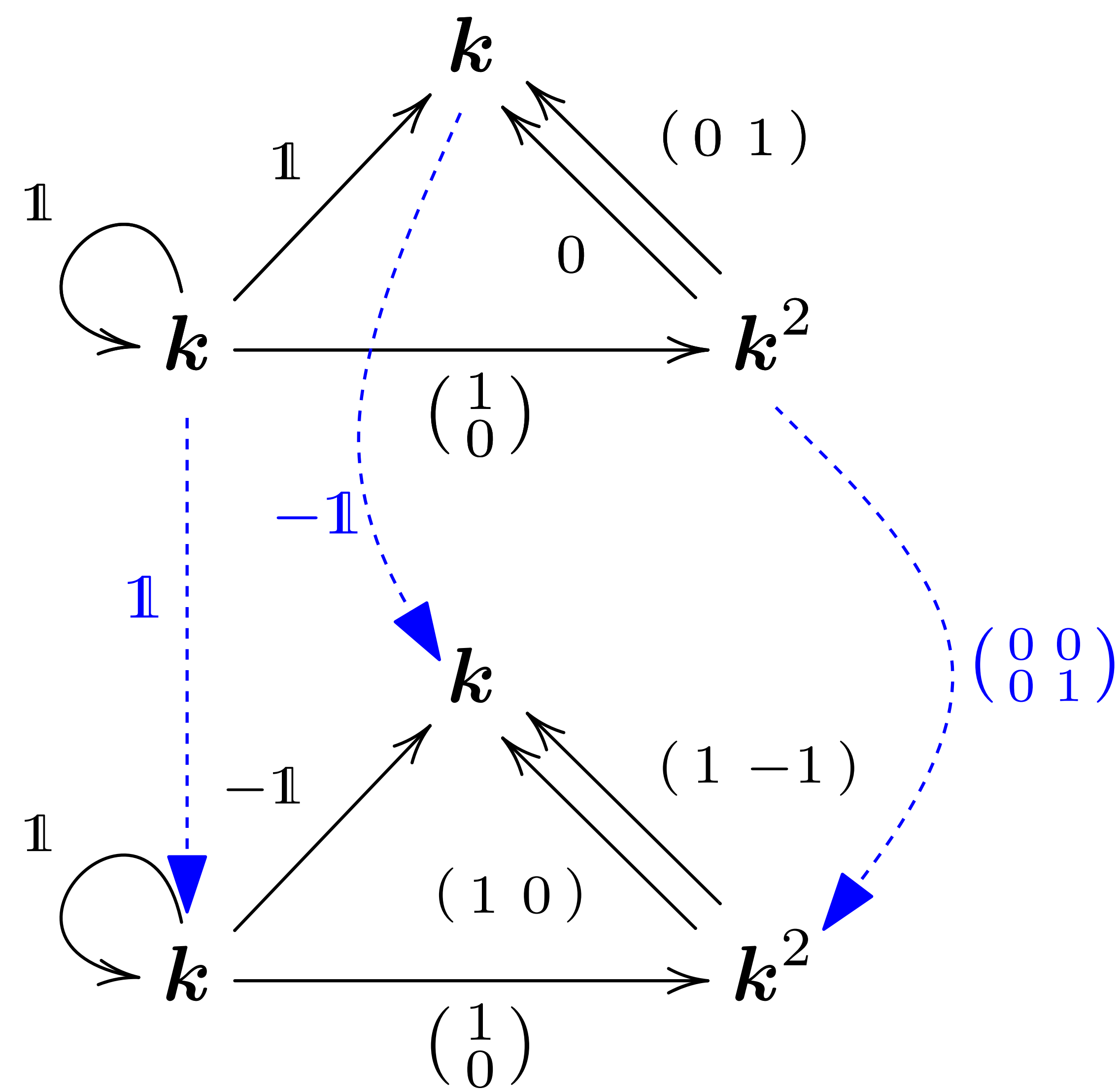

$\in \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$

The Category of Representations

The representations of a quiver $\mathtt{Q} = (Q_0, Q_1)$, together with their morphisms, form a category called $\mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$. This category is **abelian**:

- zero object: the trivial representation

$$0 \longrightarrow 0 \longleftarrow 0 \longleftarrow 0 \longrightarrow 0$$

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- internal and external direct sums, defined *pointwise*:
Given $\mathbb{V}, \mathbb{W}$, $\mathbb{V} \oplus \mathbb{W}$ has spaces $V_i \oplus W_i$ for $i \in Q_0$ and maps $v_a \oplus w_a = \begin{pmatrix} v_a & 0 \\ 0 & w_a \end{pmatrix}$ for $a \in Q_1$

$$\boldsymbol{k} \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \boldsymbol{k}^2 \xleftarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \boldsymbol{k} \xleftarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} \boldsymbol{k}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \boldsymbol{k}^2$$

$$\bigoplus$$

$$\boldsymbol{k} \xrightarrow{0} 0 \xleftarrow{0} \boldsymbol{k} \xleftarrow{\mathbb{1}} \boldsymbol{k} \xrightarrow{0} 0$$

$$=$$

$$\boldsymbol{k}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \boldsymbol{k}^2 \xleftarrow{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}} \boldsymbol{k}^2 \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \boldsymbol{k}^3 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} \boldsymbol{k}^2$$

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- every morphism ϕ has a *kernel*, an *image* and a *cokernel*, defined *pointwise*.

  $\rightarrow$ ϕ monomorphism iff $\ker \phi = 0$, epimorphism iff $\operatorname{coker} \phi = 0$.

$$\begin{tikzcd}
\boldsymbol{k} \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] \arrow[d, "\mathbb{1}"'] & \boldsymbol{k}^2 \arrow[d, "0"'] & \boldsymbol{k} \arrow[l, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"'] \arrow[d, "-\mathbb{1}"'] & \boldsymbol{k}^2 \arrow[l, "(\,0\;1\,)"'] \arrow[r, "\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)"] \arrow[d, "(\,0\;-1\,)"'] & \boldsymbol{k}^2 \arrow[d, "0"'] \\
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\end{tikzcd}$$

$$\ker \phi = \quad 0 \xrightarrow{0} \boldsymbol{k}^2 \xleftarrow{0} 0 \xleftarrow{0} \boldsymbol{k} \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} \boldsymbol{k}^2 \qquad \operatorname{coker} \phi = 0$$

The Classification Problem

Goal: Classify the representations of a given quiver $\mathtt{Q} = (Q_0, Q_1)$ up to isomorphism.

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typically hard $\rightarrow$ simplifying assumptions:

- $\mathtt{Q}$ is finite and connected
- study the subcategory $\mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$ of *finite-dimensional* representations

$$\underline{\dim}\,\mathbb{V} = (\dim V_1, \cdots, \dim V_n)^\top,$$
$$\dim \mathbb{V} = \|\underline{\dim}\,\mathbb{V}\|_1 = \sum_{i=1}^{n} \dim V_i.$$

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Theorem: [Krull-Remak-Schmidt-Azumaya]
$\forall \mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$, $\exists\ \mathbb{V}_1, \cdots, \mathbb{V}_r$ *indecomposable* s.t. $\mathbb{V} \cong \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_r$.
The decomposition is unique up to isomorphism and reordering.

note: $\mathbb{V}$ indecomposable iff there are no $\mathbb{U}, \mathbb{W} \neq 0$ such that $\mathbb{V} \cong \mathbb{U} \oplus \mathbb{W}$

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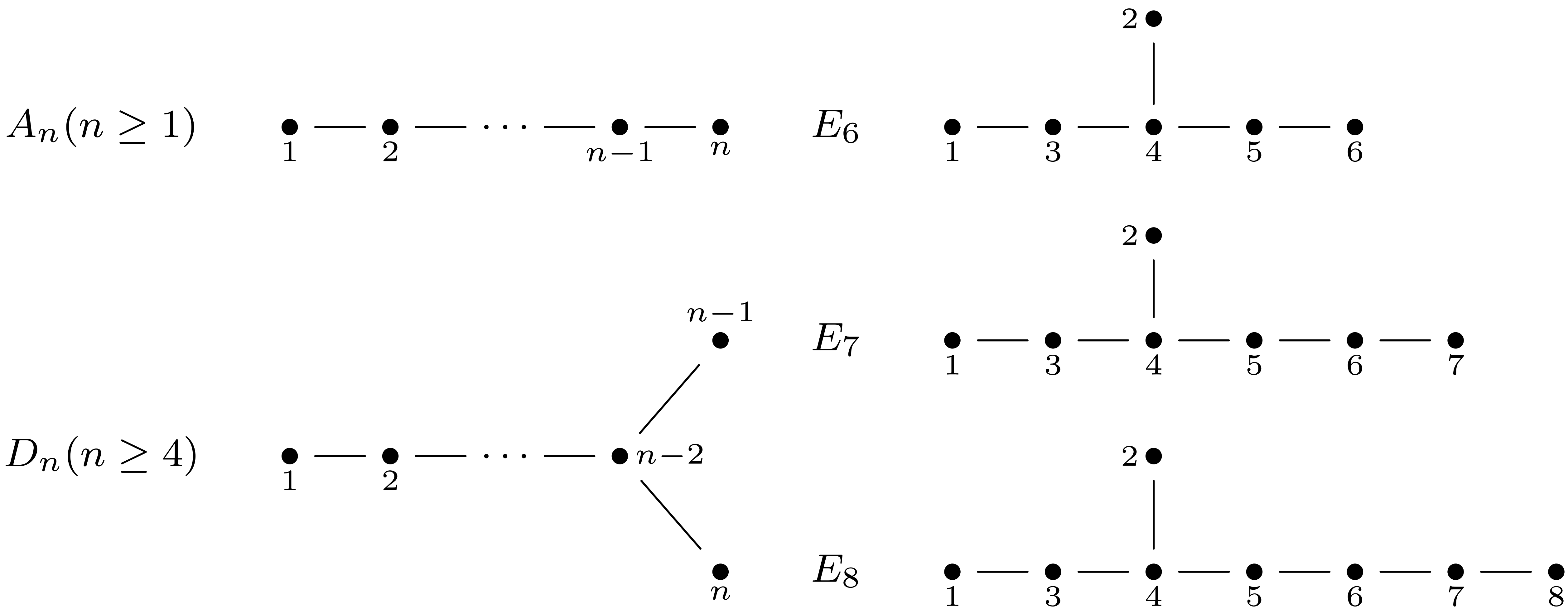
The decomposition is unique up to isomorphism and reordering.

$\rightarrow$ identify the indecomposable representations of $\mathtt{Q}$

(still non-trivial: subrepresentations may not be summands)

Gabriel's Theorem

Theorem: [Gabriel I]
Assuming Q is finite and connected, there are finitely many isomorphism classes of indecomposable representations in $\mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$ iff Q is Dynkin.



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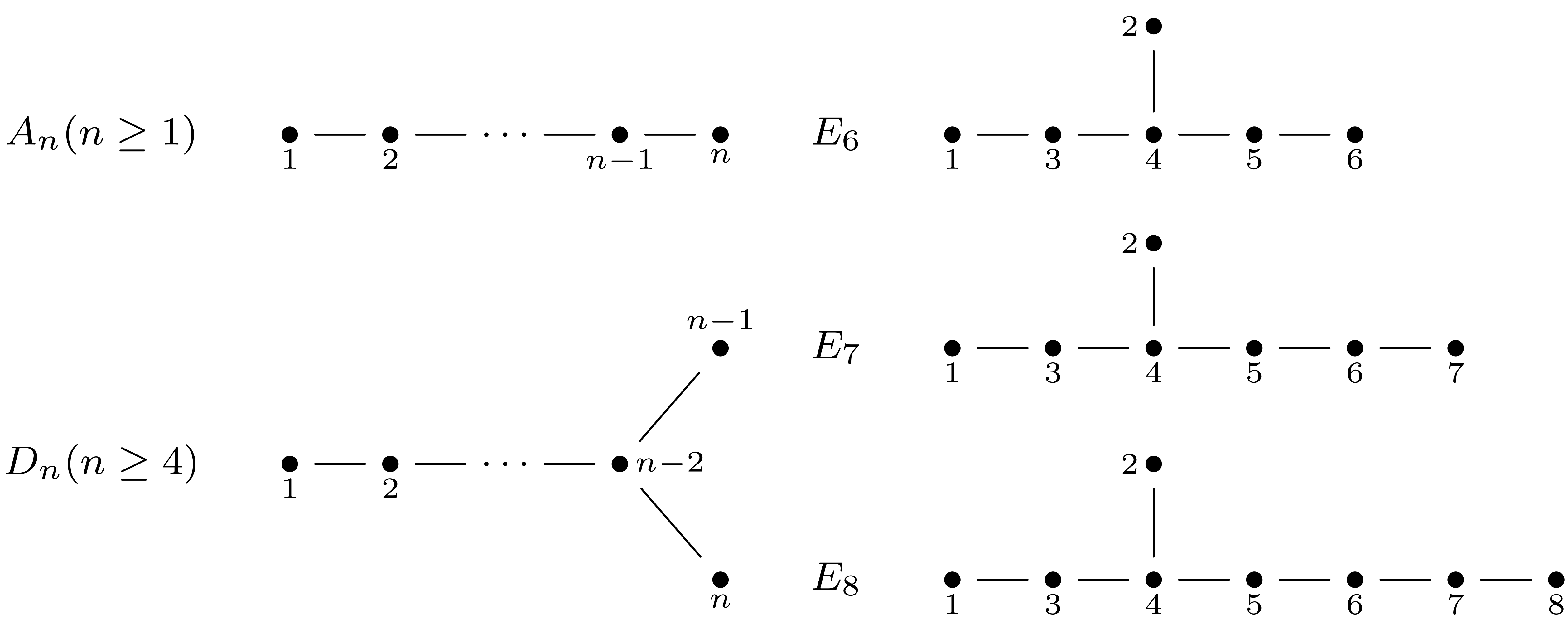

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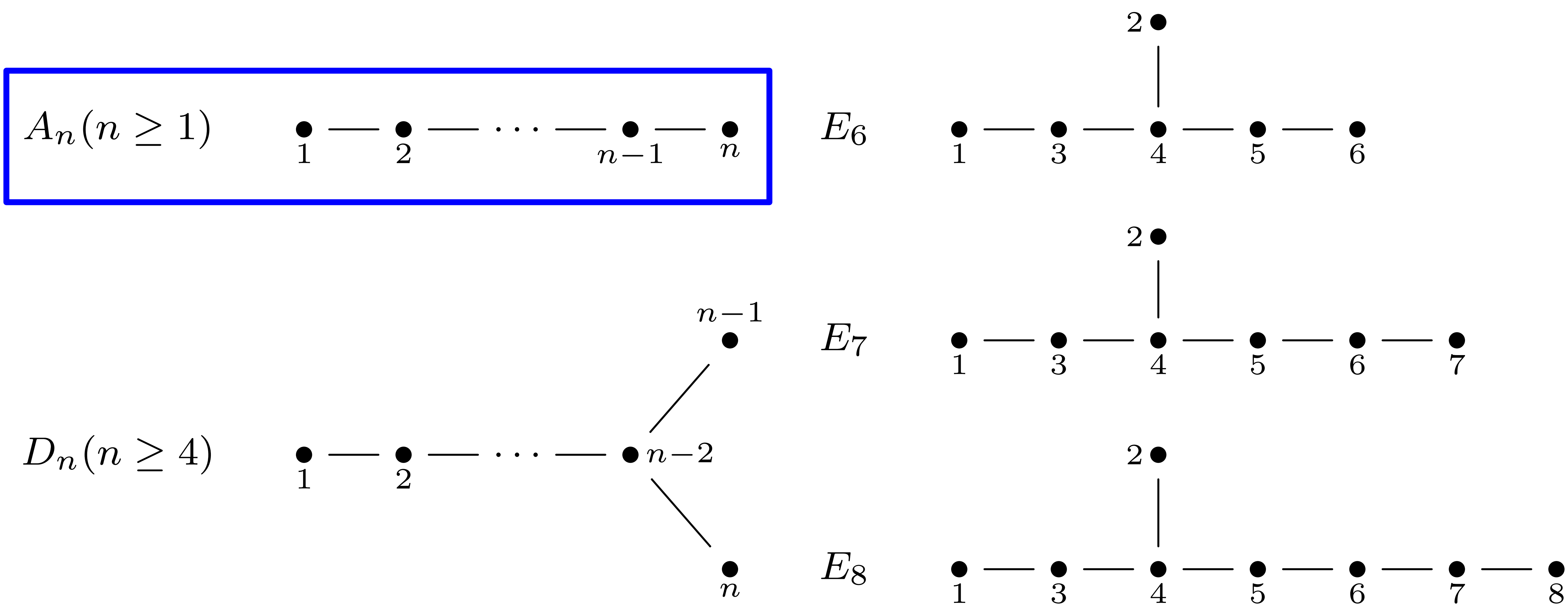

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Q given that $\mathtt{Q}$ is Dynkin, how to identify indecomposable representations?

Theorem: [Gabriel II]
Assuming $\mathtt{Q}$ is Dynkin with n vertices, the map $\mathbb{V} \mapsto \underline{\dim}\,\mathbb{V}$ induces a bijection between the set of isomorphism classes of indecomposable representations of $\mathtt{Q}$ and the set of *positive roots* of the *Tits form* of $\mathtt{Q}$.

(isom. classes of indecomposables are fully characterized by their dim. vectors)

Gabriel's Theorem

example: $\mathtt{Q}$ of type A_n:

$$\underset{1}{\bullet} \text{———} \underset{2}{\bullet} \text{———} \cdots \text{———} \underset{n-1}{\bullet} \text{———} \underset{n}{\bullet}$$

$$\begin{aligned} q_{\mathtt{Q}}(x) &= \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1} \\ &= \sum_{i=1}^{n-1} \frac{1}{2}(x_i - x_{i+1})^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_n^2 \\ &= 1 \text{ iff } x = (0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0) \end{aligned}$$

the corresponding indecomp. representations are isomorphic to $\mathbb{I}_{\mathtt{Q}}[b,d]$:

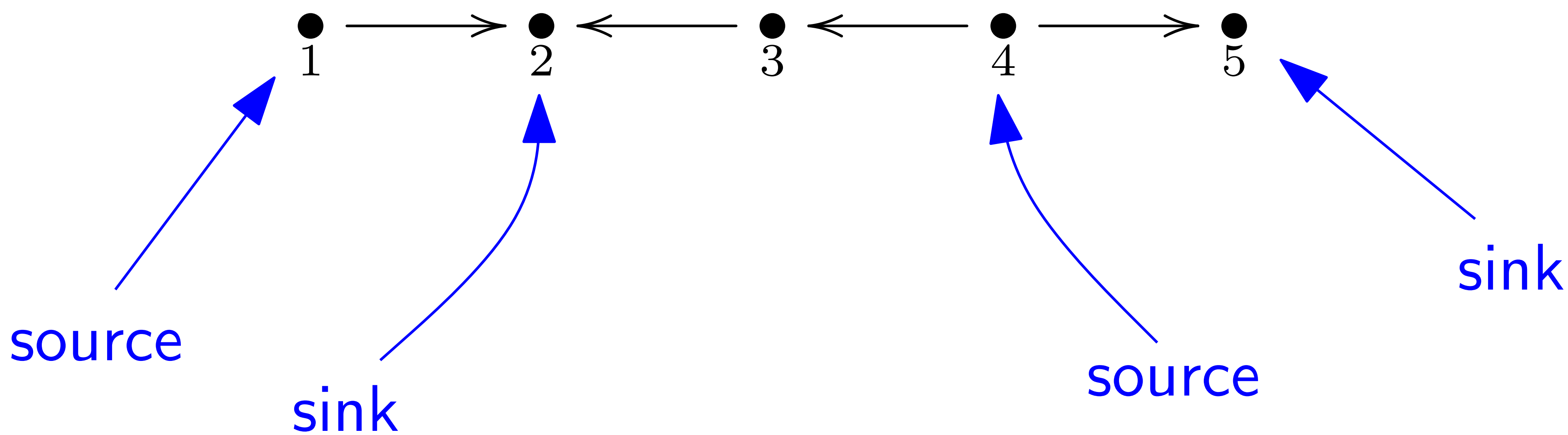
$$\underbrace{0 \xrightarrow{\;0\;} \cdots \xrightarrow{\;0\;} 0}_{[1,\,b-1]} \xrightarrow{\;0\;} \underbrace{\boldsymbol{k} \xrightarrow{\;\mathbb{1}\;} \cdots \xrightarrow{\;\mathbb{1}\;} \boldsymbol{k}}_{[b,\,d]} \xrightarrow{\;0\;} \underbrace{0 \xrightarrow{\;0\;} \cdots \xrightarrow{\;0\;} 0}_{[d+1,\,n]}$$

Reflection Functors

Feature: explain why arrow orientations are irrelevant to the classification problem (indecomposable representations are fully determined by their dimension vectors).

idea: modify quivers by reversing arrows, and study the effect on their representations (peeling off summands).

Reflection Functors



Definition: sink = only incoming arrows; source = only outgoing arrows

Reflection Functors

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet} \longleftarrow \underset{4}{\bullet} \longrightarrow \underset{5}{\bullet}$$

$$s_1\mathtt{Q}: \quad \underset{1}{\bullet} \longleftarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet} \longleftarrow \underset{4}{\bullet} \longrightarrow \underset{5}{\bullet}$$

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$$s_4\mathtt{Q}: \quad \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet} \longrightarrow \underset{4}{\bullet} \longleftarrow \underset{5}{\bullet}$$

$$s_5\mathtt{Q}: \quad \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet} \longleftarrow \underset{4}{\bullet} \longleftarrow \underset{5}{\bullet}$$

Definition: reflection s_i = reverse all arrows incident to sink/source i

Reflection Functors

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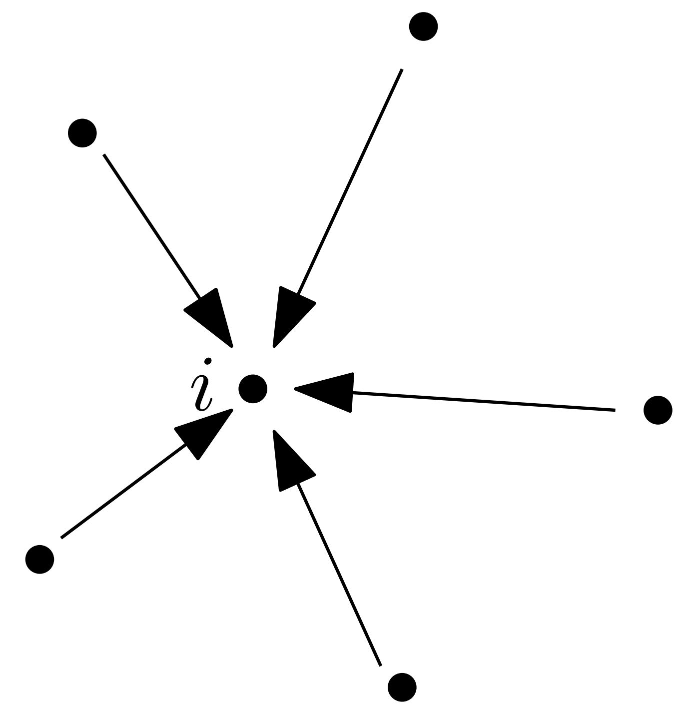
Definition: reflection functor $\mathcal{R}_i^{\pm}$ = functor $\mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q}) \to \mathrm{Rep}_{\boldsymbol{k}}(s_i\mathtt{Q})$

Reflection Functors

Let $\mathbb{V} = (V_i, v_a) \in \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$, let i be a sink



Reflection Functors

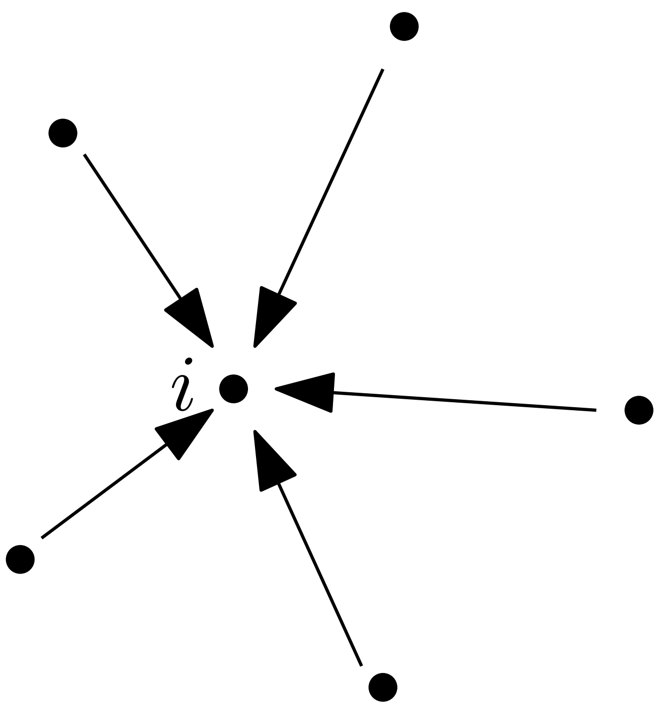
Let $\mathbb{V} = (V_i, v_a) \in \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$, let i be a sink

Definition: $\mathcal{R}_i^+ \mathbb{V} = (W_i, w_a)$ is defined by :

- $W_j = V_j$ for all $j \neq i$

(arrows incident to i)

- $w_a = v_a$ for all $a \notin Q_1^i$



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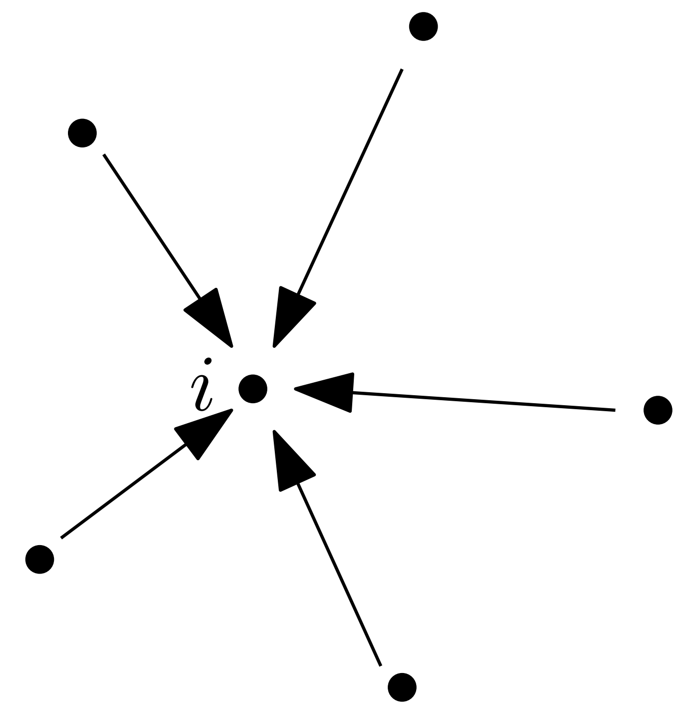

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- $W_i = \ker \xi_i :$
$$\begin{array}{|rcl} \displaystyle\bigoplus_{a \in Q_1^i} V_{s_a} & \longrightarrow & V_i \\ (x_{s_a})_{a \in Q_1^i} & \longmapsto & \displaystyle\sum_{a \in Q_1^i} v_a(x_{s_a}) \end{array}$$

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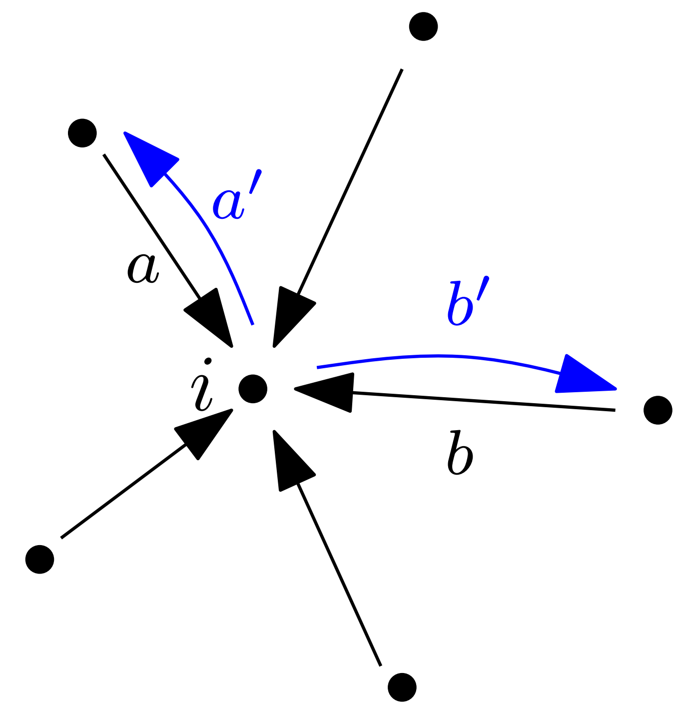

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- for $a \in Q_1^i$, let b be the opposite arrow, and let w_b be the composition:

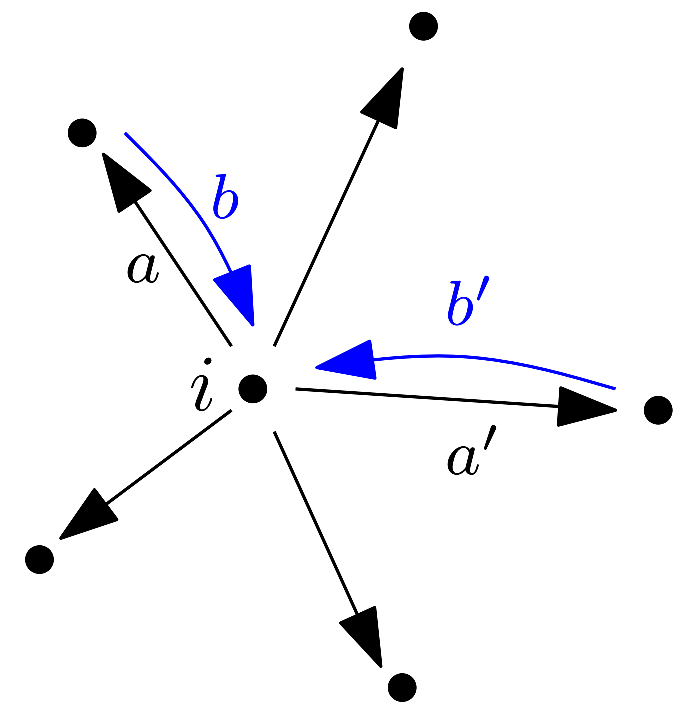
$$W_{s_b} = W_i = \ker \xi_i \hookrightarrow \bigoplus\nolimits_{c \in Q_1^i} V_{s_c} \longrightarrow V_{s_a} = W_{s_a} = W_{t_b}$$

(projection to component V_{s_a})

Reflection Functors

Let $\mathbb{V} = (V_i, v_a) \in \mathrm{Rep}_{\boldsymbol{k}}(\mathtt{Q})$, let i be a ~~sink~~ source

Definition: $\cancel{\mathcal{R}_i^+\mathbb{V}}$ $\mathcal{R}_i^-\mathbb{V} = (W_i, w_a)$ is defined by :



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(arrows incident to i)

- $w_a = v_a$ for all $a \notin Q_1^i$

- $W_i = \cancel{\ker \zeta_i}$ $\operatorname{coker} \zeta_i$: $\displaystyle\bigoplus_{a \in Q_1^i} V_{s_a} \longleftarrow V_i$

  $x_i \longmapsto (v_a(x_i))_{a \in Q_1^i}$

- for $a \in Q_1^i$, let b be the opposite arrow, and let w_b be the composition:

$$W_{s_b} = W_{t_a} = V_{t_a} \hookrightarrow \bigoplus_{c \in Q_1^i} V_{t_c} \longrightarrow \operatorname{coker} \zeta_i = W_i = W_{t_b}$$

(quotient modulo $\mathrm{im}\,\zeta_i$)

Reflection Functors

$$\mathbb{V}: \qquad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{v_d} V_5$$

$$\mathcal{R}_5^+\mathbb{V}: \qquad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \hookleftarrow \ker v_d$$

$$\mathcal{R}_5^-\mathcal{R}_5^+\mathbb{V}: \qquad V_1 \xrightarrow{v_a} V_2 \xleftarrow{v_b} V_3 \xleftarrow{v_c} V_4 \xrightarrow{\bmod \ \ker v_d} V_4/\ker v_d$$

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet} \longleftarrow \underset{4}{\bullet} \longrightarrow \underset{5}{\bullet}$$

Reflection Functors

$$\mathbb{V}: \quad \begin{tikzcd} V_1 \arrow[r, "v_a"] & V_2 & V_3 \arrow[l, "v_b"'] & V_4 \arrow[l, "v_c"'] \arrow[r, "v_d"] & V_5 \end{tikzcd}$$

$$\mathcal{R}_5^+\mathbb{V}: \quad \begin{tikzcd} V_1 \arrow[r, "v_a"] & V_2 & V_3 \arrow[l, "v_b"'] & V_4 \arrow[l, "v_c"'] & \ker v_d \arrow[l, hook'] \end{tikzcd}$$

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$\cong \operatorname{im} v_d$

$\mathbb{V} \cong \mathcal{R}_5^-\mathcal{R}_5^+\mathbb{V} \oplus \mathbb{S}_5^r$, where $r = \dim \operatorname{coker} v_d$

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} & \underset{3}{\bullet} \arrow[l] & \underset{4}{\bullet} \arrow[l] \arrow[r] & \underset{5}{\bullet} \end{tikzcd}$$

Reflection Functors

Theorem: [Bernstein, Gelfand, Ponomarev]
Let $\mathtt{Q}$ be a finite connected quiver and let $\mathbb{V}$ be a representation of $\mathtt{Q}$. If $\mathbb{V} \cong \mathbb{U} \oplus \mathbb{W}$, then for any source or sink $i \in Q_0$, $\mathcal{R}_i^{\pm}\mathbb{V} \cong \mathcal{R}_i^{\pm}\mathbb{U} \oplus \mathcal{R}_i^{\pm}\mathbb{W}$.

If now $\mathbb{V}$ is indecomposable:
1. If $i \in Q_0$ is a sink, then two cases are possible:

- $\mathbb{V} \cong \mathbb{S}_i$: in this case, $\mathcal{R}_i^{+}\mathbb{V} = 0$.
- $\mathbb{V} \ncong \mathbb{S}_i$: in this case, $\mathcal{R}_i^{+}\mathbb{V}$ is nonzero and indecomposable, $\mathcal{R}_i^{-}\mathcal{R}_i^{+}\mathbb{V} \cong \mathbb{V}$, and the dimension vectors x of $\mathbb{V}$ and y of $\mathcal{R}_i^{+}\mathbb{V}$ are related to each other by the following formula:

$$y_j = \begin{cases} x_j & \text{if } j \neq i; \\ -x_i + \sum\limits_{\substack{a \in Q_1 \\ t_a = i}} x_{s_a} & \text{if } j = i. \end{cases}$$

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2. If $i \in Q_0$ is a source, then two cases are possible:

- $\mathbb{V} \cong \mathbb{S}_i$: in this case, $\mathcal{R}_i^{-}\mathbb{V} = 0$.
- $\mathbb{V} \not\cong \mathbb{S}_i$: in this case, $\mathcal{R}_i^{-}\mathbb{V}$ is nonzero and indecomposable, $\mathcal{R}_i^{+}\mathcal{R}_i^{-}\mathbb{V} \cong \mathbb{V}$, and the dimension vectors x of $\mathbb{V}$ and y of $\mathcal{R}_i^{-}\mathbb{V}$ are related to each other by the following formula:

$$y_j = \begin{cases} x_j & \text{if } j \neq i; \\ -x_i + \sum_{\substack{a \in Q_1 \\ s_a = i}} x_{t_a} & \text{if } j = i. \end{cases}$$

Reflection Functors

Example: $\mathtt{Q}$ of type A_n, i sink, $\mathbb{V} \cong \bigoplus_{j=1}^{r} \mathbb{I}_{\mathtt{Q}}[b_j, d_j] \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$:

$$\begin{tikzcd}
 & & & V_i & & & & \mathbb{V} \\
V_1 \arrow[r, dash] & \cdots \arrow[r, dash] & V_{i-1} \arrow[ur] & & V_{i+1} \arrow[ul] \arrow[r, dash] & \cdots \arrow[r, dash] & V_n & \\
 & & & W_i \arrow[ul] \arrow[ur] & & & & \mathcal{R}_i^+\mathbb{V}
\end{tikzcd}$$

$\mathcal{R}_i^+\mathbb{V} \cong \bigoplus_{j=1}^{r} \mathcal{R}_i^+\mathbb{I}_{\mathtt{Q}}[b_j, d_j]$, where

$$\mathcal{R}_i^+\mathbb{I}_{\mathtt{Q}}[b_j, d_j] = \begin{cases} 0 & \text{if } i = b_j = d_j; \\ \mathbb{I}_{s_i\mathtt{Q}}[i+1, d_j] & \text{if } i = b_j < d_j; \\ \mathbb{I}_{s_i\mathtt{Q}}[i, d_j] & \text{if } i+1 = b_j \leq d_j; \\ \mathbb{I}_{s_i\mathtt{Q}}[b_j, i-1] & \text{if } b_j < d_j = i; \\ \mathbb{I}_{s_i\mathtt{Q}}[b_j, i] & \text{if } b_j \leq d_j = i-1; \\ \mathbb{I}_{s_i\mathtt{Q}}[b_j, d_j] & \text{otherwise.} \end{cases}$$

Reflection Functors

Example: $\mathtt{Q}$ of type A_n, i sink, $\mathbb{V} \cong \bigoplus_{j=1}^{r} \mathbb{I}_{\mathtt{Q}}[b_j, d_j] \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$:

Diamond (forced exact by $\mathcal{R}_i^+$)

$\mathbb{V}$

$$
\begin{tikzcd}
 & & & V_i & & & \\
V_1 \arrow[r, dash] & \cdots \arrow[r, dash] & V_{i-1} \arrow[ur] & & V_{i+1} \arrow[ul] \arrow[r, dash] & \cdots \arrow[r, dash] & V_n \\
 & & & W_i \arrow[ul] \arrow[ur] & & &
\end{tikzcd}
$$

$\mathcal{R}_i^+\mathbb{V}$

Note: $\mathcal{R}_i^+\mathbb{V}$ is *de facto* interval-decomposable

$\mathcal{R}_i^+\mathbb{V} \cong \bigoplus_{j=1}^{r} \mathcal{R}_i^+\mathbb{I}_{\mathtt{Q}}[b_j, d_j]$, where

$$
\mathcal{R}_i^+\mathbb{I}_{\mathtt{Q}}[b_j, d_j] = \begin{cases} 0 & \text{if } i = b_j = d_j; \\ \mathbb{I}_{s_i\mathtt{Q}}[i+1, d_j] & \text{if } i = b_j < d_j; \\ \mathbb{I}_{s_i\mathtt{Q}}[i, d_j] & \text{if } i+1 = b_j \le d_j; \\ \mathbb{I}_{s_i\mathtt{Q}}[b_j, i-1] & \text{if } b_j < d_j = i; \\ \mathbb{I}_{s_i\mathtt{Q}}[b_j, i] & \text{if } b_j \le d_j = i-1; \\ \mathbb{I}_{s_i\mathtt{Q}}[b_j, d_j] & \text{otherwise.} \end{cases}
$$

Exact Diamond Principle [Carlsson, de Silva]

Algorithm to decompose A_n representations

We are currently able to turn indecomposable representations of `Q` into indecomposable representations of reflections of `Q` (or zero)

$\rightarrow$ idea: turn `Q` into itself via sequences of reflections, and observe the evolution of the indecomposables through the reflection functors

Algorithm to decompose A_n representations

Special case: linear quiver $\mathtt{L}_n$: $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet}$

Let $\mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{L}_n)$ indecomposable, $\underline{\dim}\,\mathbb{V} = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

$\to$ apply reflections $s_1 s_2 \cdots s_{n-1} s_n \mathtt{L}_n$ and observe evolution of $\underline{\dim}\,\mathbb{V}$

Algorithm to decompose A_n representations

Special case: linear quiver $\mathtt{L}_n$: $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet}$

Let $\mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathsf{L}_n)$ indecomposable, $\underline{\dim}\,\mathbb{V} = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

$$\underline{\dim}\,\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (x_1, x_2, \cdots, x_{n-1}, x_{n-1} - x_n)^\top$$

$$\underline{\dim}\,\mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (x_1, x_2, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$$

$$\cdots$$

$$\underline{\dim}\,\mathcal{R}_2^+ \cdots \mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (x_1, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$$

$$\underline{\dim}\,\mathcal{R}_1^+\mathcal{R}_2^+ \cdots \mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (-x_n, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$$

Algorithm to decompose A_n representations

Special case: linear quiver $\mathtt{L}_n$:

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet}$$

Let $\mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathsf{L}_n)$ indecomposable, $\underline{\dim}\,\mathbb{V} = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

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$$\cdots$$

$$\underline{\dim}\,\mathcal{R}_2^+\cdots\mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (x_1, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$$

$$\underline{\dim}\,\mathcal{R}_1^+\mathcal{R}_2^+\cdots\mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (\overset{\leq 0}{-x_n}, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$$

$$\Longrightarrow \mathcal{C}^+\mathbb{V} = \mathcal{R}_1^+\mathcal{R}_2^+\cdots\mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } x_n = 0$$

Algorithm to decompose A_n representations

Special case: linear quiver L_n: $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet}$

Let $\mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathrm{L}_n)$ indecomposable, $\underline{\dim}\,\mathbb{V} = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

$$\underline{\dim}\,\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (x_1, x_2, \cdots, x_{n-1}, x_{n-1} - x_n)^\top$$

$$\underline{\dim}\,\mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (x_1, x_2, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$$

$$\cdots$$

if $\mathbb{V}$ is arbitrary, then at this stage all summands ending at n have been peeled off $\mathbb{V}$
all other summands have been shifted

$$\underline{\dim}\,\mathcal{R}_2^+\cdots\mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (x_1, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$$

$$\underline{\dim}\,\mathcal{R}_1^+\mathcal{R}_2^+\cdots\mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } (\overset{\leq 0}{-x_n}, x_1 - x_n, \cdots, x_{n-2} - x_n, x_{n-1} - x_n)^\top$$

$$\Longrightarrow \mathcal{C}^+\mathbb{V} = \mathcal{R}_1^+\mathcal{R}_2^+\cdots\mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} = 0 \text{ or } x_n = 0$$

Algorithm to decompose A_n representations

Special case: linear quiver L_n: $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n-1}{\bullet} \longrightarrow \underset{n}{\bullet}$

Let $\mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathrm{L}_n)$ indecomposable, $\underline{\dim}\,\mathbb{V} = (x_1, x_2, \cdots, x_{n-1}, x_n)^\top$

$\underline{\dim}\,\mathcal{C}^+\mathbb{V} = 0$ or $(0, x_1, x_2, \cdots, x_{n-2}, x_{n-1})^\top$ $\left(\begin{array}{l}\text{peel off } [b, n] \\ \text{shift}\end{array}\right.$

$\underline{\dim}\,\mathcal{C}^+\mathcal{C}^+\mathbb{V} = 0$ or $(0, 0, x_1, \cdots, x_{n-3}, x_{n-2})^\top$ $\left(\begin{array}{l}\text{peel off } [b, n-1] \\ \text{shift}\end{array}\right.$

$\cdots$

$\underline{\dim}\,\underbrace{\mathcal{C}^+ \cdots \mathcal{C}^+}_{n-1 \text{ times}}\mathbb{V} = 0$ or $(0, 0, 0, \cdots, 0, x_1)^\top$ $\left(\begin{array}{l}\text{peel off } [b, 2] \\ \text{shift}\end{array}\right.$

$\underline{\dim}\,\underbrace{\mathcal{C}^+ \cdots \mathcal{C}^+}_{n \text{ times}}\mathbb{V} = 0$ $\left(\begin{array}{l}\text{peel off } [1, 1] \\ \text{[end of algo.]}\end{array}\right.$

Algorithm to decompose A_n representations

A_n-type quiver $\mathtt{Q}$:

$$\underset{1}{\bullet} \text{———} \underset{2}{\bullet} \text{———} \cdots \text{———} \underset{n-1}{\bullet} \text{———} \underset{n}{\bullet}$$

$\rightarrow$ goal: find a sequence of indices $i_1, i_2, \cdots, i_{s-1}, i_s$ s.t.

$$\mathcal{R}^+_{i_s} \mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2} \mathcal{R}^+_{i_1} \mathbb{V} = 0 \text{ for all } \mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$$

Algorithm to decompose A_n representations

A_n-type quiver $\mathtt{Q}$: $\underset{1}{\bullet} \longrightarrow\!\!\!\!\!\!- \;\; \underset{2}{\bullet}$

$$\underset{1}{\bullet} \;\text{———}\; \underset{2}{\bullet} \;\text{———}\; \cdots \;\text{———}\; \underset{n-1}{\bullet} \;\text{———}\; \underset{n}{\bullet}$$

$\to$ goal: find a sequence of indices $i_1, i_2, \cdots, i_{s-1}, i_s$ s.t.

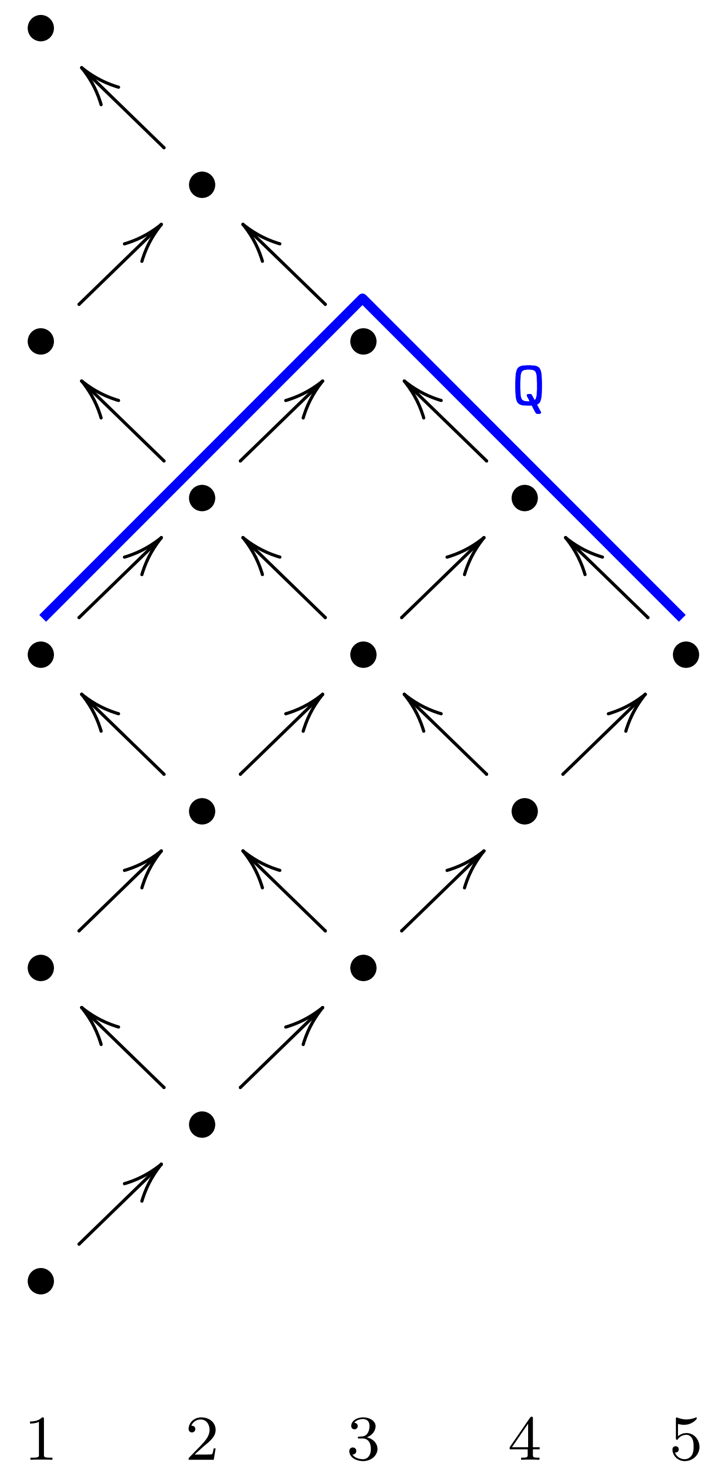
$$\mathcal{R}^+_{i_s}\mathcal{R}^+_{i_{s-1}} \cdots \mathcal{R}^+_{i_2}\mathcal{R}^+_{i_1}\mathbb{V} = 0 \text{ for all } \mathbb{V} \in \mathrm{rep}_{\boldsymbol{k}}(\mathtt{Q})$$

$\to$ idea: turn $\mathtt{Q}$ into $\mathtt{L}_n$, then use the same sequence a before

Algorithm to decompose A_n representations

A_n-type quiver Q:

- embed Q in a giant pyramid



Algorithm to decompose A_n representations

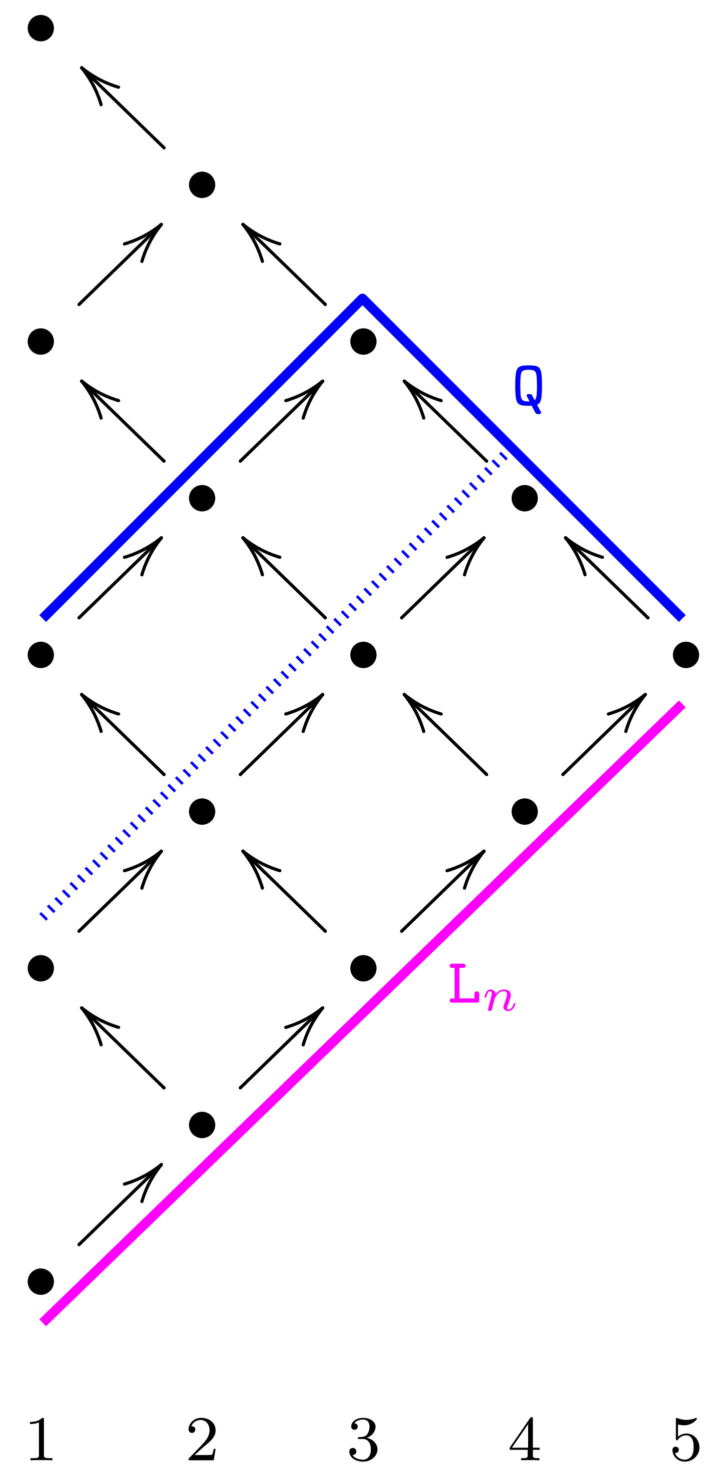
A_n-type quiver `Q`:

- embed `Q` in a giant pyramid

- travel down the pyramid to its bottom $\mathtt{L}_n$

  $\longrightarrow$ travelling one level down reverses the leftmost backward arrow

  e.g. $s_1 s_2 s_3$ reverses $\underset{3}{\bullet} \longleftarrow \underset{4}{\bullet}$



Algorithm to decompose A_n representations

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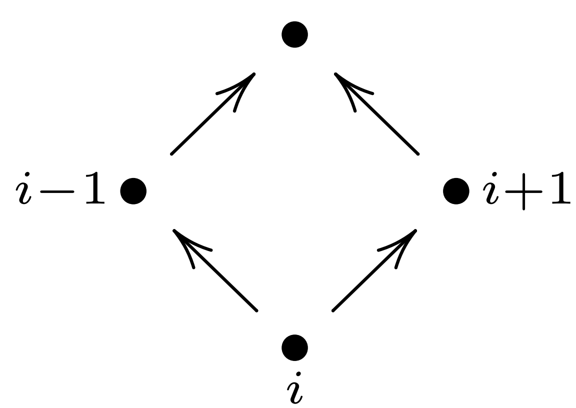
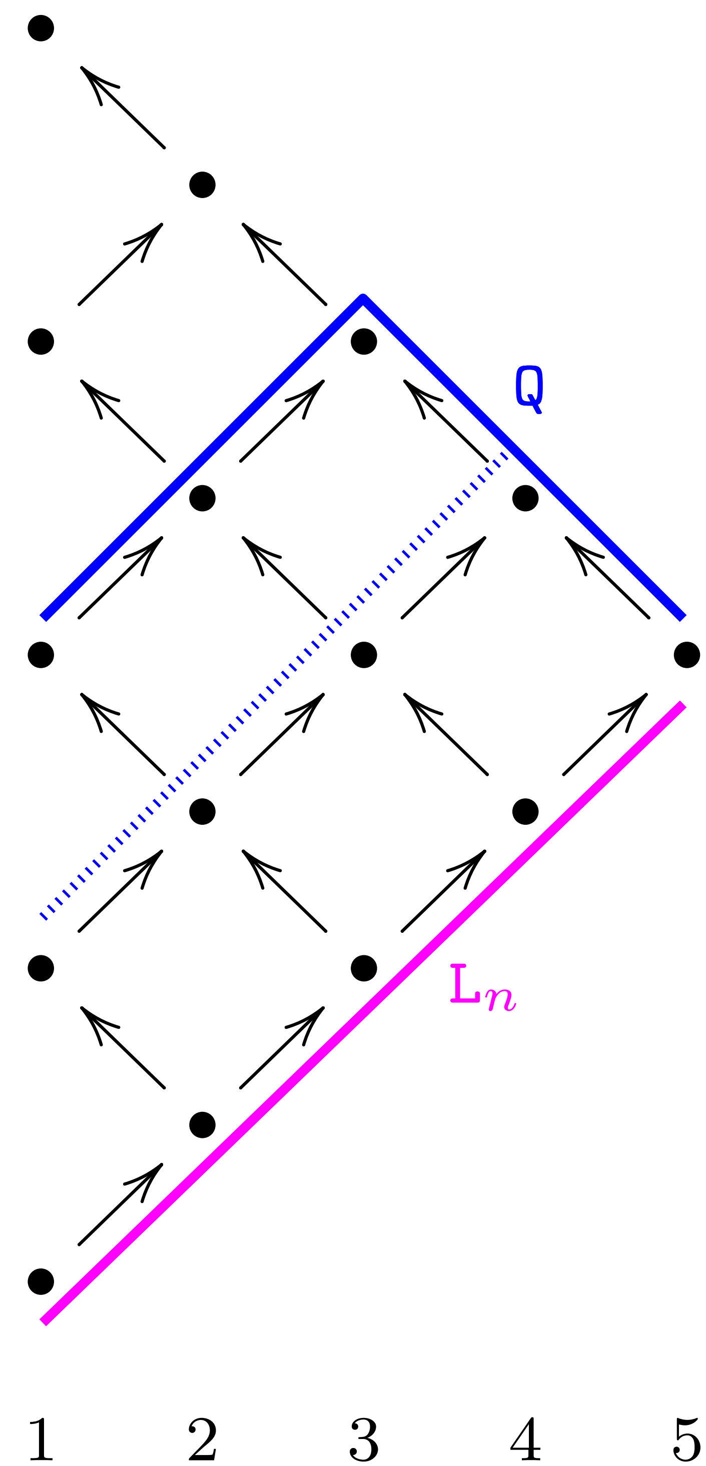
  → travelling one level down reverses the leftmost backward arrow

  e.g. $s_1 s_2 s_3$ reverses $\underset{3}{\bullet} \longleftarrow \underset{4}{\bullet}$

- each diamond



  is travelled down using $\mathcal{R}_i^+$



Algorithm to decompose A_n representations

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- embed `Q` in a giant pyramid

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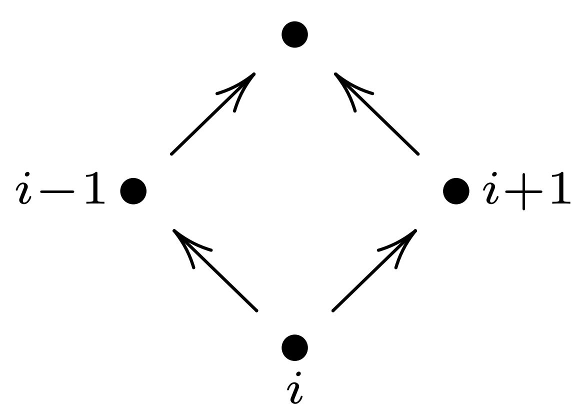
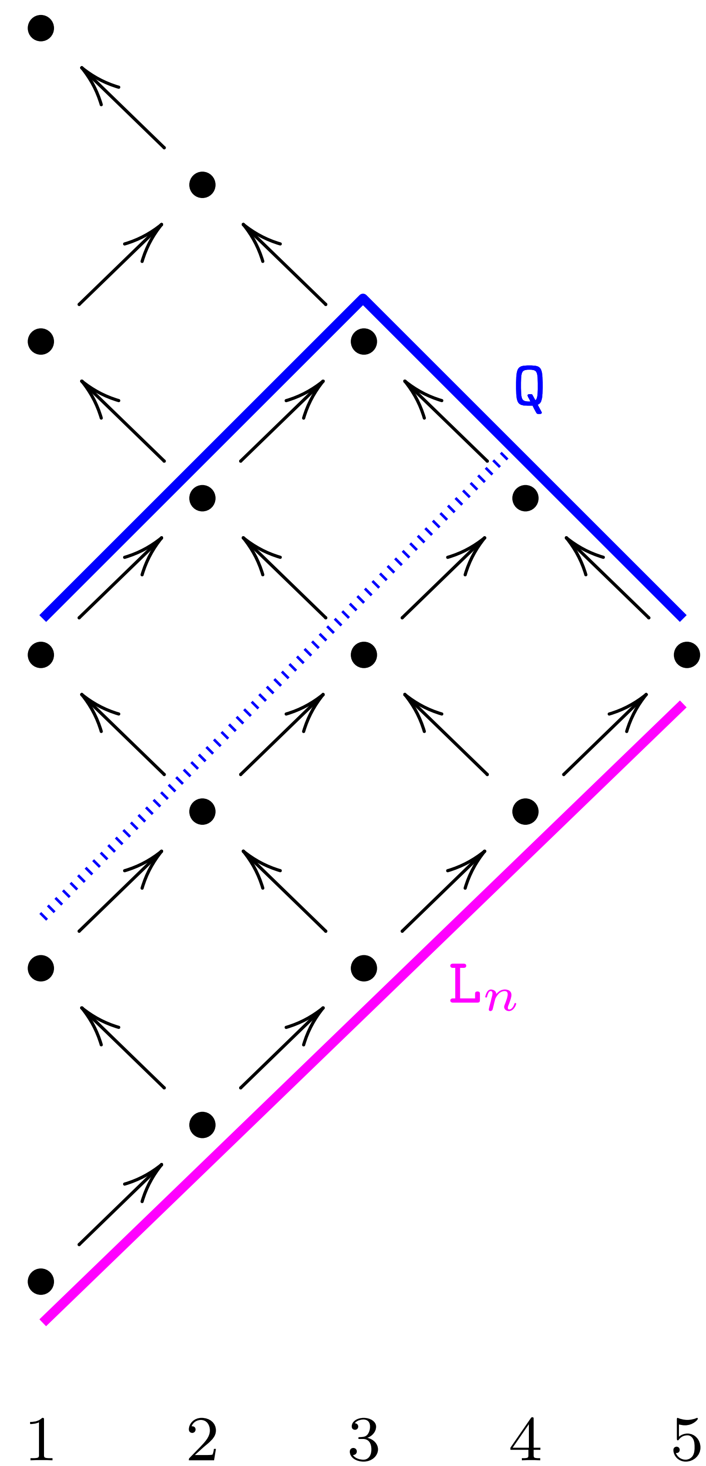


is travelled down using $\mathcal{R}_i^+$



$\rightarrow$ algo. to compute zigzag persistence

(at the algebraic level $\rightarrow$ maintain bases)

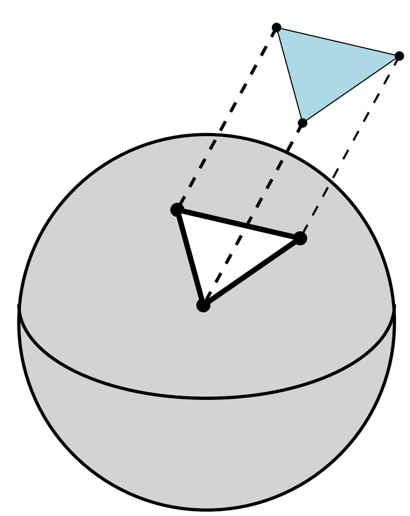
Application: Persistence Computation

$$\begin{tikzcd}
K_1 \arrow[r, dash] \arrow[d, dotted] & \cdots \arrow[r, dash] & K_i \arrow[r, dash, "\sigma"] \arrow[d, dotted] & K_{i+1} \arrow[r, dash] \arrow[d, dotted] & \cdots \arrow[r, dash] & K_n \arrow[d, dotted] \\
\mathsf{H}(K_1) \arrow[r, dash] & \cdots \arrow[r, dash] & \mathsf{H}(K_i) \arrow[r, dash, "f"] & \mathsf{H}(K_{i+1}) \arrow[r, dash] & \cdots \arrow[r, dash] & \mathsf{H}(K_n)
\end{tikzcd}$$

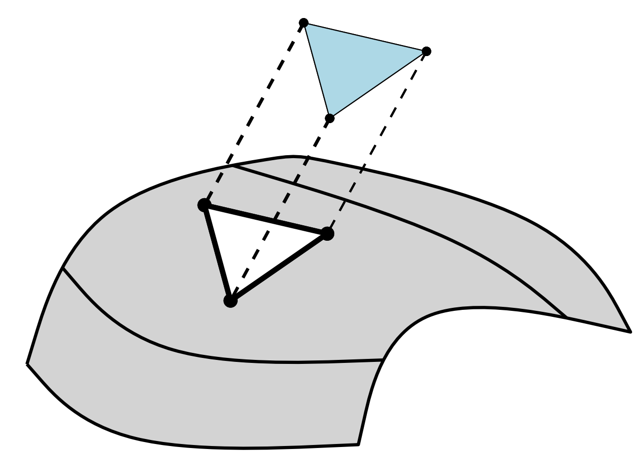
- every horizontal map is either forward or backward
- the K_i are simplicial complexes, the inclusions are *elementary*
- the $\mathsf{H}(K_i)$ are vector spaces connected by linear maps (quiver representation)

$$\begin{tikzcd}
K \arrow[r, "\sigma"] \arrow[d, dotted] & K \cup \{\sigma\} \arrow[d, dotted] \\
\mathsf{H}(K) \arrow[r, "f"] & \mathsf{H}(K \cup \{\sigma\})
\end{tikzcd}$$

$$\overbrace{\ker f = [\partial\sigma]}$$



f inj. of corank 1



f surj. of nullity 1

Application: Persistence Computation

$$\begin{tikzcd}
K_1 \arrow[r, dash] \arrow[d, dotted] & \cdots \arrow[r, dash] & K_i \arrow[r, dash, "\sigma"] \arrow[d, dotted] & K_{i+1} \arrow[r, dash] \arrow[d, dotted] & \cdots \arrow[r, dash] & K_n \arrow[d, dotted] \\
\mathsf{H}(K_1) \arrow[r, dash] & \cdots \arrow[r, dash] & \mathsf{H}(K_i) \arrow[r, dash, "f"] & \mathsf{H}(K_{i+1}) \arrow[r, dash] & \cdots \arrow[r, dash] & \mathsf{H}(K_n)
\end{tikzcd}$$

- every horizontal map is either forward or backward
- the K_i are simplicial complexes, the inclusions are *elementary*
- the $\mathsf{H}(K_i)$ are vector spaces connected by linear maps (quiver representation)

Algorithms for when all maps are forward:

- Gaussian elimination: worst-case $O(n^3)$, highly optimized in practice
- Fast matrix multiplication: worst-case $O(n^\omega)$, not implemented

Algorithms for when maps can be forward or backward:

- Gaussian elimination + *right filtration* functor: worst-case $O(n^3)$,
- Fast matrix multiplication: worst-case $O(n^\omega)$, not implemented

Application: Persistence Computation

We compute the persistent homology of:

$$K_1 \text{ --- } K_2 \text{ --- } \cdots \text{ --- } K_i \overset{\sigma}{\text{ --- }} K_{i+1} \text{ --- } \cdots \text{ --- } K_{n-1} \text{ --- } K_n$$

Application: Persistence Computation

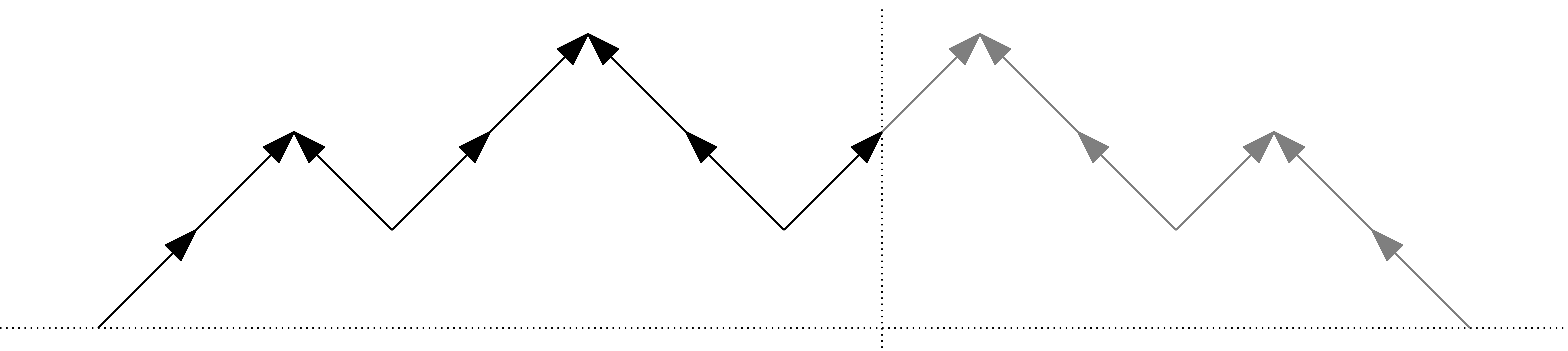
We compute the persistent homology of:

$$K_1 \text{———} K_2 \text{———} \cdots \text{———} K_i \xrightarrow{\ \sigma\ } K_{i+1} \text{———} \cdots \text{———} K_{n-1} \text{———} K_n$$

by maintaining a compatible homology basis for

$$\underbrace{K_1 \text{———} \cdots \text{———} K_i}_{\mathbb{K}[1;i]}$$

[Carlsson, de Silva '10],[C,deS, Morozov '09]



Application: Persistence Computation

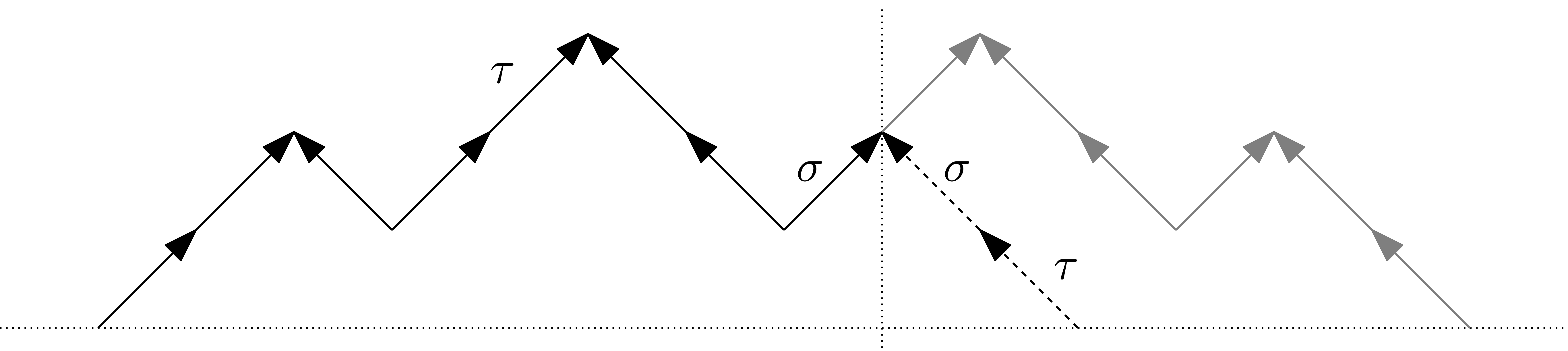
We compute the persistent homology of:

$$K_1 \text{———} K_2 \text{———} \cdots \text{———} K_i \overset{\sigma}{\text{———}} K_{i+1} \text{———} \cdots \text{———} K_{n-1} \text{———} K_n$$

by maintaining a compatible homology basis for

[Maria, O. '15]

$$\underbrace{K_1 \text{———} \cdots \text{———} K_i}_{\mathbb{K}[1;i]} \quad = \quad K'_m \xleftarrow{\tau_m} K'_{m-1} \xleftarrow{\tau_{m-1}} K'_{m-2} \xleftarrow{\tau_{m-2}} \cdots \xleftarrow{\tau_1} \emptyset$$



Application: Persistence Computation

We compute the persistent homology of:

$$K_1 \longleftrightarrow K_2 \longleftrightarrow \cdots \longleftrightarrow K_i \overset{\sigma}{\longleftrightarrow} K_{i+1} \longleftrightarrow \cdots \longleftrightarrow K_{n-1} \longleftrightarrow K_n$$

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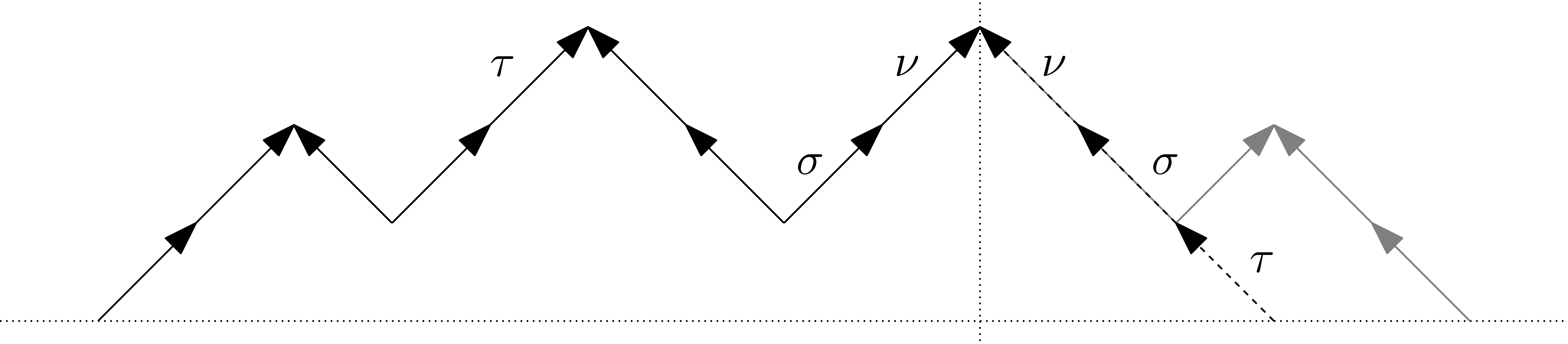

Application: Persistence Computation

We compute the persistent homology of:

$$K_1 \longrightarrow K_2 \longrightarrow \cdots \longrightarrow K_i \overset{\sigma}{\longrightarrow} K_{i+1} \longrightarrow \cdots \longrightarrow K_{n-1} \longrightarrow K_n$$

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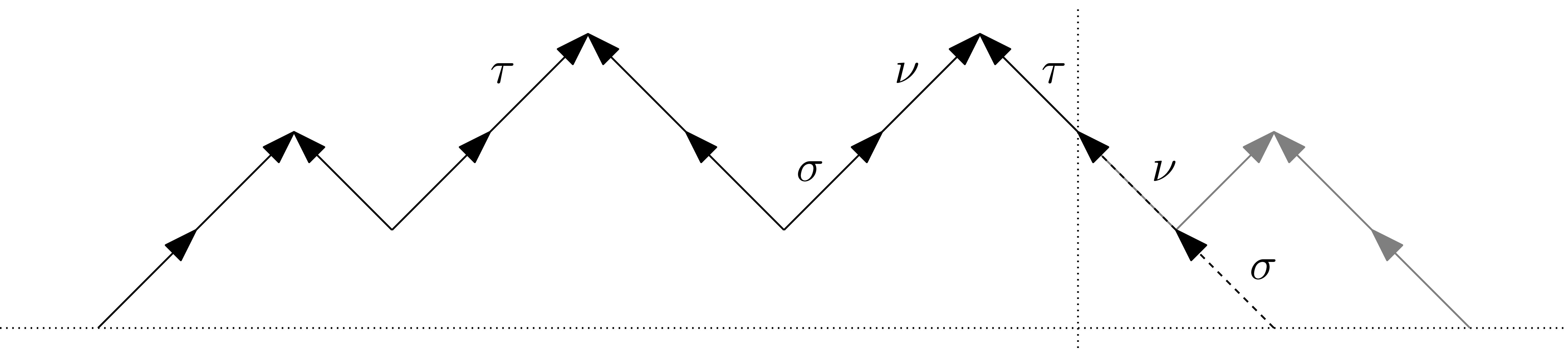


Application: Persistence Computation

We compute the persistent homology of:

$$K_1 \longleftrightarrow K_2 \longleftrightarrow \cdots \longleftrightarrow K_i \overset{\sigma}{\longleftrightarrow} K_{i+1} \longleftrightarrow \cdots \longleftrightarrow K_{n-1} \longleftrightarrow K_n$$

by maintaining a compatible homology basis for

[Maria, O. '15]

$$\underbrace{K_1 \longleftrightarrow \cdots \longleftrightarrow K_i}_{\mathbb{K}[1;i]} \quad = \quad K'_m \xleftarrow{\tau_m} K'_{m-1} \xleftarrow{\tau_{m-1}} K'_{m-2} \xleftarrow{\tau_{m-2}} \cdots \xleftarrow{\tau_1} \emptyset$$

- arrow reflection if $\xrightarrow{\sigma}$ is forward

$$\begin{tikzcd}
 & & K \cup \{\sigma\} & & \\
\cdots \arrow[r, dash] & K \arrow[ur, "\sigma"] & & K \arrow[ul, "\sigma"'] \arrow[r, dash] & \cdots \\
 & & K \arrow[ul, "\mathbb{1}"] \arrow[ur, "\mathbb{1}"'] & &
\end{tikzcd}$$

- arrow transposition if $\xleftarrow{\sigma}$ is backward

$$\begin{tikzcd}
 & & K \cup \{\tau\} \arrow[dll, "\sigma"'] & & \\
\cdots \arrow[r, dash] & K \cup \{\sigma, \tau\} & & & K \arrow[ull, "\tau"'] \arrow[dll, "\sigma"] \arrow[r, dash] & \cdots \\
 & & K \cup \{\sigma\} \arrow[ull, "\tau"] & &
\end{tikzcd}$$

Application: Persistence Computation

We compute the persistent homology of:

$$K_1 \;\frac{\quad\quad}{}\; K_2 \;\frac{\quad\quad}{}\; \cdots \;\frac{\quad\quad}{}\; K_i \;\frac{\;\sigma\;}{}\; K_{i+1} \;\frac{\quad\quad}{}\; \cdots \;\frac{\quad\quad}{}\; K_{n-1} \;\frac{\quad\quad}{}\; K_n$$

by maintaining a compatible homology basis for

[Maria, O. '15]

$$\underbrace{K_1 \;\frac{\quad\quad}{}\; \cdots \;\frac{\quad\quad}{}\; K_i}_{\mathbb{K}[1;i]} \quad = \quad K'_m \xleftarrow{\tau_m} K'_{m-1} \xleftarrow{\tau_{m-1}} K'_{m-2} \xleftarrow{\tau_{m-2}} \cdots \xleftarrow{\tau_1} \emptyset$$

- arrow reflection if $\xrightarrow{\sigma}$ is forward

$$\begin{tikzcd}
& & K \cup \{\sigma\} & & \\
\cdots \arrow[r, dash] & K \arrow[ur, "\sigma"] & & K \arrow[ul, "\sigma"'] \arrow[r, dash] & \cdots \\
& & K \arrow[ul, "\mathbb{1}"] \arrow[ur, "\mathbb{1}"'] & &
\end{tikzcd}$$

various cases depending on whether σ, τ are positive/negative

- arrow transposition if $\xleftarrow{\sigma}$ is backward

$$\begin{tikzcd}
& & K \cup \{\tau\} \arrow[dl, "\sigma"'] & & \\
\cdots \arrow[r, dash] & K \cup \{\sigma, \tau\} & & K \arrow[ul, "\tau"'] \arrow[dl, "\sigma"] \arrow[r, dash] & \cdots \\
& & K \cup \{\sigma\} \arrow[ul, "\tau"] & &
\end{tikzcd}$$

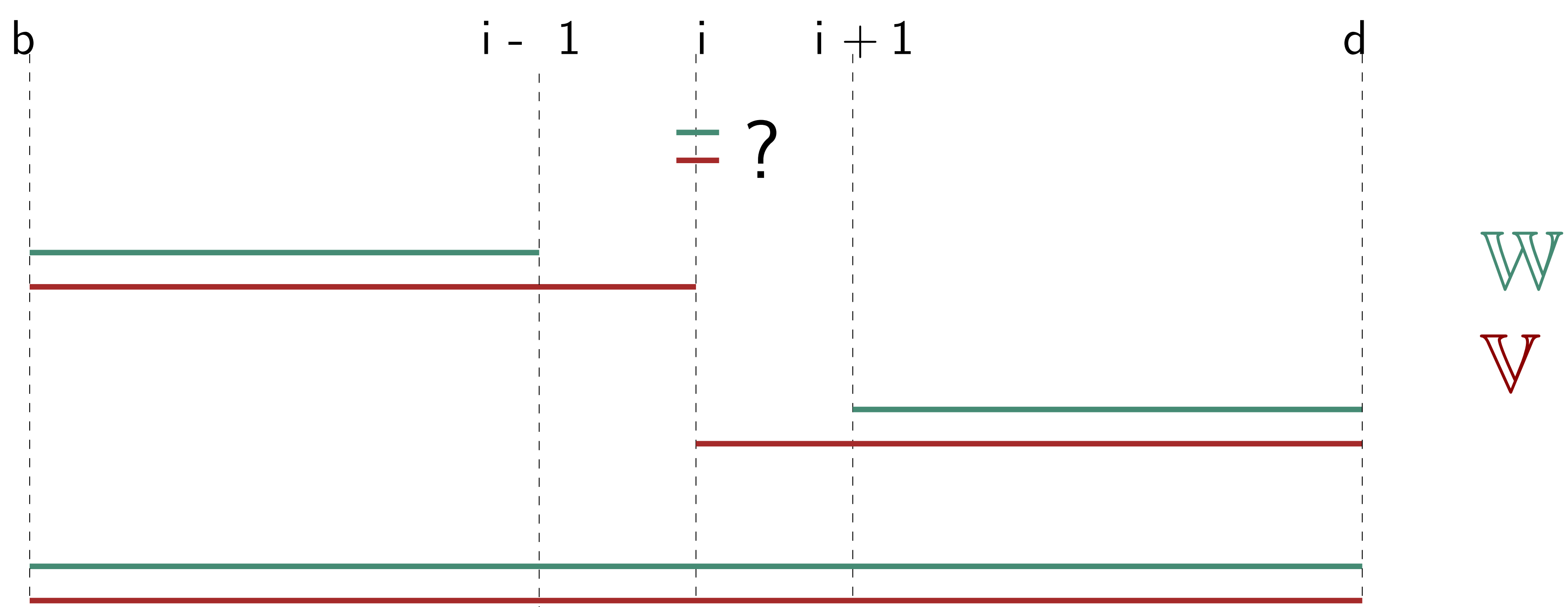
Application: Persistence Computation

Theorem: Exact Diamond Principle [Carlsson, de Silva '10]

Under the *exactness* hypothesis on the diamond:

$$\begin{tikzcd}[column sep=small]
\mathbb{W} := & & & & & W_i & & & & \\
& V_1 \arrow[r, dash] & \cdots \arrow[r, dash] & V_{i-1} \arrow[urr, "b"] & & & & V_{i+1} \arrow[ull, "d"'] \arrow[r, dash] & \cdots \arrow[r, dash] & V_n \\
\mathbb{V} := & & & & & V_i \arrow[ull, "a"] \arrow[urr, "c"'] & & & &
\end{tikzcd}$$

Interval decompositions of $\mathbb{V}, \mathbb{W}$ are related as follows:

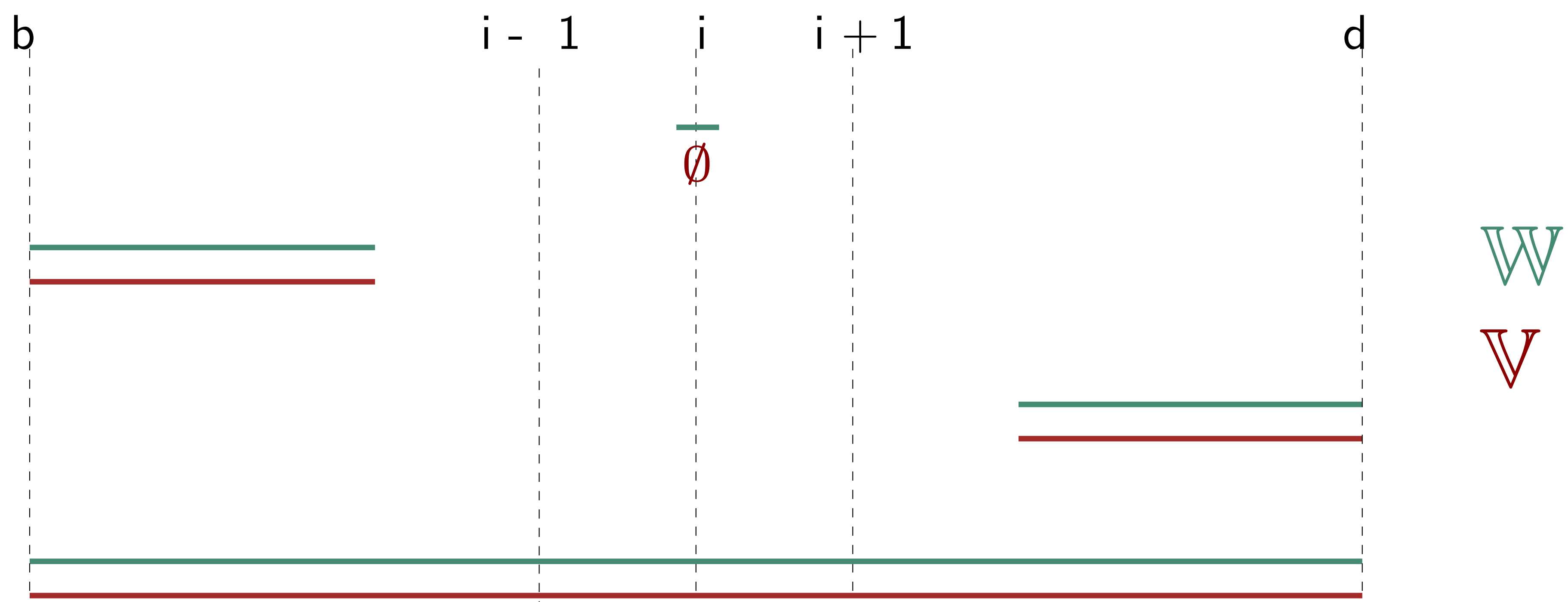


Application: Persistence Computation

Corollary: Injective Diamond Principle [Maria, O. '15]
For f injective of corank 1, the diamond is exact:

$$\begin{tikzcd}[row sep=small]
\mathbb{W} := & & & & W_i & & & \\
& V_1 \arrow[r, dash] & \cdots \arrow[r, dash] & V_{i-1} \arrow[ur, "f"] & & V_{i+1} \arrow[ul, "f"'] \arrow[r, dash] & \cdots \arrow[r, dash] & V_n \\
\mathbb{V} := & & & & V_i \arrow[ul, "\mathbb{1}"] \arrow[ur, "\mathbb{1}"'] & & &
\end{tikzcd}$$

Interval decompositions of $\mathbb{V}, \mathbb{W}$ are related as follows:

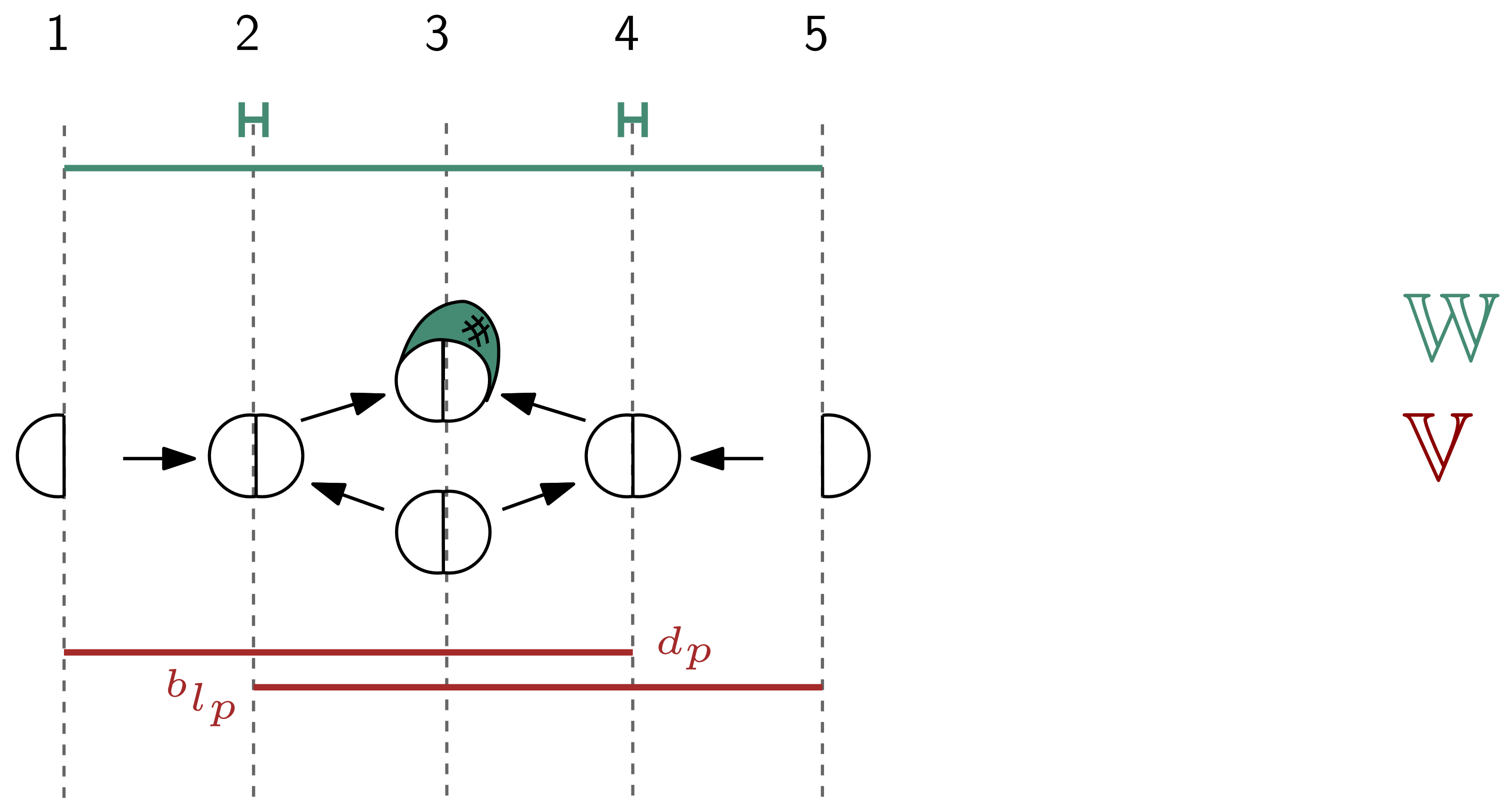


Application: Persistence Computation

Theorem: Surjective Diamond Principle [Morozov et al. '06] [Maria, O. '15]
For f surjective of nullity 1:

$$\begin{tikzcd}[column sep=small]
\mathbb{W} := & & & & W_i & & & & \\
& V_1 \arrow[r, dash] & \cdots \arrow[r, dash] & V_{i-1} \arrow[ur, "f"] & & V_{i+1} \arrow[ul, "f"'] \arrow[r, dash] & \cdots \arrow[r, dash] & V_n \\
\mathbb{V} := & & & & V_i \arrow[ul, "\mathbb{1}"] \arrow[ur, "\mathbb{1}"'] & & & &
\end{tikzcd}$$

Interval decompositions of $\mathbb{V}, \mathbb{W}$ are related through some *greedy rule*.



Application: Persistence Computation

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For f surjective of nullity 1:

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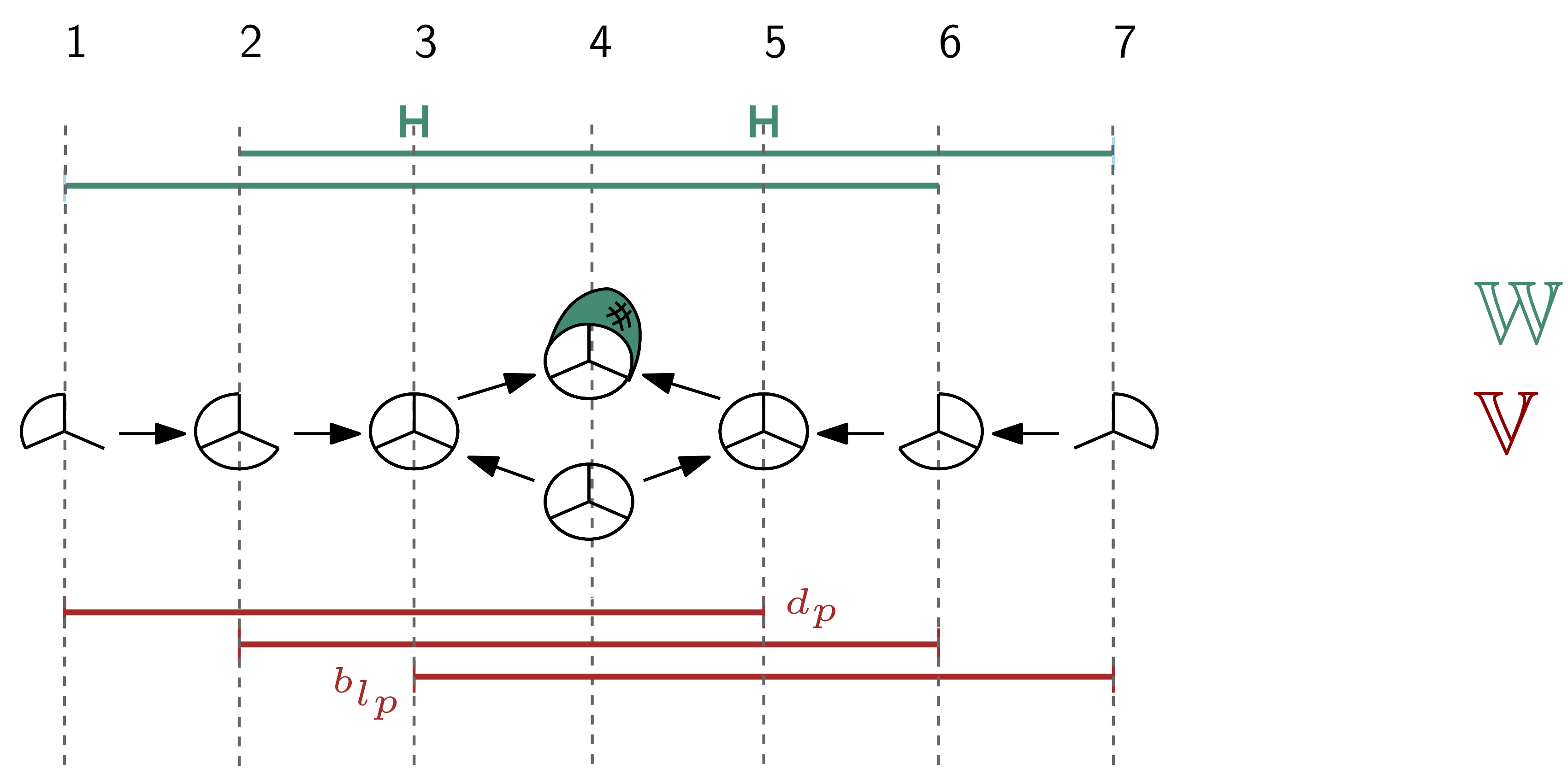
\mathbb{V} := & & & & V_i \arrow[ul, "\mathbb{1}"] \arrow[ur, "\mathbb{1}"'] & & & &
\end{tikzcd}$$

Interval decompositions of $\mathbb{V}, \mathbb{W}$ are related through some *greedy rule*.

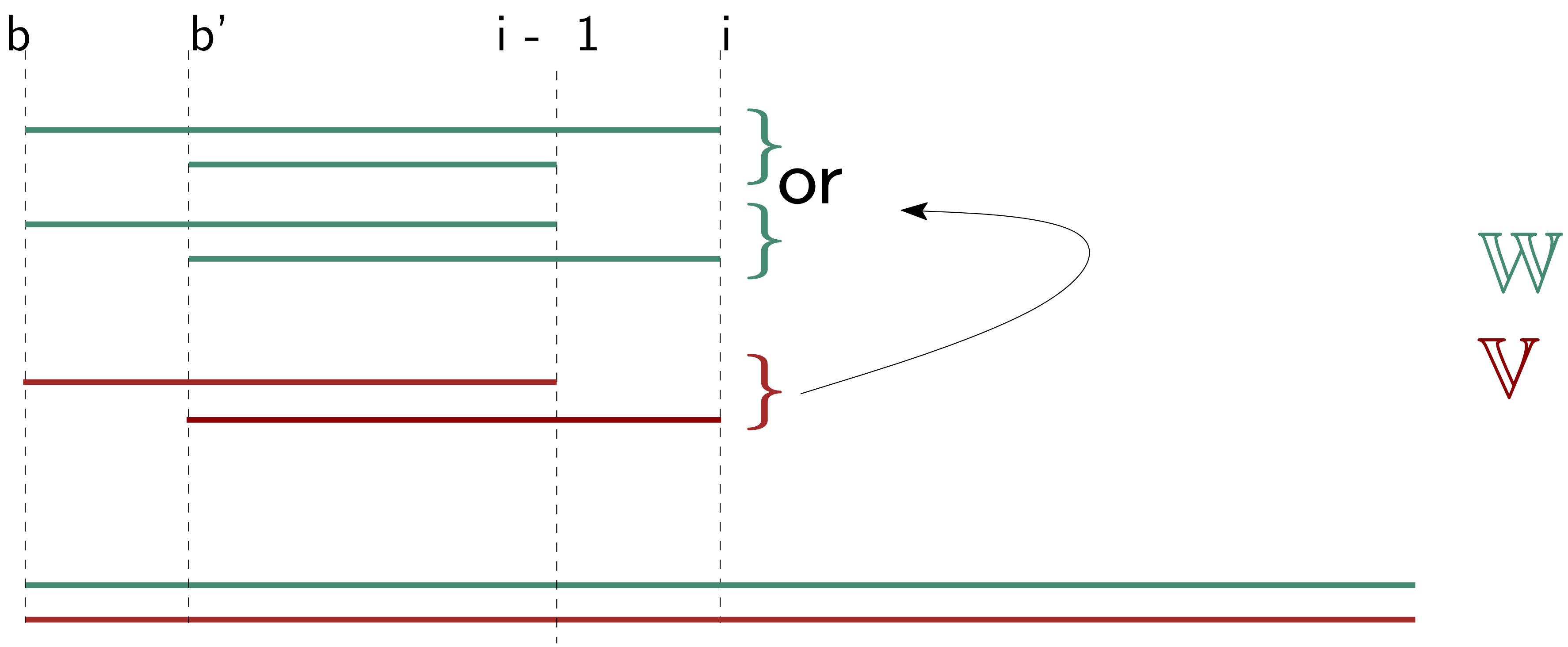


Application: Persistence Computation

Theorem: Transposition Diamond Principle [Maria, O. '15]
For an *exact* diamond + morphisms inj. of corank 1 or surj. of nullity 1:

$$
\begin{tikzcd}[row sep=small]
\mathbb{W} := & & & & W_i \arrow[dl, "b"'] & & & \\
& V_1 \arrow[r, dash] & \cdots \arrow[r, dash] & V_{i-1} & & V_{i+1} \arrow[ul, "d"'] \arrow[dl, "c"] \arrow[r, dash] & \cdots \arrow[r, dash] & V_n \\
\mathbb{V} := & & & & V_i \arrow[ul, "a"] & & &
\end{tikzcd}
$$

Interval decompositions of $\mathbb{V}, \mathbb{W}$ are related as in vineyards.



Application: Persistence Computation

Complexity: at iteration i of algo., with suffix of length h:

a) insertion:

- determine sign of σ: $O(h^2)$ (column reduction)
- injective diamond: $O(1)$ (single extra summand)
- surjective diamond:
  - compute kernel of f: $O(h^2)$ (column reduction)
  - greedy rule: $O(h)$

b) deletion:

- single transposition: $O(h)$ (cf. vineyards)
- $O(h)$ elementary transpositions

Application: Persistence Computation

Wrap-up:

- extensions of Exact Diamond Principle / Reflection Functors

  (cf. injective/surjective diamonds and transposition diamonds)

- same asymptotic complexity: $O(nh^2)$ in the worst case

- better performances than [CdSM'09] in practice

- extension to cohomology $\rightarrow$ significant improvement expected