TANGENTIAL DELAUNAY COMPLEXES IN PRACTICE

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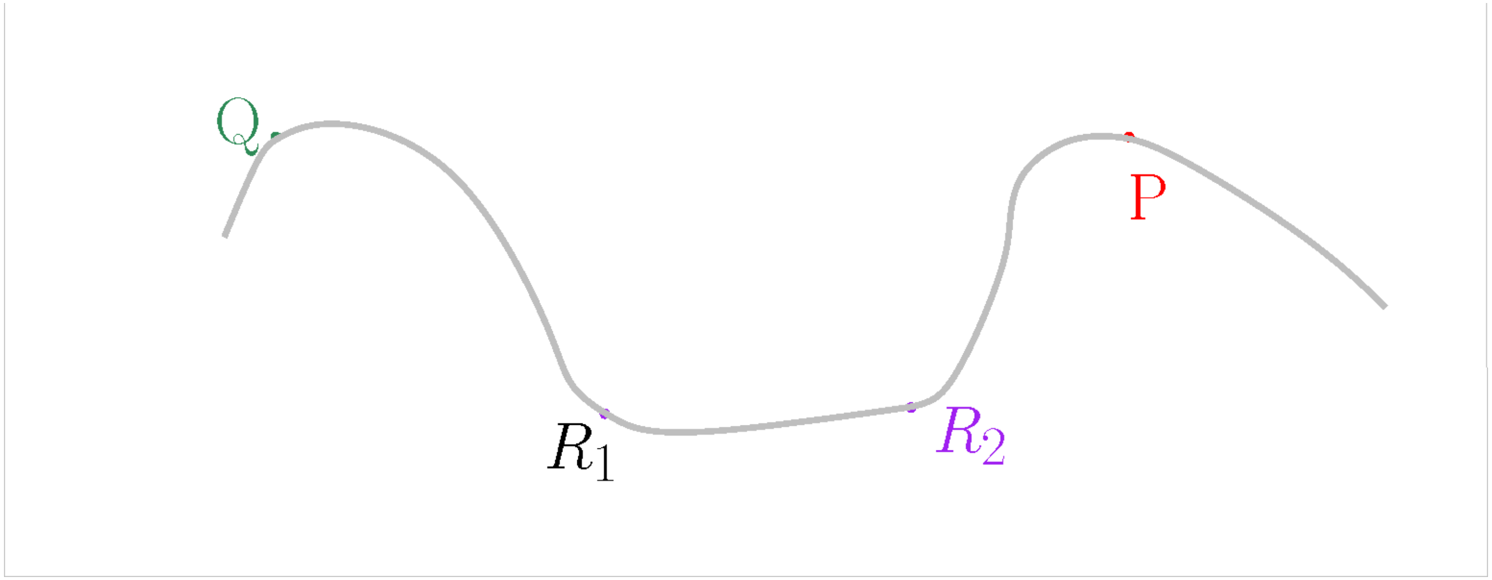
THE TANGENTIAL DELAUNAY COMPLEX

❖ Reconstructing manifolds from point clouds

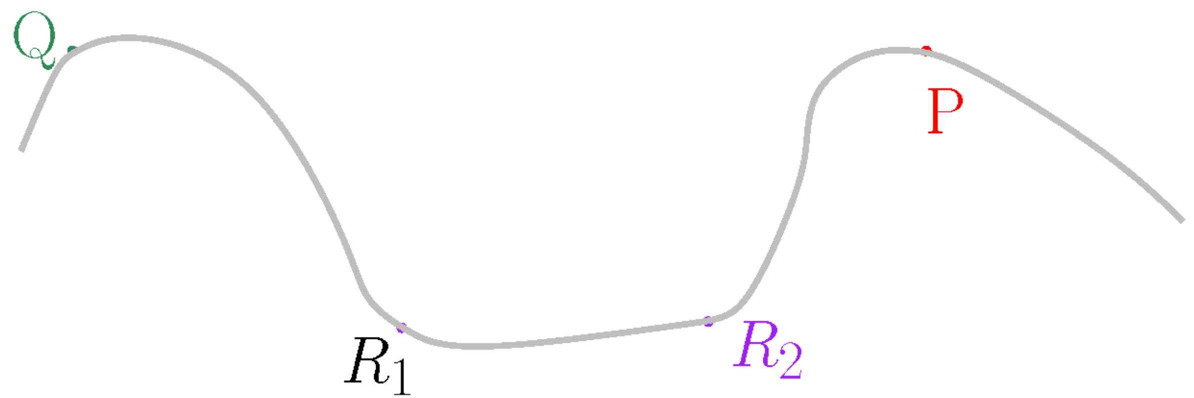
- Geometrization : Data = points + distances between points
- Hypothesis : Data lie close to a structure of "small" intrinsic dimension
- Problem : Infer the structure from the data



THE TANGENTIAL DELAUNAY COMPLEX

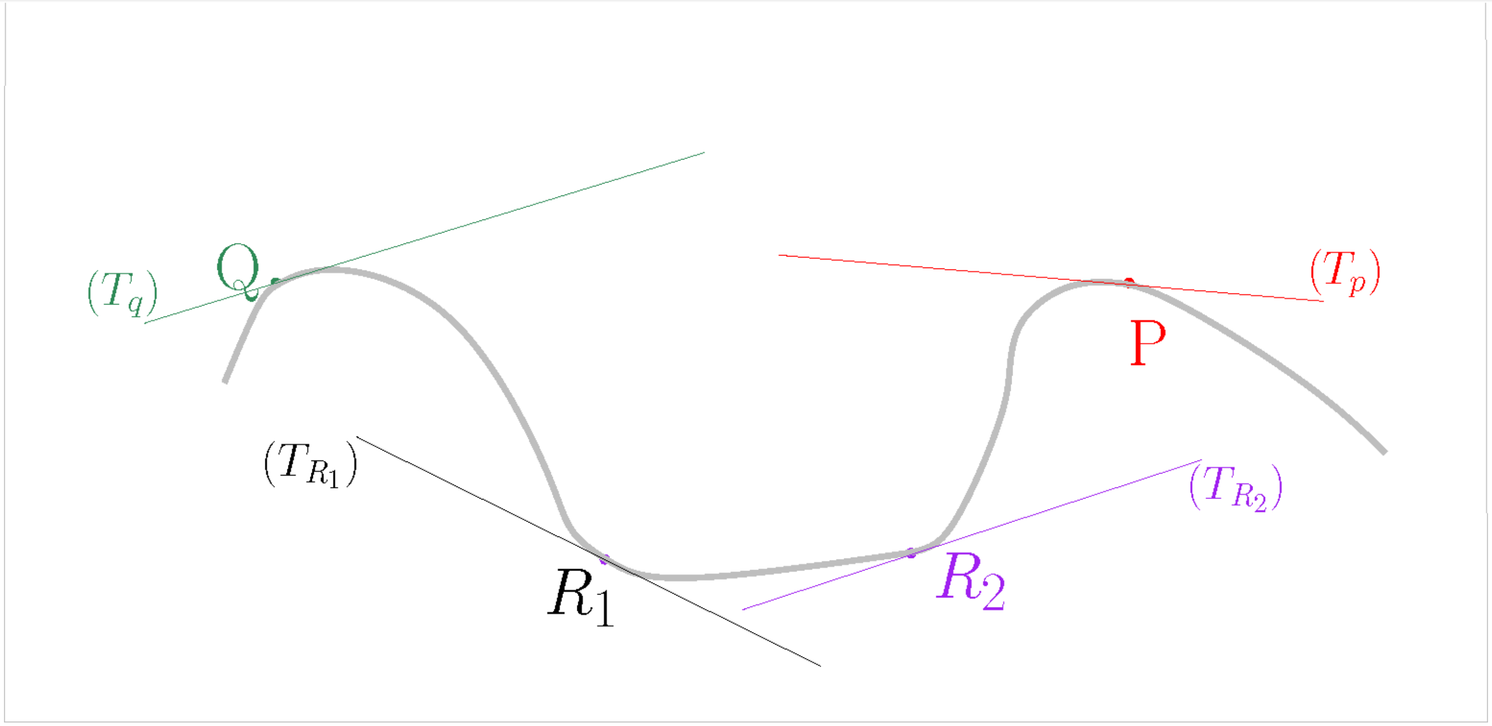
Reconstructing manifolds from point clouds

- Input: point samples coming from an unknown manifold
- Goal: reconstruct a *k*-dimensional smooth manifold embedded in *d*-dimensional Euclidian space



THE TANGENTIAL DELAUNAY COMPLEX

1. Construct the star of $p \in \mathcal{P}$ in the Delaunay triangulation $\mathrm{Del}_{Tp}(\mathcal{P})$ of $\mathcal{P}$ restricted to T_p
2. $\mathrm{Del}_{T\mathbb{M}}(\mathcal{P}) = \bigcup_{p \in \mathcal{P}} \mathrm{star}(p)$



THE TANGENTIAL DELAUNAY COMPLEX

1. Construct the star of $p \in \mathcal{P}$ in the Delaunay triangulation $\mathrm{Del}_{T_p}(\mathcal{P})$ of $\mathcal{P}$ restricted to T_p
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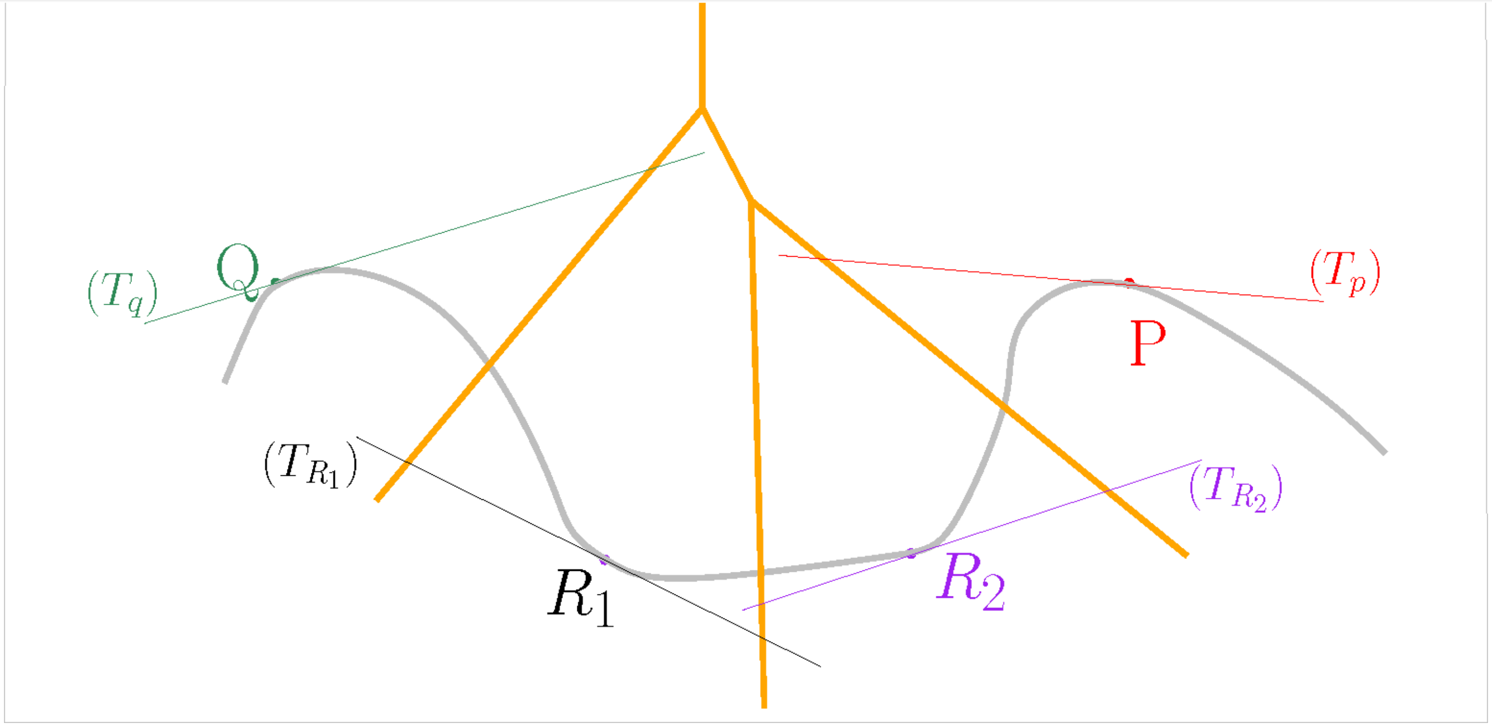

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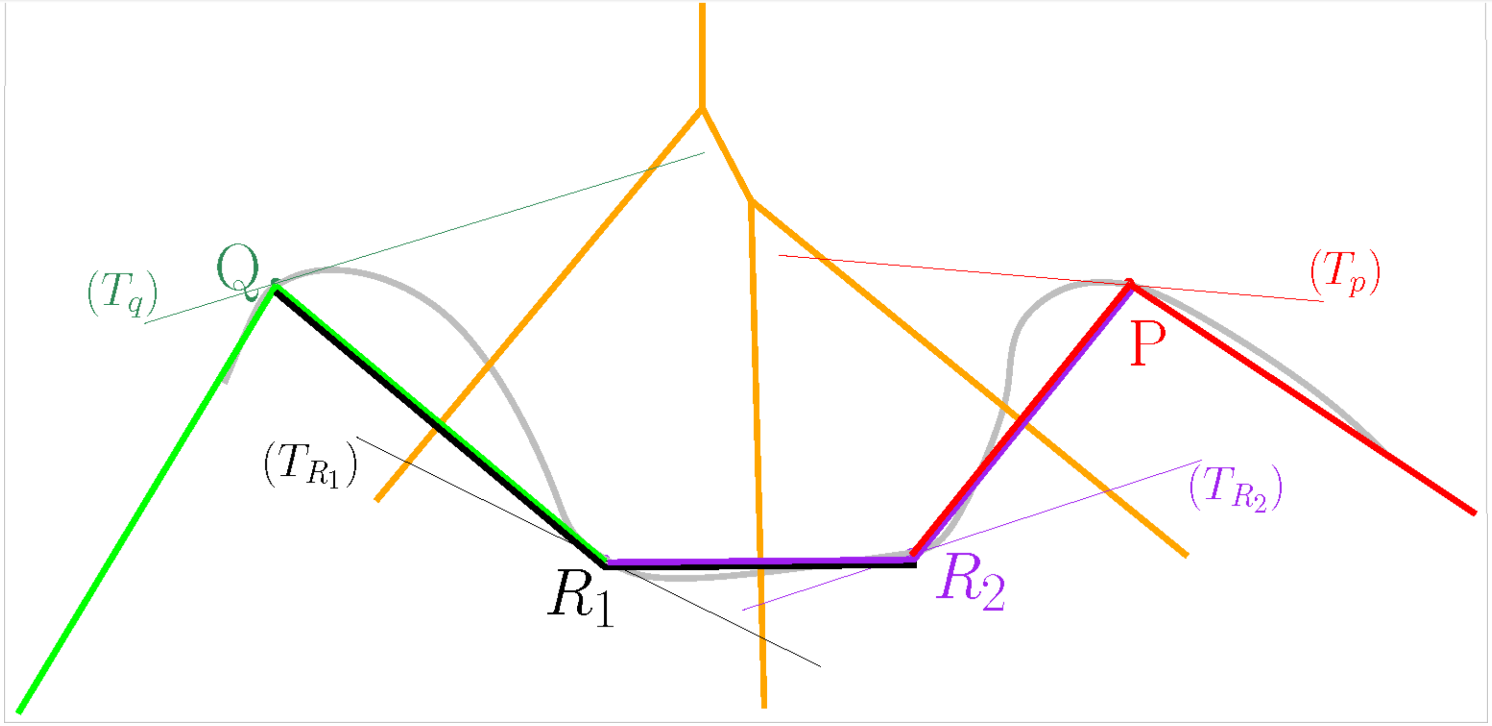

THE TANGENTIAL DELAUNAY COMPLEX

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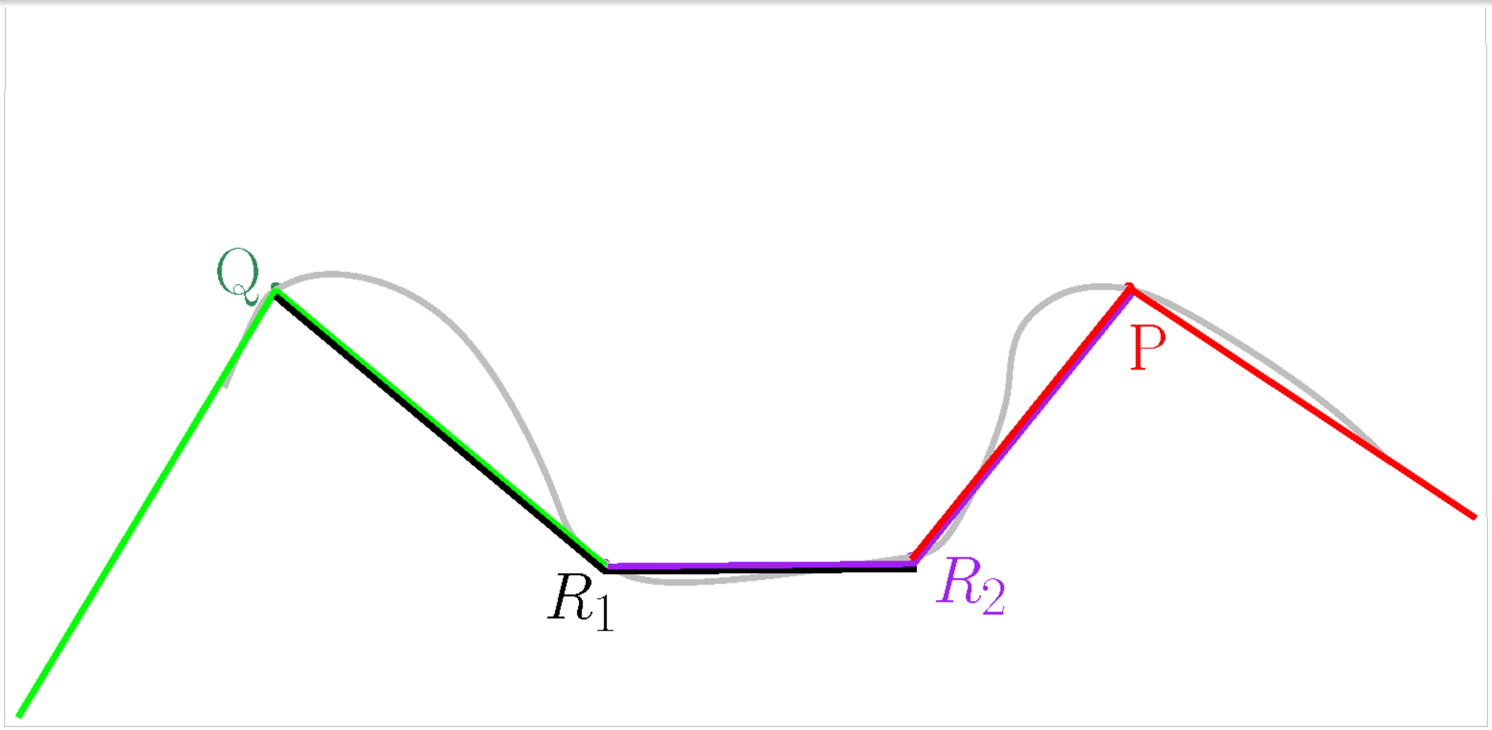
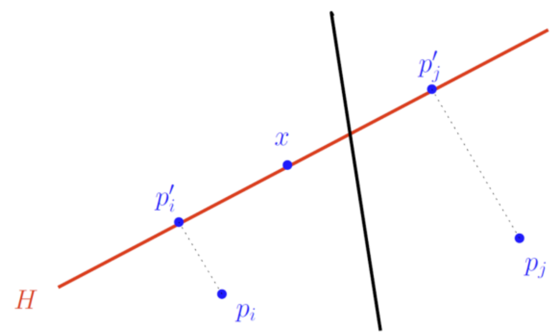


CONSTRUCTION OF $\mathrm{Del}_{T_p}(\mathcal{P})$

Given a d-flat $H \subset \mathbb{R}$, $\mathrm{Vor}(\mathcal{P}) \cap H$ is a weighted Voronoi diagram in H

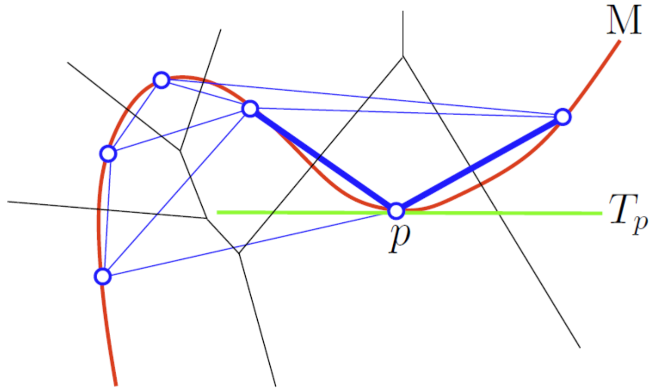


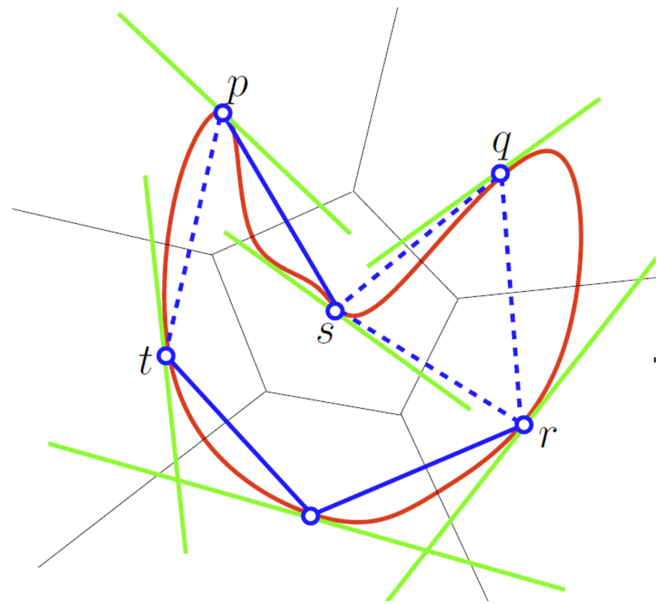
Corollary: construction of Del_{T_p}

$$\psi_p(p_i) = (p'_i, -\|p_i - p'_i\|^2) \qquad \text{(weighted point)}$$

1. project $\mathcal{P}$ onto T_p which requires $O(Dn)$ time
2. construct $\mathrm{star}(\psi_p(p_i))$ in $\mathrm{Del}(\psi_p(p_i)) \subset T_{p_i}$
3. $\mathrm{star}(p_i) \approx \mathrm{star}(\psi_p(p_i))$ (isomorphic)

THE TANGENTIAL DELAUNAY COMPLEX

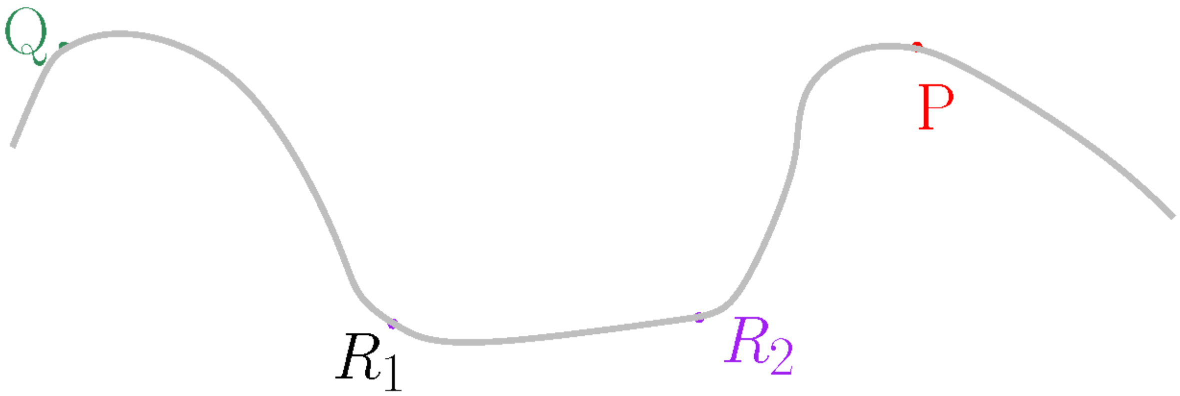




+ $\mathrm{Del}_{T\mathbb{M}}(\mathcal{P}) \subset \mathrm{Del}(\mathcal{P})$
+ $\mathrm{star}(p)$, $\mathrm{Del}_{T_p}(\mathcal{P})$ and therefore $\mathrm{Del}_{T\mathbb{M}}(\mathcal{P})$ can be computed without computing $\mathrm{Del}(\mathcal{P})$
– $\mathrm{Del}_{T\mathbb{M}}(\mathcal{P})$ is not necessarily a triangulated manifold

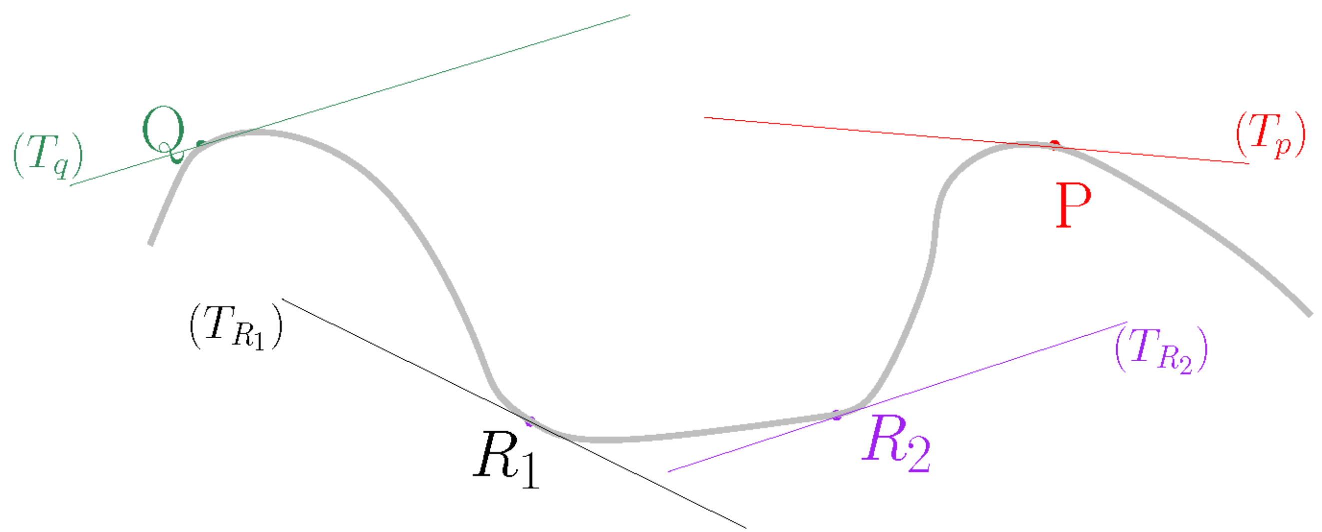
INCONSISTENCIES IN THE TANGENTIAL COMPLEX

Sometimes, things are not so smooth…



INCONSISTENCIES IN THE TANGENTIAL COMPLEX

Before…



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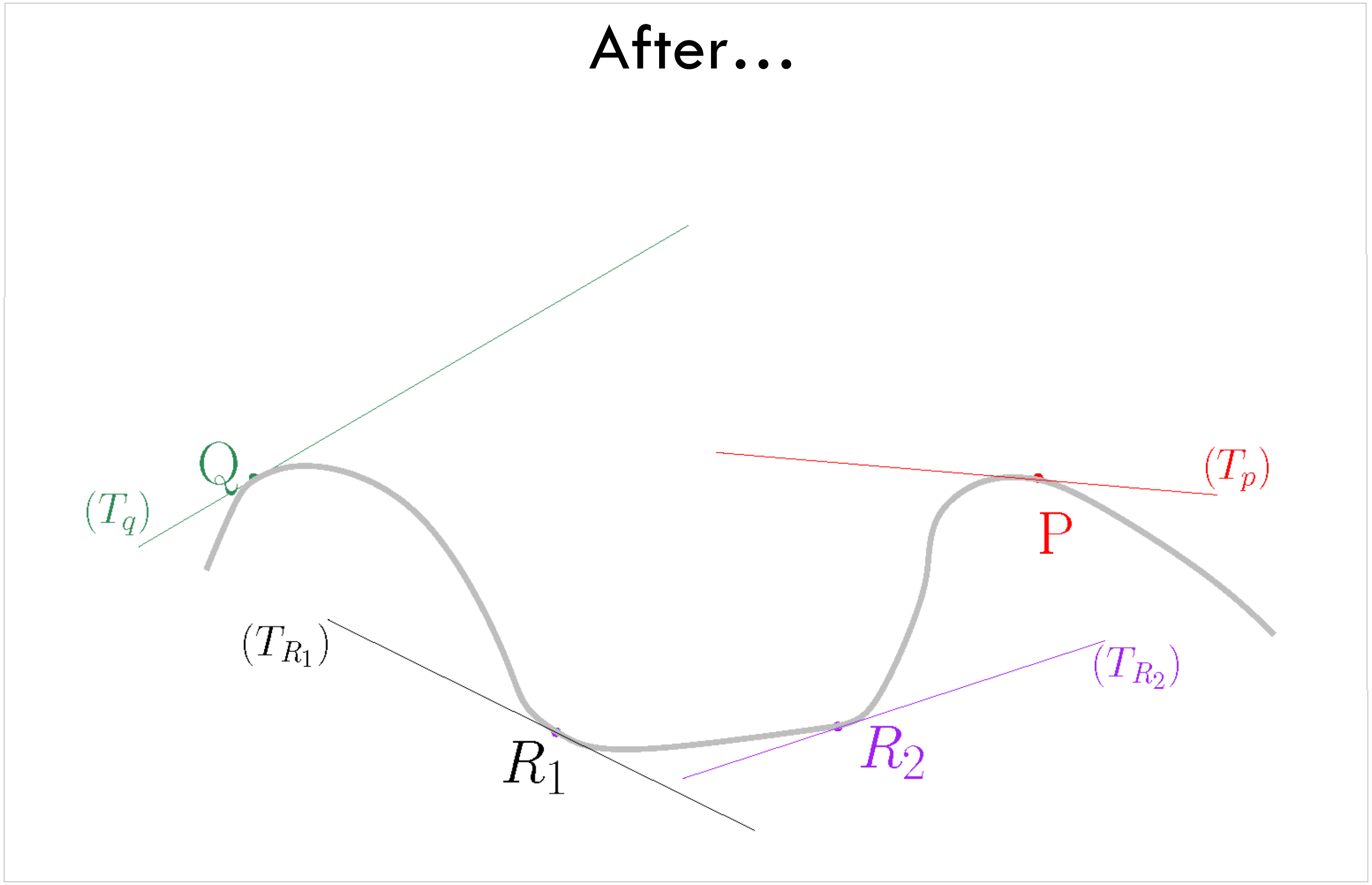

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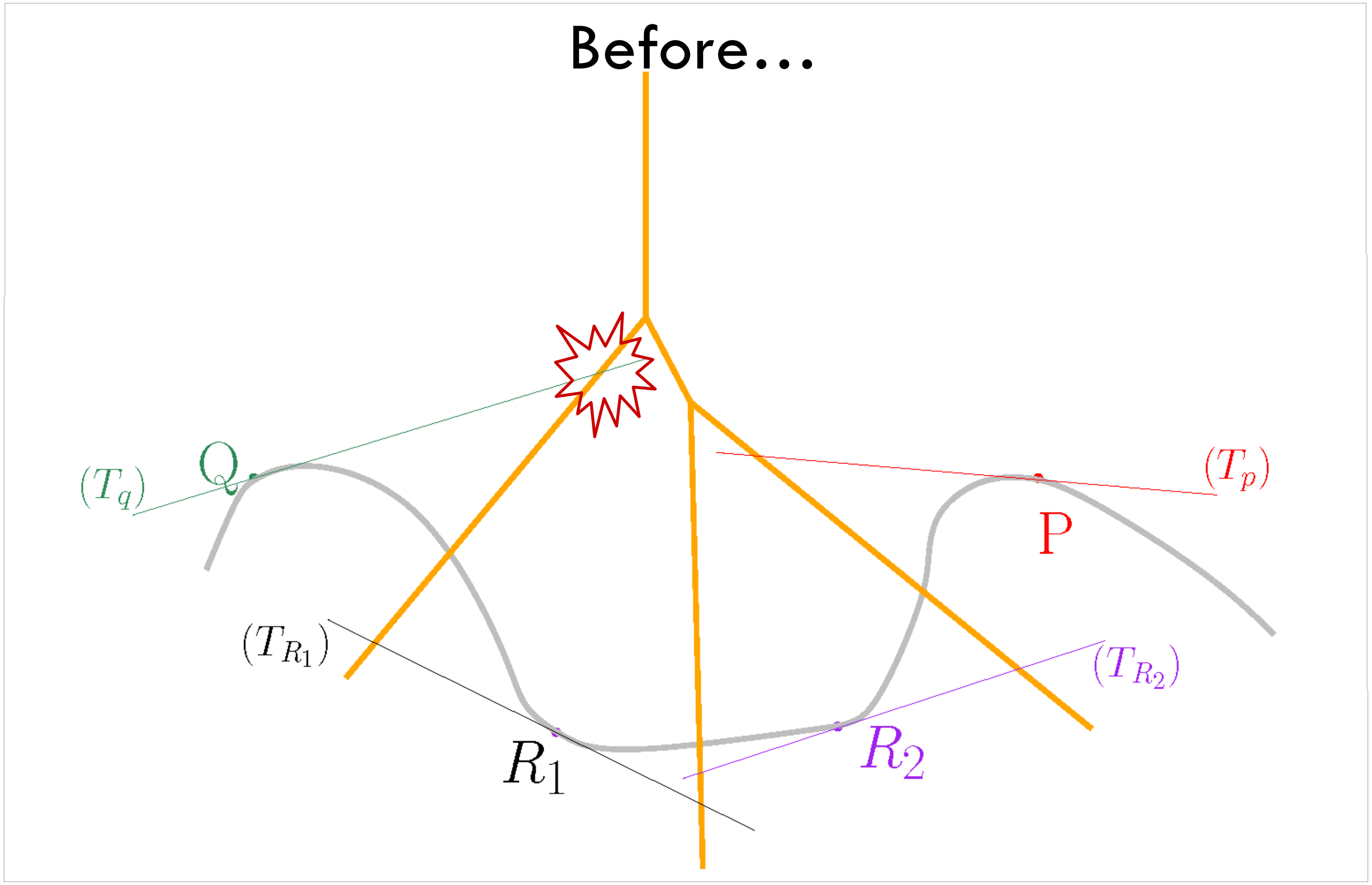

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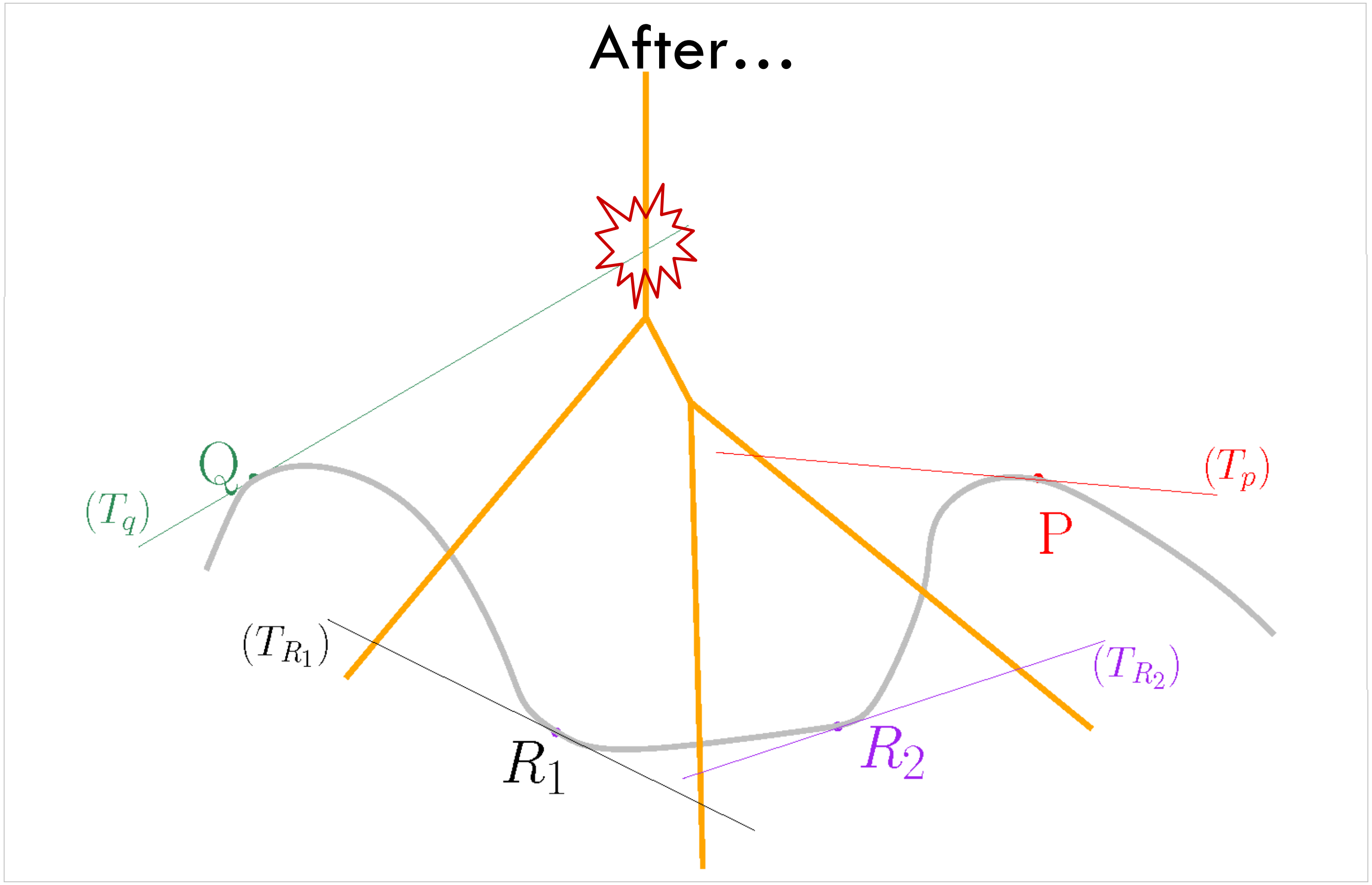

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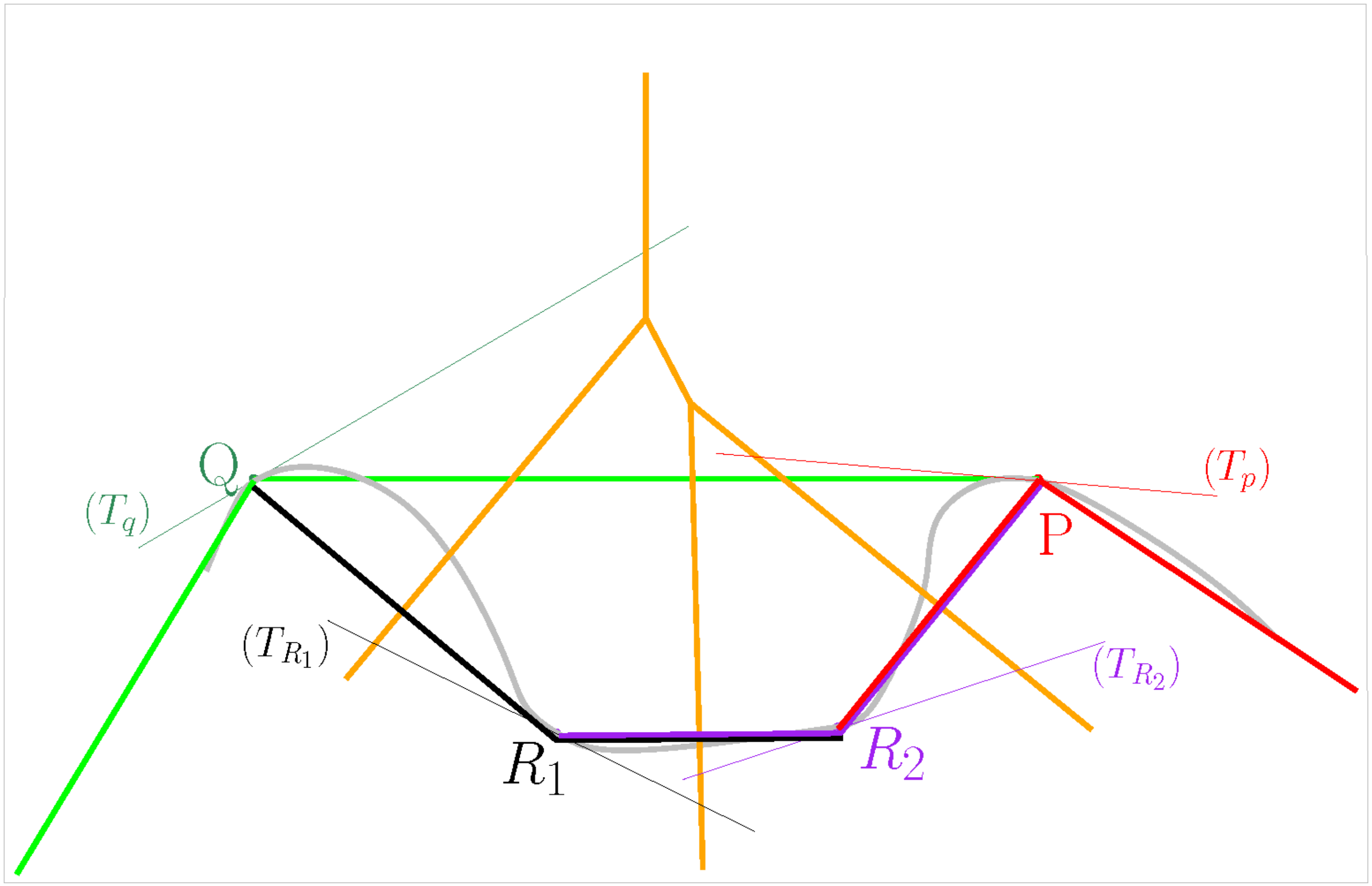
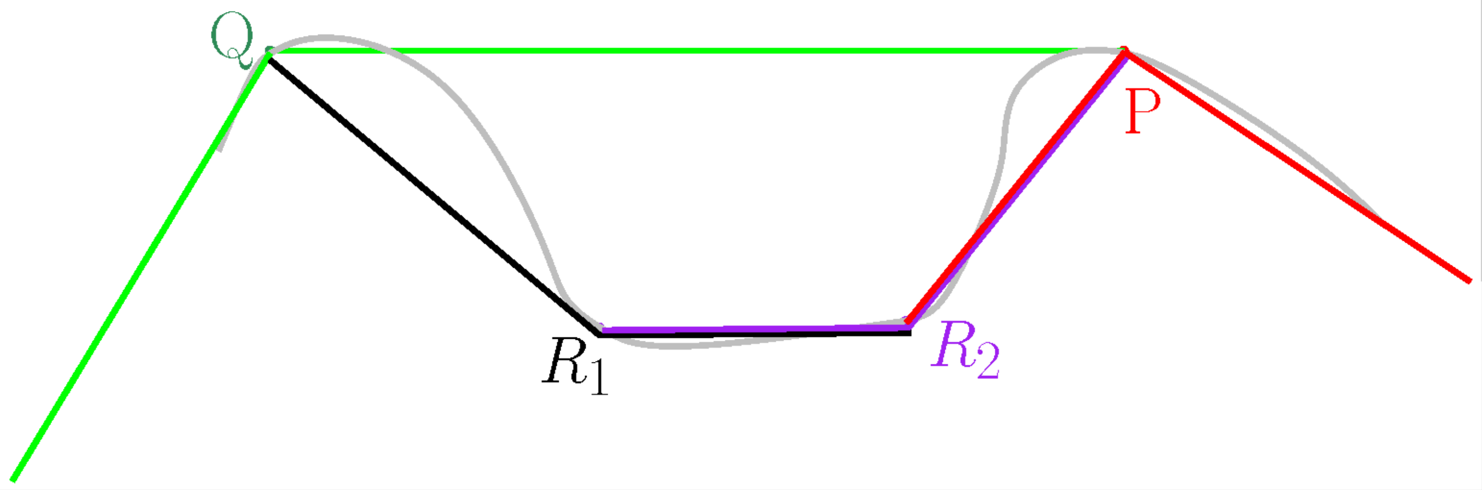


INCONSISTENCIES IN THE TANGENTIAL COMPLEX

Inconsistency: when a simplex is not in the star of all its vertices



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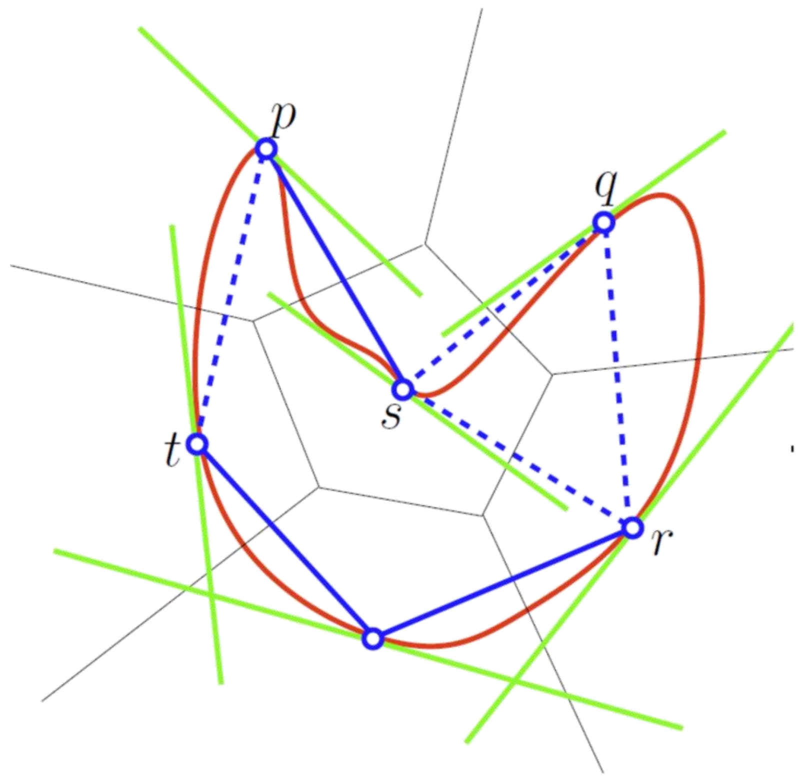
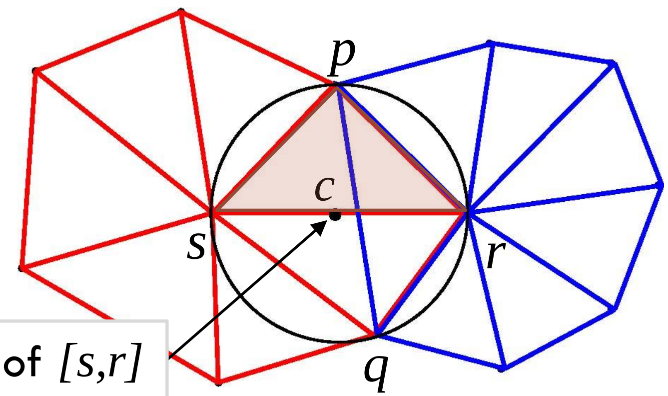


SOLVING INCONSISTENCIES

- Perturb the points…
  1. by weighting the points of P
  2. by moving points in their tangent space
- Option 2 is a bit more effective (more degrees of freedom)

SOLVING INCONSISTENCIES



Power center of *[s,r]*

r is in the star of *s*,
but *s* is not the star of *r*
➔ Simplex *[s,p,r]* is inconsistent

❖ Which points?

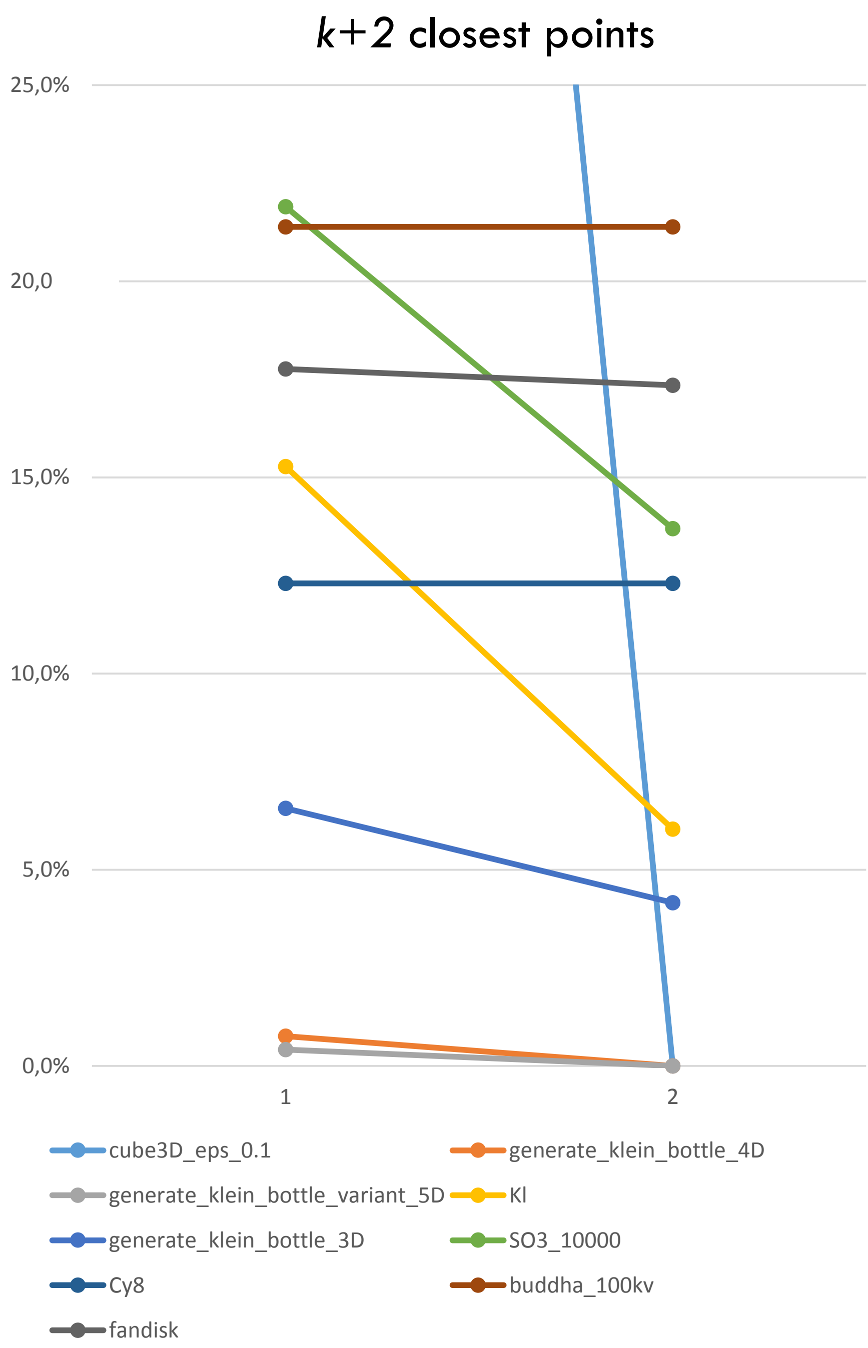
1. Center point only: s
2. One random point of the simplex: s or p or r
3. Simplex points only: s, p, r
4. k+2 closest points to c: s, p, r, q
5. Points of the 1-star of s: s, p, r, q, *etc.*

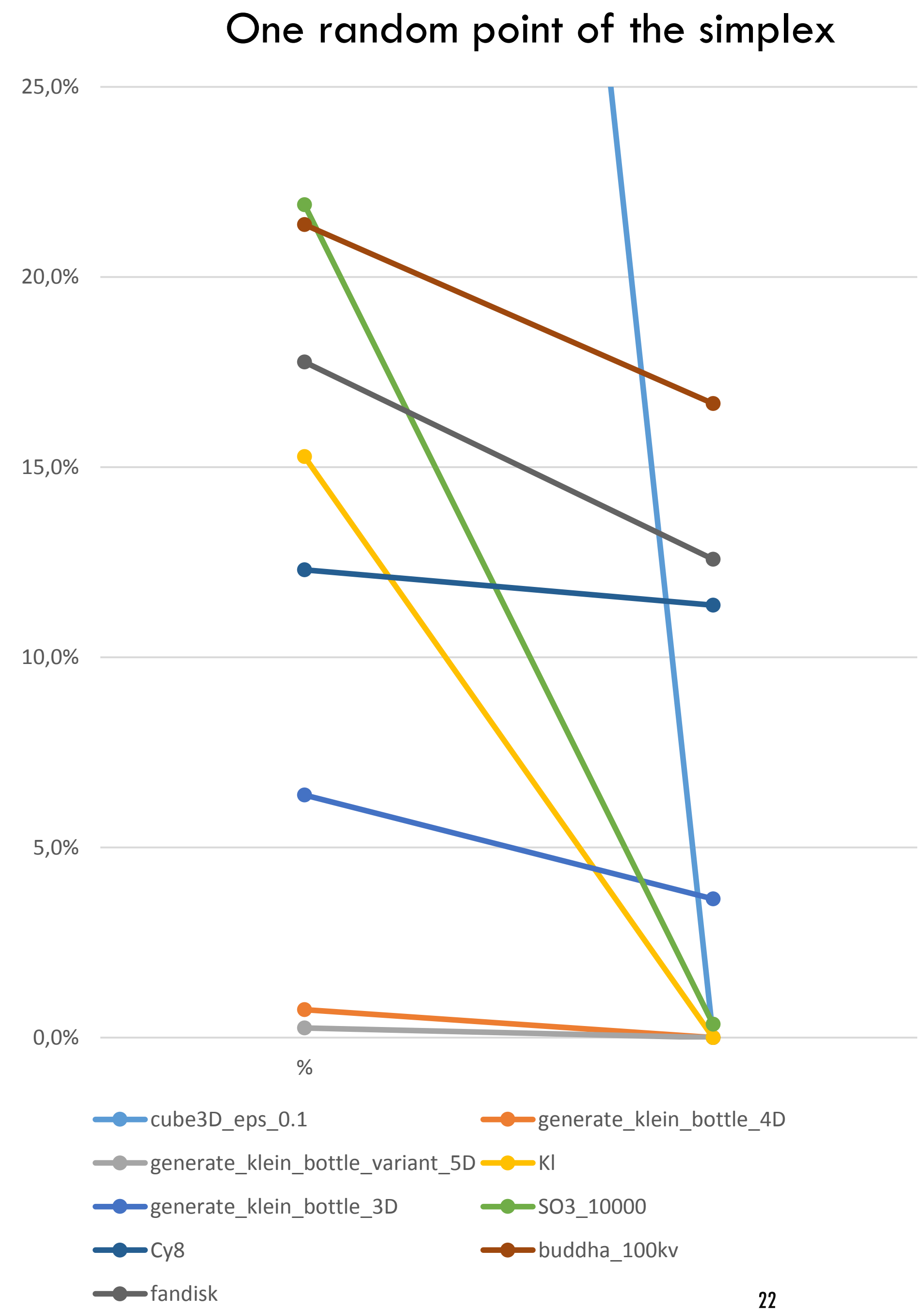
K+2 CLOSEST POINTS

Input	Intrinsic_dim	Ambient_dim	Sparsity	Num_points_in_input	Num_points	Initial_num_inconsistent_local_tr	%	Best_num_inconsistent_local_tr	%	Final_num_inconsistent_local_tr	%	Init_time	Comput_time	Fix_time	Fix_steps
S3	3	4	0.05	2000	2000	0	0.0%	0	0.0%	0	0.0%	0.04	0.42	N/A	N/A
generate_sphere_d	1	2	0.005	30000	2678	0	0.0%	0	0.0%	0	0.0%	0.06	0.01	N/A	N/A
generate_sphere_d	2	3	0.005	30000	29678	0	0.0%	0	0.0%	0	0.0%	0.20	1.06	N/A	N/A
generate_sphere_d	3	4	0.05	30000	29588	0	0.0%	0	0.0%	0	0.0%	0.32	12.85	N/A	N/A
generate_plane	2	3	0.005	30000	28632	0	0.0%	0	0.0%	0	0.0%	0.16	17.95	N/A	N/A
generate_moment_curve	1	6	0.005	30000	405	0	0.0%	0	0.0%	0	0.0%	0.07	0.01	N/A	N/A
cube3D_eps_0.1	2	3	0.05	1350	1350	1348	99.9%	0	0.0%	0	0.0%	0.02	0.09	2.40	19
generate_klein_bottle_4D	2	4	0.05	10000	9014	68	0.8%	0	0.0%	0	0.0%	0.07	0.32	2.67	3
generate_klein_bottle_variant_5D	2	5	0.05	30000	24989	103	0.4%	0	0.0%	0	0.0%	0.25	0.87	10.24	4
Kl	2	5	0.05	4900	4792	732	15.3%	289	6.0%	463	9.7%	0.05	0.13	3000.35	7628
generate_klein_bottle_3D	2	3	0.05	30000	24179	1588	6.6%	1005	4.2%	1121	4.6%	0.17	0.85	3002.44	1213
SO3_10000	3	9	0.05	10000	10000	2190	21.9%	1369	13.7%	1707	17.1%	0.17	2.74	3007.64	214
Cy8	2	24	0.1	6040	6040	743	12.3%	743	12.3%	1260	20.9%	0.14	0.29	3000.10	4495
buddha_100kv	2	3	0.005	99678	40921	8749	21.4%	8749	21.4%	16754	40.9%	0.39	1.11	3002.36	958
fandisk	2	3	0.01	6475	6417	1140	17.8%	1113	17.3%	1610	25.1%	0.05	0.19	3000.36	6000

ONE RANDOM POINT OF THE SIMPLEX

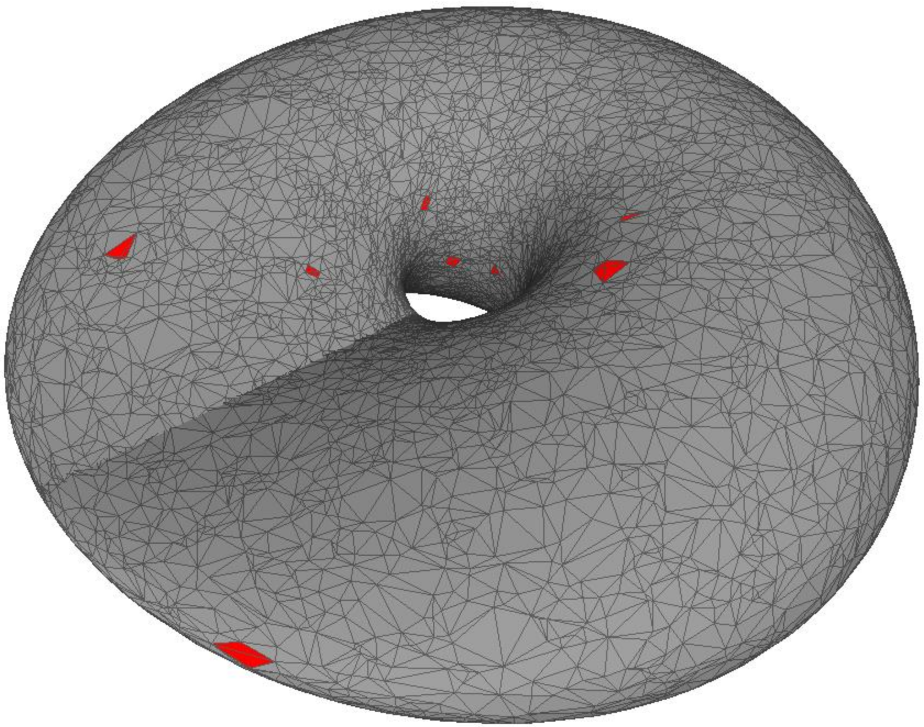
Input	Intrinsic_dim	Ambient_dim	Sparsity	Num_points_in_input	Num_points	Initial_num_inconsistent_local_tr	%	Best_num_inconsistent_local_tr	%	Final_num_inconsistent_local_tr	%	Init_time	Comput_time	Fix_time	Fix_steps
S3	3	4	0.05	2000	2000	0	0.0%	0	0.0%	0	0.0%	0.04	0.43	N/A	N/A
generate_sphere_d	1	2	0.005	30000	2655	0	0.0%	0	0.0%	0	0.0%	0.06	0.01	N/A	N/A
generate_sphere_d	2	3	0.005	30000	29691	0	0.0%	0	0.0%	0	0.0%	0.21	1.05	N/A	N/A
generate_sphere_d	3	4	0.05	30000	29590	0	0.0%	0	0.0%	0	0.0%	0.34	12.80	N/A	N/A
generate_plane	2	3	0.005	30000	28644	0	0.0%	0	0.0%	0	0.0%	0.17	17.33	N/A	N/A
generate_moment_curve	1	6	0.005	30000	409	0	0.0%	0	0.0%	0	0.0%	0.08	0.00	N/A	N/A
cube3D_eps_0.1	2	3	0.05	1350	1350	1348	99.9%	0	0.0%	0	0.0%	0.02	0.09	1.26	9
generate_klein_bottle_4D	2	4	0.05	10000	9029	66	0.7%	0	0.0%	0	0.0%	0.08	0.33	2.89	3
generate_klein_bottle_variant_5D	2	5	0.05	30000	25057	64	0.3%	0	0.0%	0	0.0%	0.28	0.87	5.26	2
Kl	2	5	0.05	4900	4792	732	15.3%	0	0.0%	0	0.0%	0.06	0.13	376.49	895
generate_klein_bottle_3D	2	3	0.05	30000	24194	1544	6.4%	882	3.6%	968	4.0%	0.19	0.87	3001.06	1140
SO3_10000	3	9	0.05	10000	10000	2190	21.9%	35	0.4%	250	2.5%	0.19	2.77	3007.12	189
Cy8	2	24	0.1	6040	6040	743	12.3%	687	11.4%	878	14.5%	0.16	0.29	3000.13	4342
buddha_100kv	2	3	0.005	99678	40921	8749	21.4%	6824	16.7%	8602	21.0%	0.44	1.12	3002.86	892
fandisk	2	3	0.01	6475	6417	1140	17.8%	807	12.6%	1054	16.4%	0.05	0.20	3000.22	5617


k+2 closest points
25,0%
20,0
15,0%
10,0%
5,0%
0,0%
1
2
cube3D_eps_0.1
generate_klein_bottle_4D
generate_klein_bottle_variant_5D
Kl
generate_klein_bottle_3D
SO3_10000
Cy8
buddha_100kv
fandisk


One random point of the simplex
25,0%
20,0%
15,0%
10,0%
5,0%
0,0%
%
cube3D_eps_0.1
generate_klein_bottle_4D
generate_klein_bottle_variant_5D
Kl
generate_klein_bottle_3D
SO3_10000
Cy8
buddha_100kv
fandisk

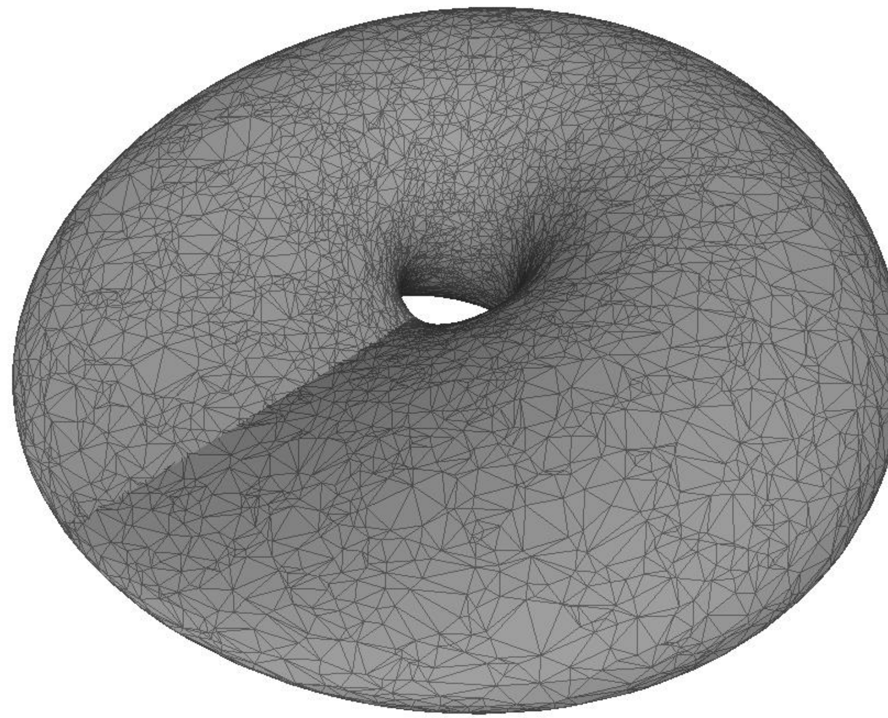
SNAPSHOTS: GENERATED 4D KLEIN BOTTLE

BEFORE
9k points – Dim 2 in R^4



0.7 %

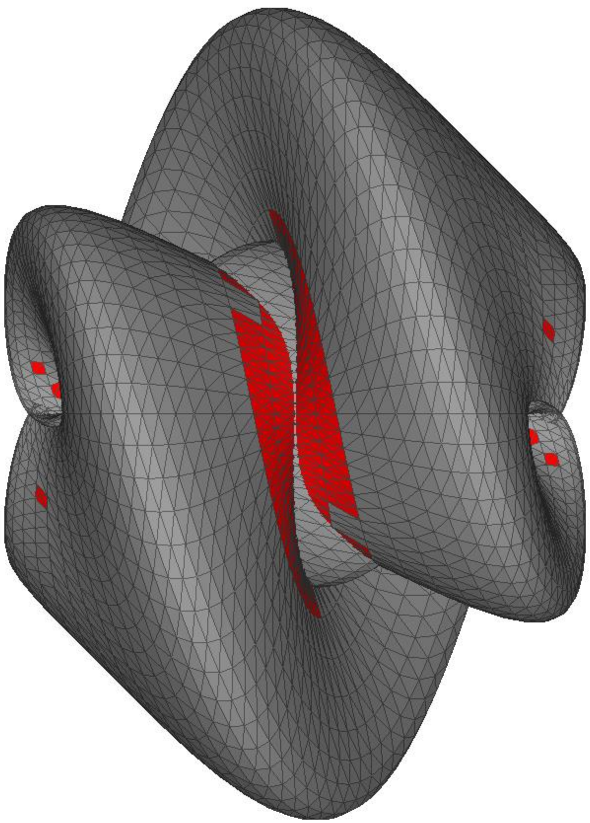
AFTER



0 %

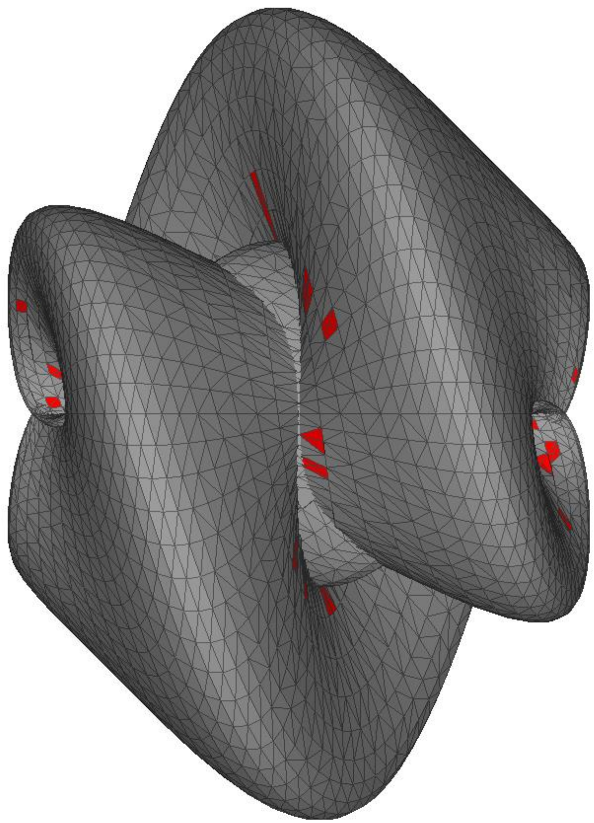
SNAPSHOTS: 5D KLEIN BOTTLE

5k points – Dim 2 in R^5



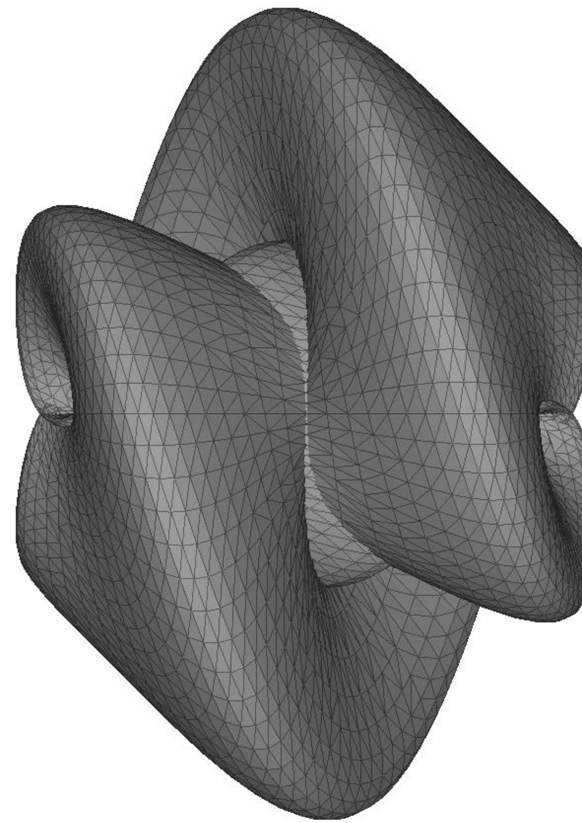
15.3 %

k+2 closest points



9.7 %

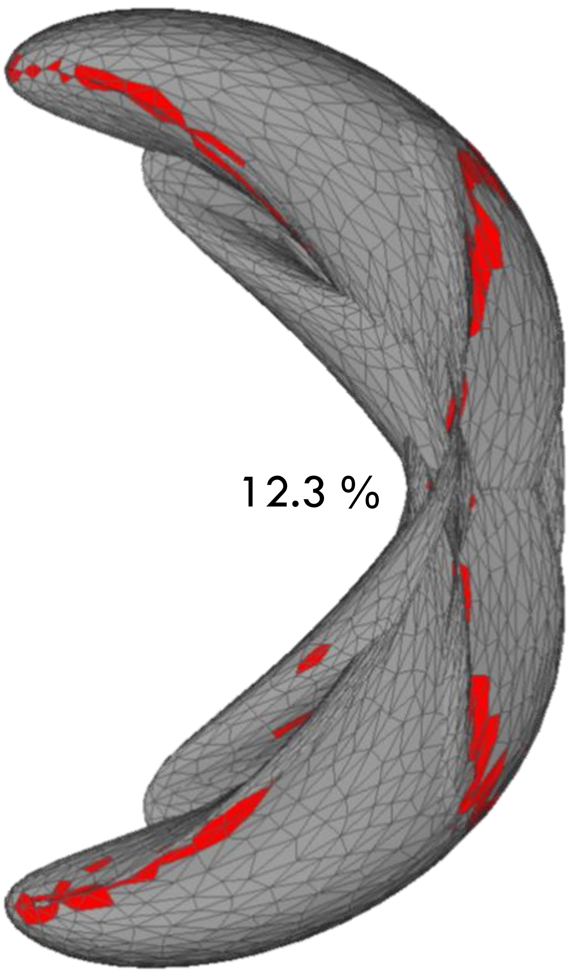
One random point of the simplex



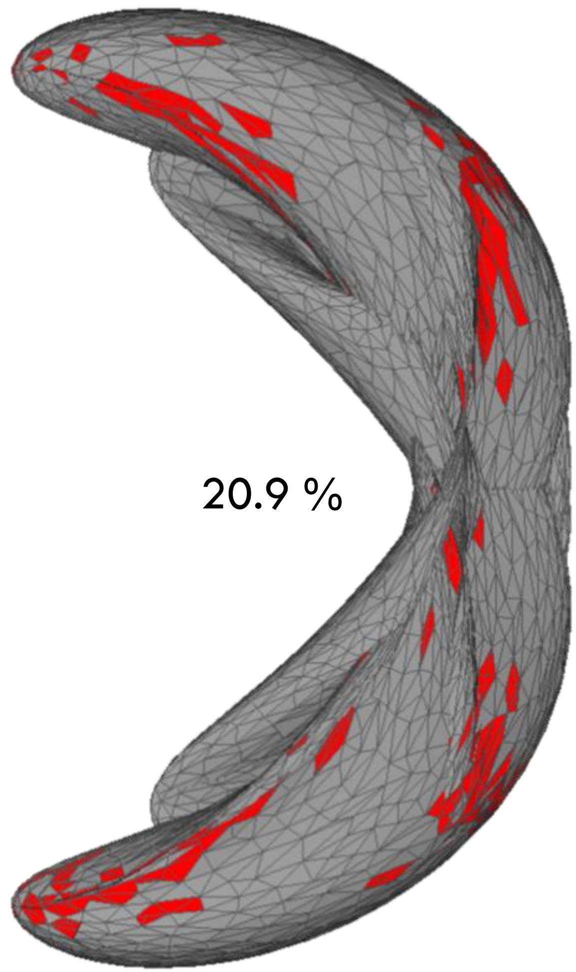
0 %

SNAPSHOTS: CYCLO-OCTANE

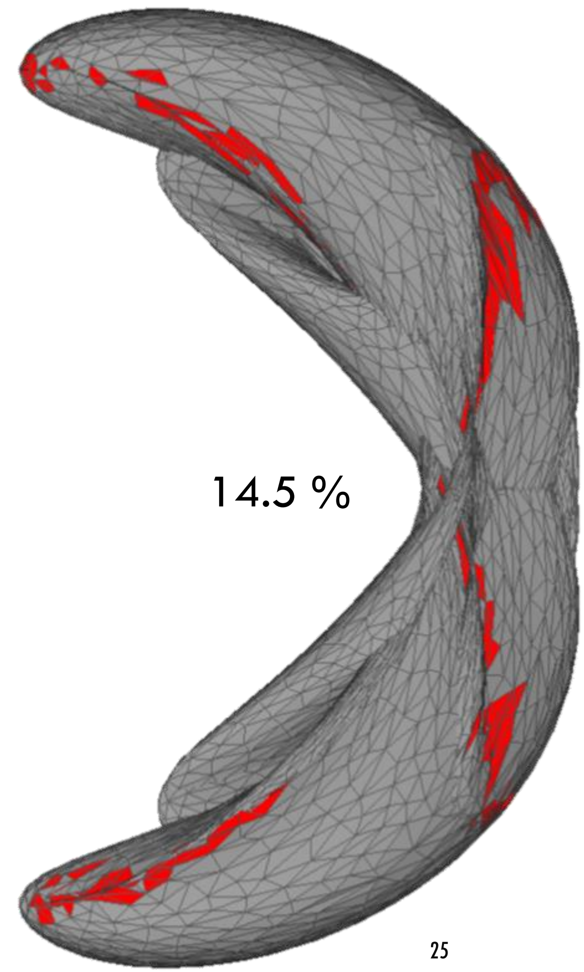
6k points – Dim 2 in R^{24}



k+2 closest points

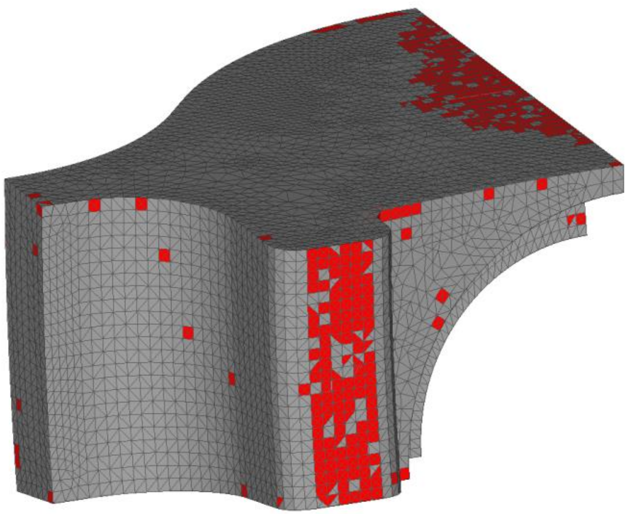


One random point
of the simplex



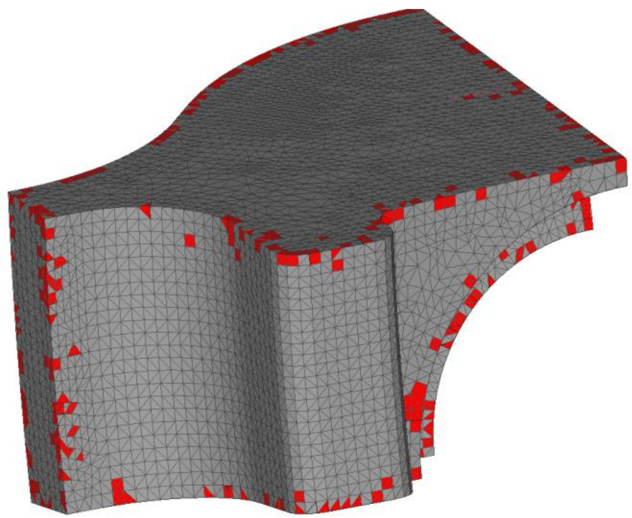
SNAPSHOTS: FANDISK

6k points – Dim 2 in R^3



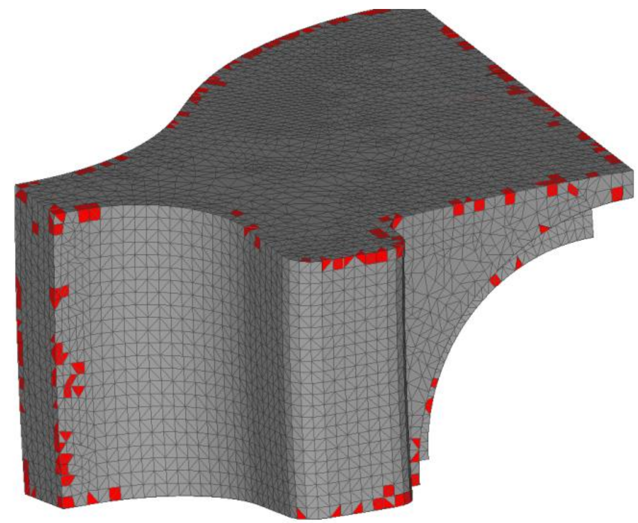
17.8 %

$k+2$ closest points



25.1 %

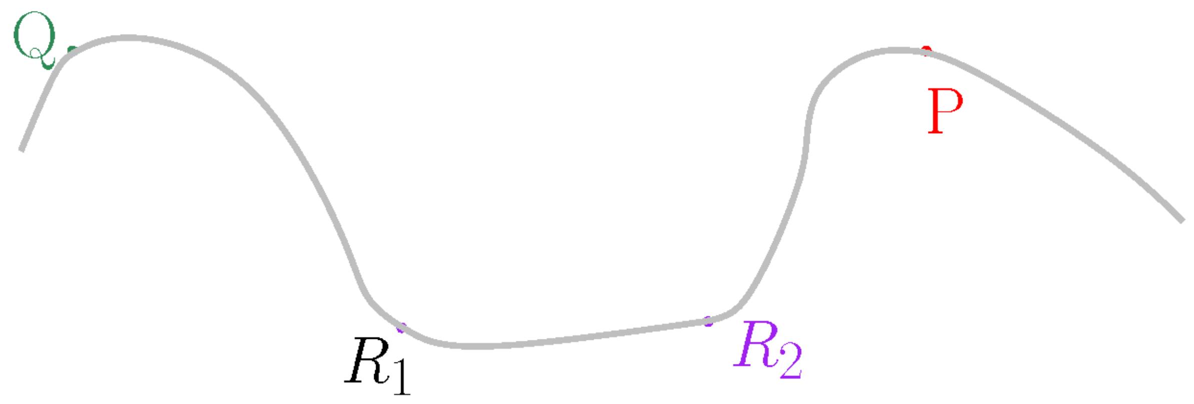
One random point of the simplex



16.4 %

WIP: ALPHA TANGENTIAL COMPLEX

What if we thicken the tangent planes by α?



WIP: ALPHA TANGENTIAL COMPLEX

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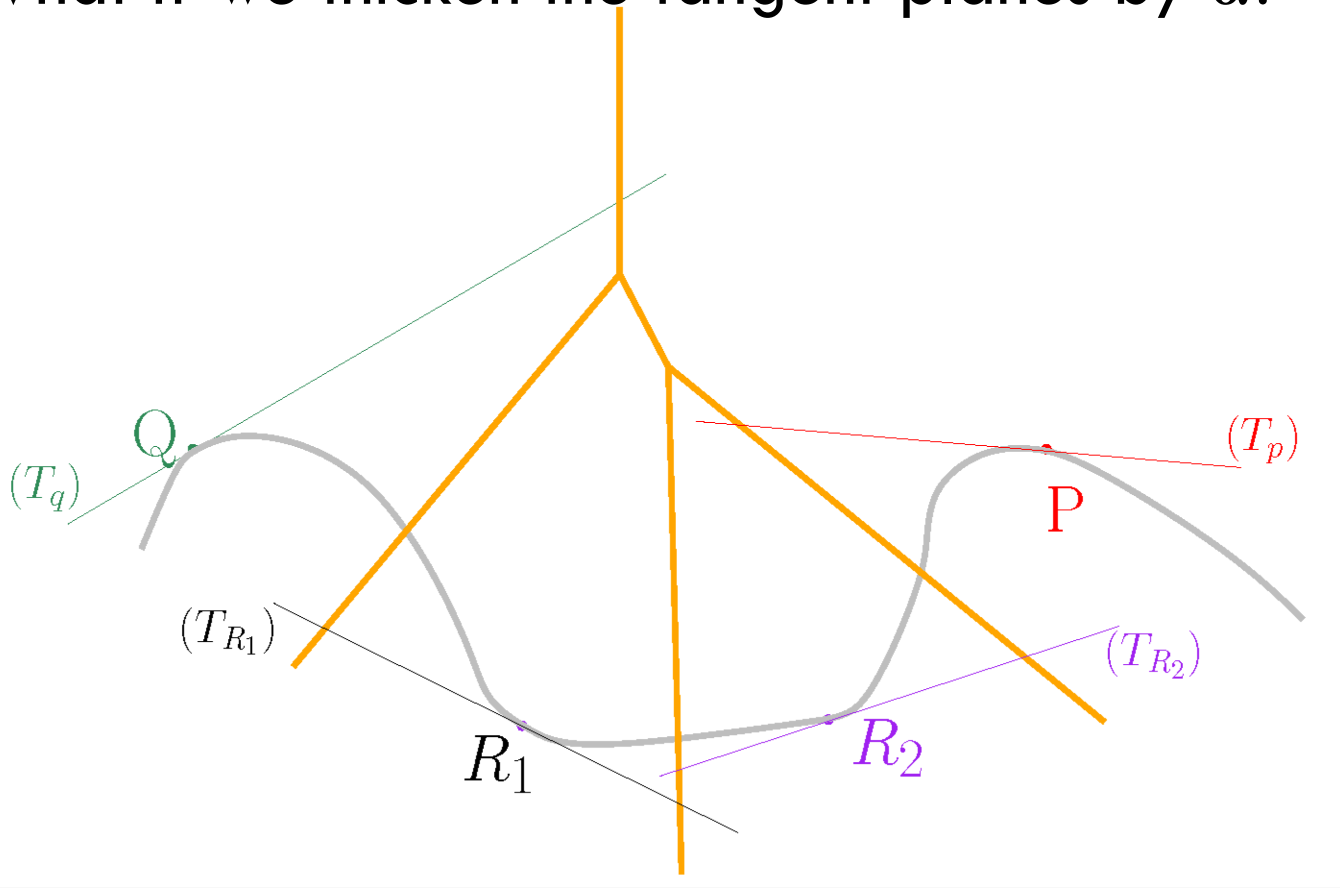
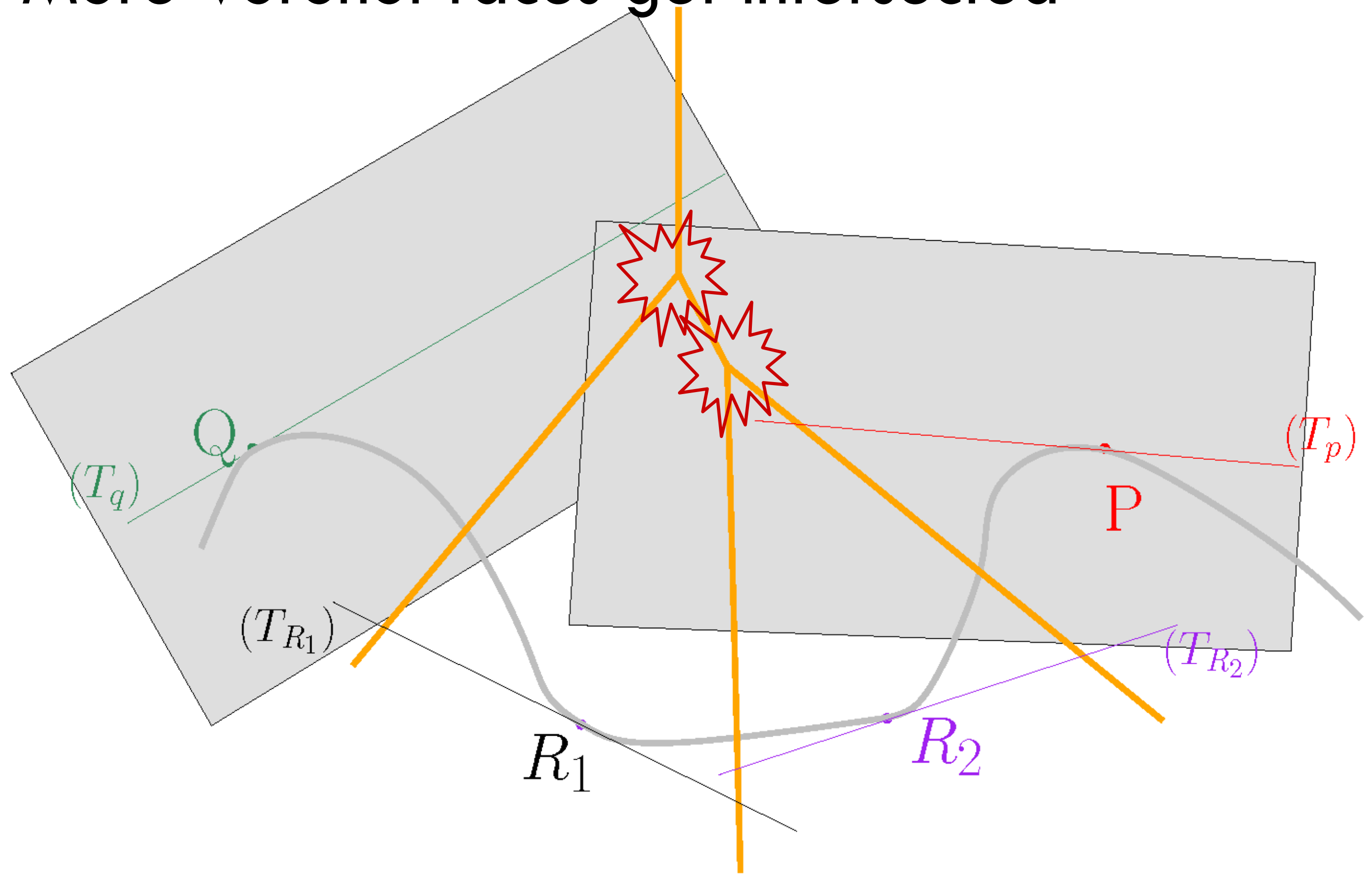


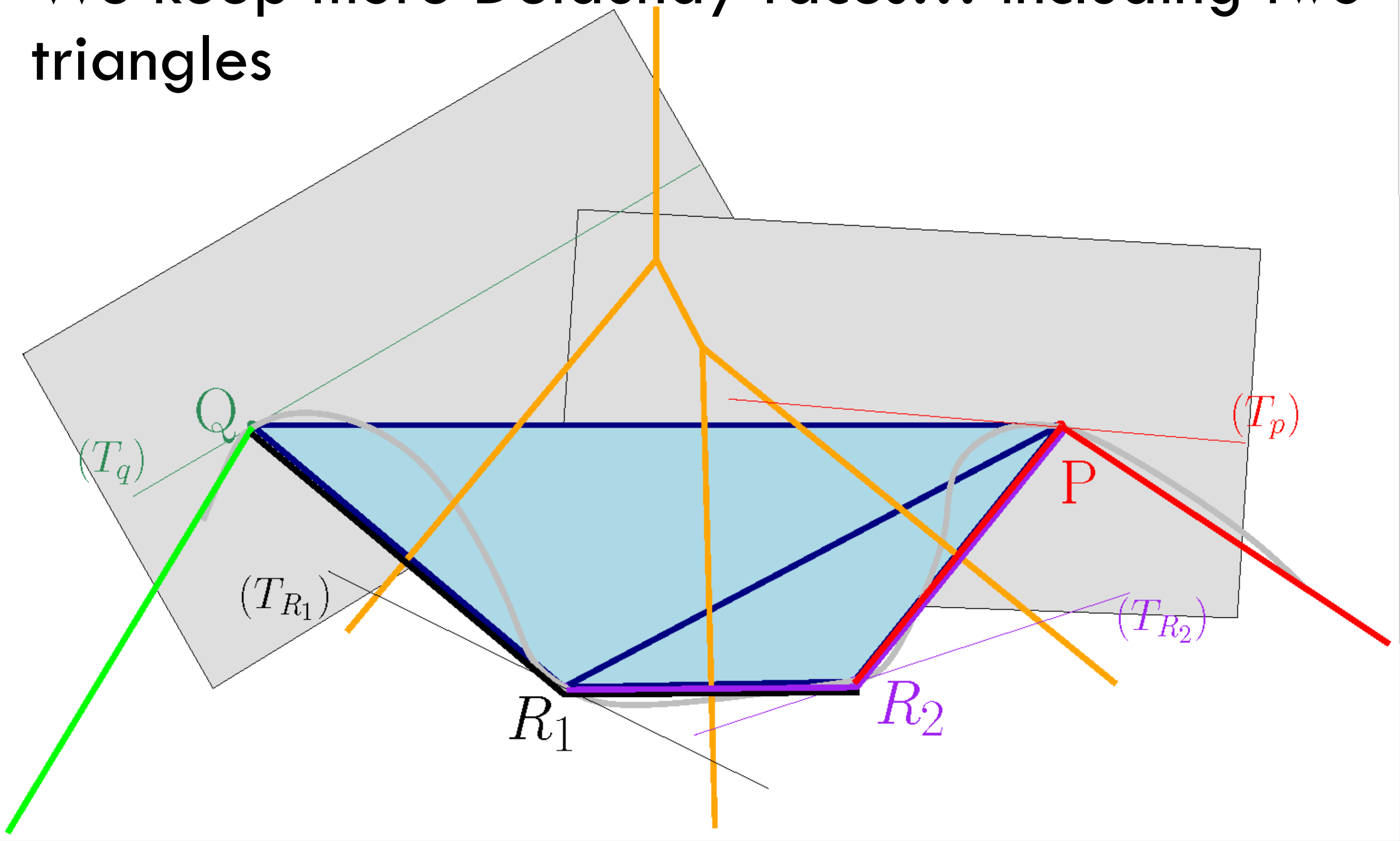
WIP: ALPHA TANGENTIAL COMPLEX

More Voronoi faces get intersected



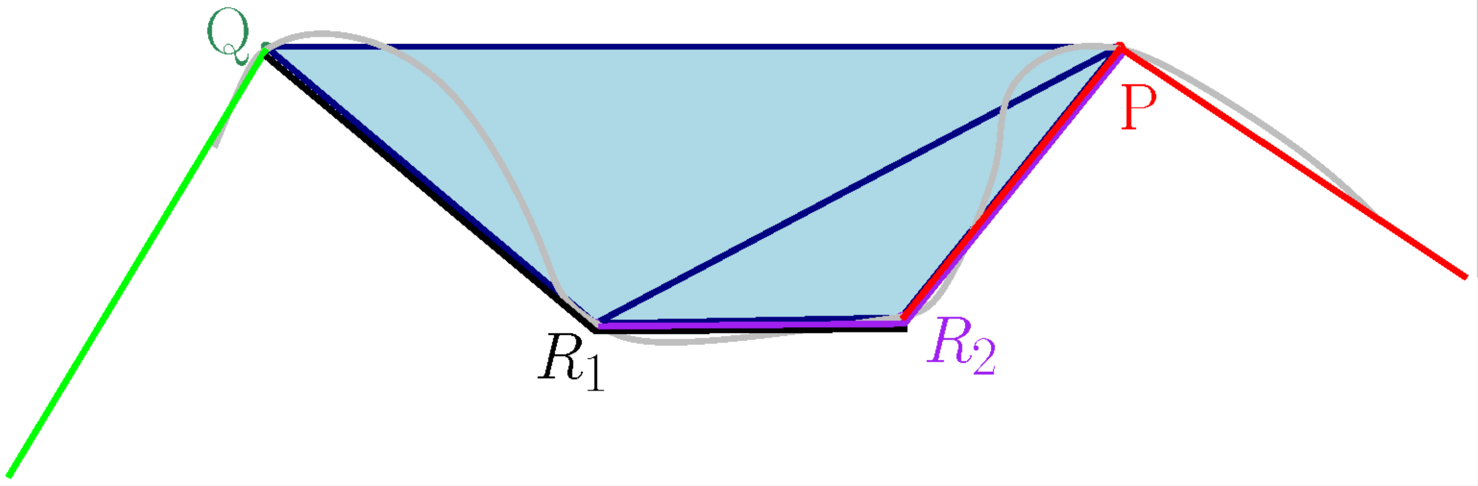
WIP: ALPHA TANGENTIAL COMPLEX

We keep more Delaunay faces… including two triangles



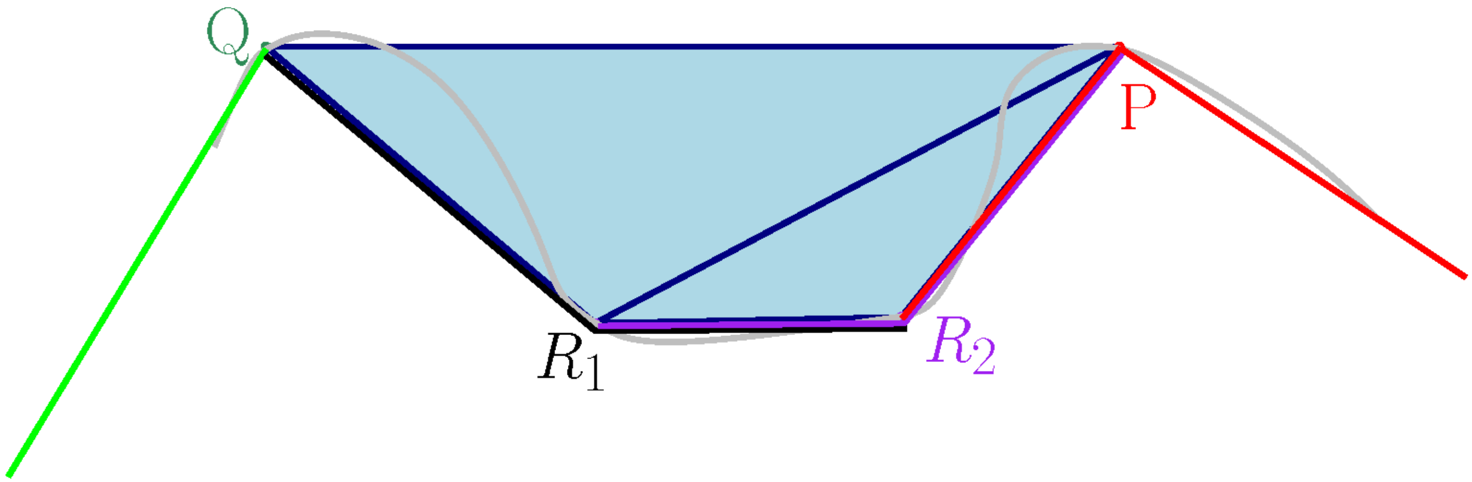
WIP: ALPHA TANGENTIAL COMPLEX

We end up with simplices of different dimensions



WIP: ALPHA TANGENTIAL COMPLEX

Collapse: when a simplex S has only one co-face C, we can remove S and C without changing the topology.



WIP: ALPHA TANGENTIAL COMPLEX

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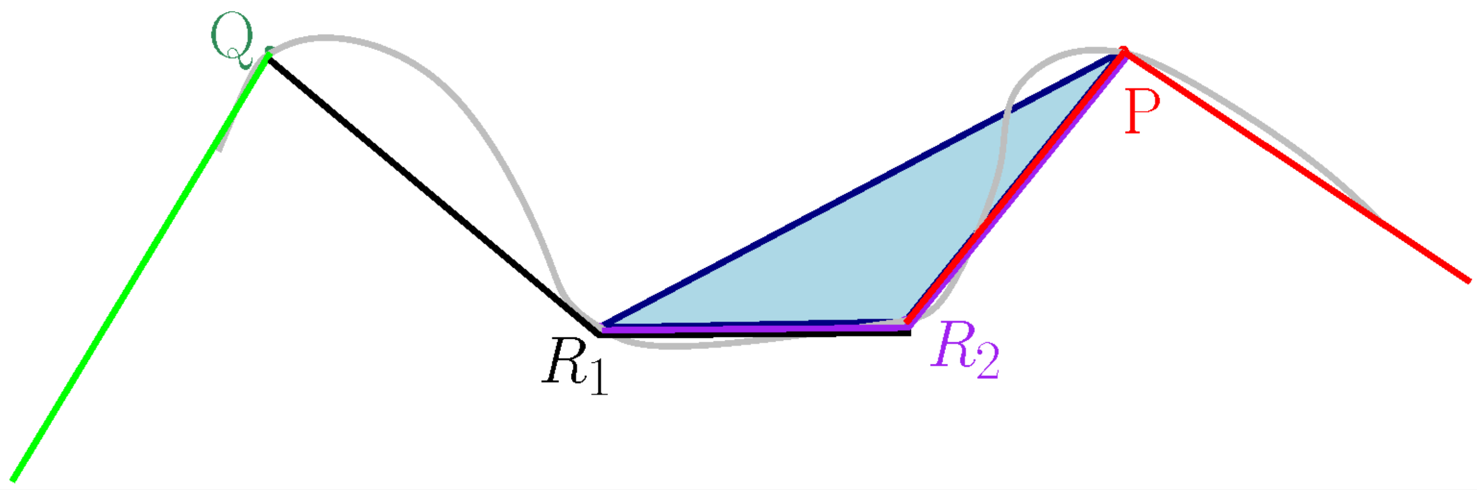
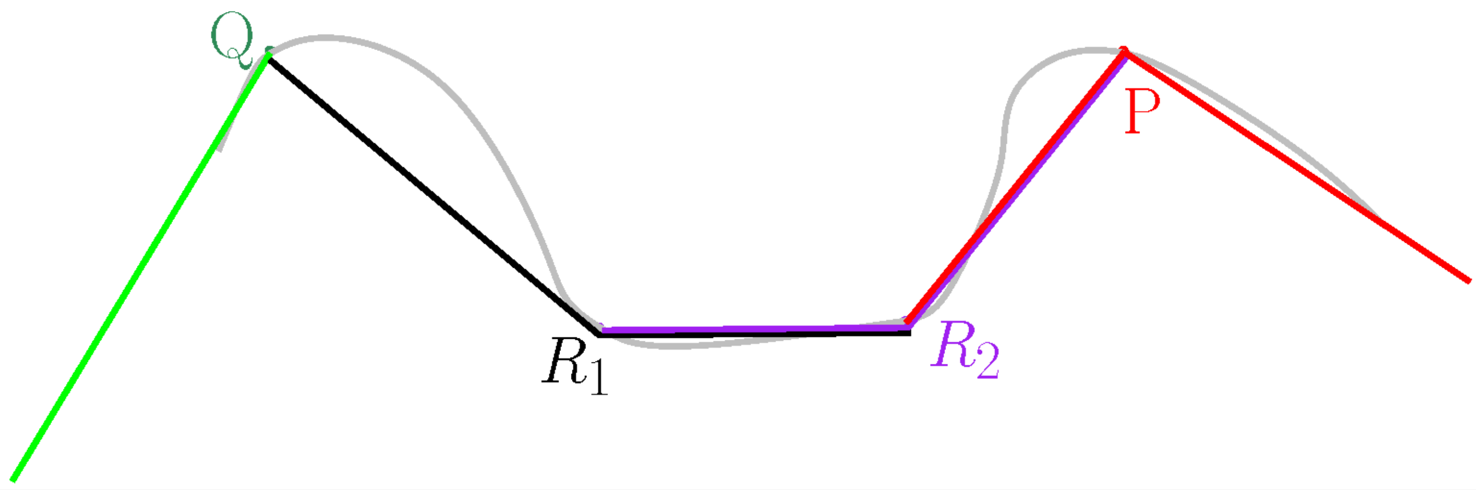


WIP: ALPHA TANGENTIAL COMPLEX

Collapse: when a simplex S has only one co-face C, we can remove S and C without changing the topology.



CONCLUSION

- Tangential Complex: reconstructing a manifold from a point cloud with a complexity depending on the intrinsic dimension.
- Solving inconsistencies by perturbing the points or the weights
  - Sometimes it works…
    - Dense sampling
    - Low intrinsic dimension
    - Best technique so far: perturb position of one random point of the simplex
  - Sometimes it doesn't
    - Sparse sampling
    - High dimensions (e.g. PCA and very high ambient dimension)
- Work in progress: alpha tangential complex