The Waist Inequality

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Theorem 1. *For any continuos map* $f\colon \mathbb{S}^d \to \mathbb{R}^k$ *there exists* $z \in \mathbb{R}^k$ *such that*

$$vol_{d-k}(f^{-1}(z)) \geq vol_{d-k}(\mathbb{S}^{d-k}).$$

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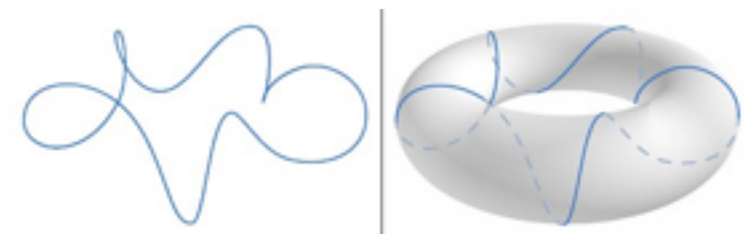
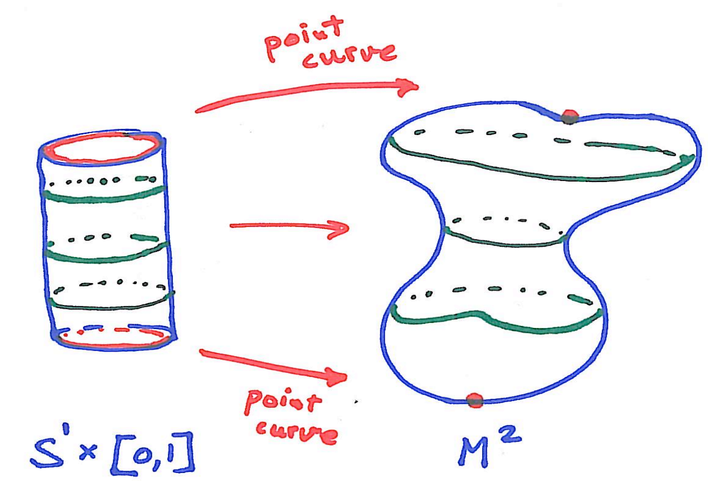
example: d=2,k=1

Q:Is there a closed geodesics in every Riemannian surface?

If curve is homotopically not trivial, it has positive length.

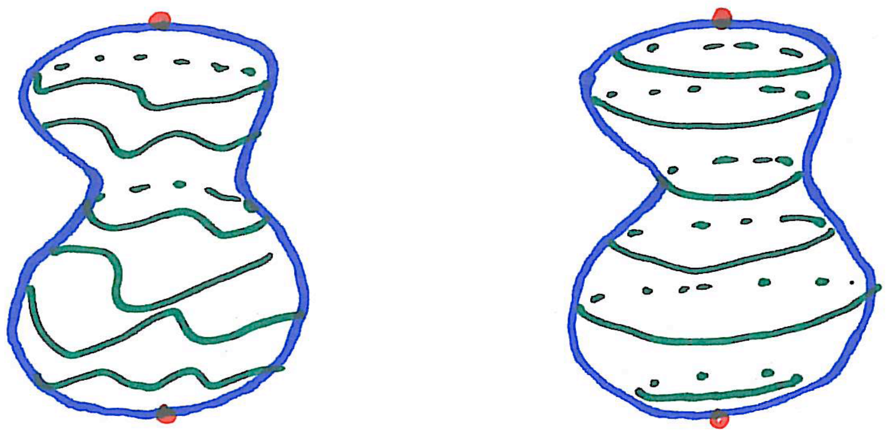




Let $\mathcal{F}$ a homotopy class of maps $F\colon \mathbb{S}^1 \times I \to M^2$

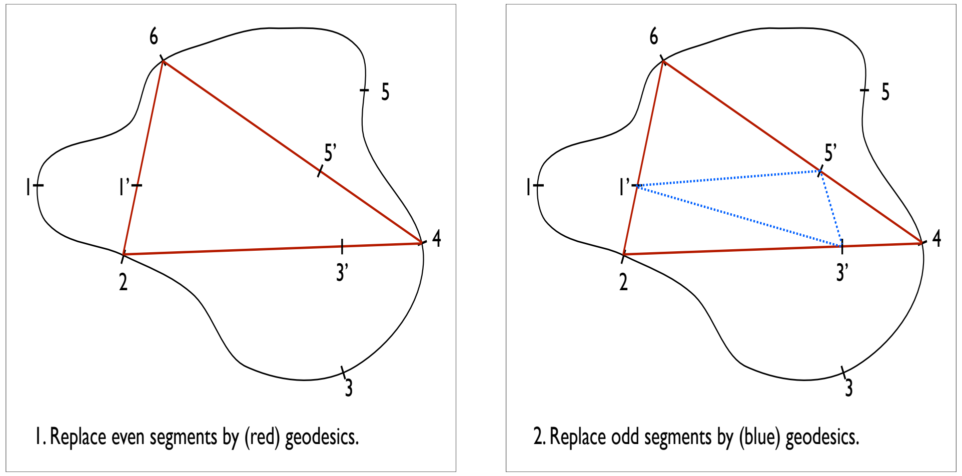
$$width(\mathcal{F}) := min_{F\in\mathcal{F}} max_z vol_1(F(z))$$

Theorem 2. *For any Riemannan surface M and any topologically non trivial class $\mathcal{F}$, $width(\mathcal{F}) > 0$, moreover, the min max is a geodesic.*



Initial sweepout　　Tightened sweepout

Sweepouts



Shortening curves: continuous, look at fixed points.

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$$vol_{d-k}(f^{-1}(z)) \geq vol_{d-k}(\mathbb{S}^{d-k}).$$

Theorem 3. *If* $A \subset \mathbb{S}^d$ *such that* $\frac{vol_d(A)}{vol_d(\mathbb{S}^d)} = \frac{1}{2}$ *then* $vol(\partial A) \geq vol(\mathbb{S}^{d-1})$

Waist is harder than isoperimetry.

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Theorem. *(Almgren) Let F be a family of k-cycles sweeping out the unit n-sphere. Then the maximal volume of F is at least the volume of the unit k-sphere.*

Gromov's short proof:

Isoperimetric inequality +
Brower fixed point theorem.

Theorem 4. *For any continuous map* $f\colon \mathbb{S}^d \to \mathbb{R}^k$ *there is a point* $z \in \mathbb{R}^k$ *such that for every* $\varepsilon > 0$, $vol(f^{-1}(z) + \varepsilon) \geq vol(\mathbb{S}^{d-k} + \varepsilon)$

Concentration of measure.

Localisation.

Gromov's short proof:

Isoperimetric inequality +
Brower fixed point theorem.

Gromov's long proof:

Brunn-Minkoswki inequality +
Borsuk-Ulam theorem.

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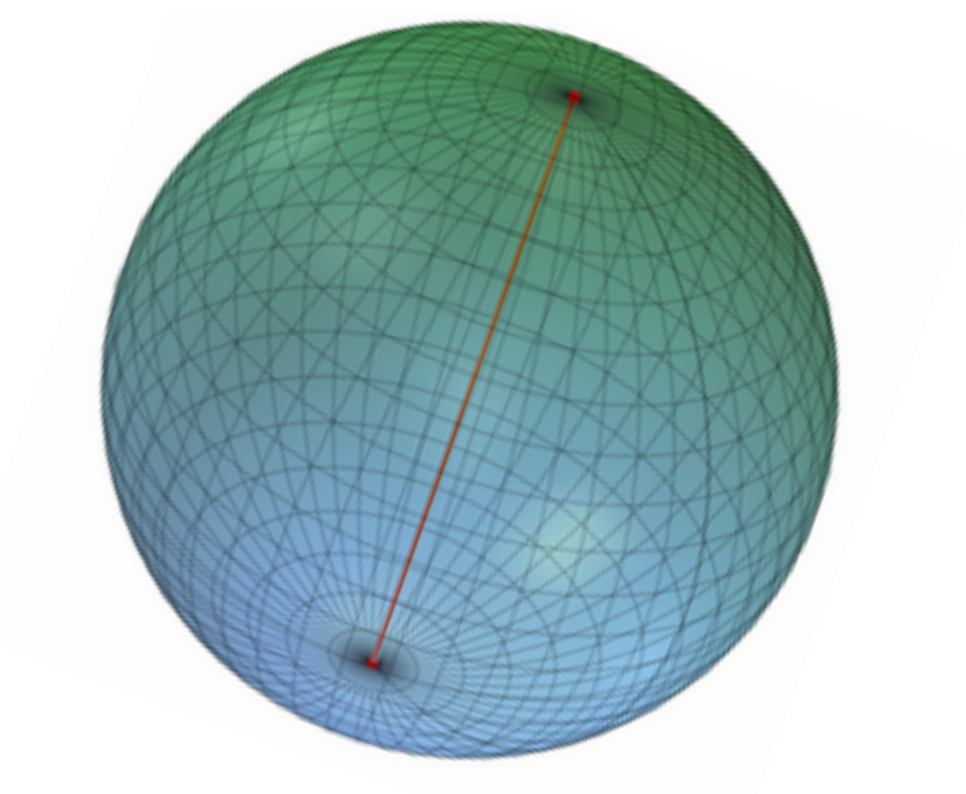
Theorem 5. *For any continuous map $f\colon \mathbb{S}^d \to \mathbb{R}^d$ there is a point $x \in \mathbb{S}^d$ such that*

$$f(x) = f(-x).$$



Brunn-Minkoswki inequality +
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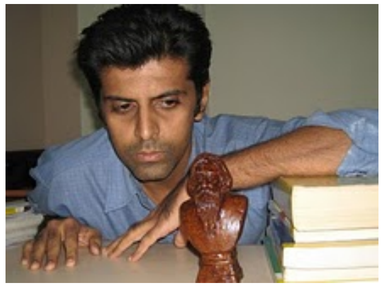
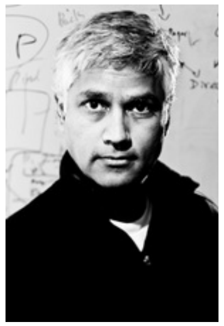
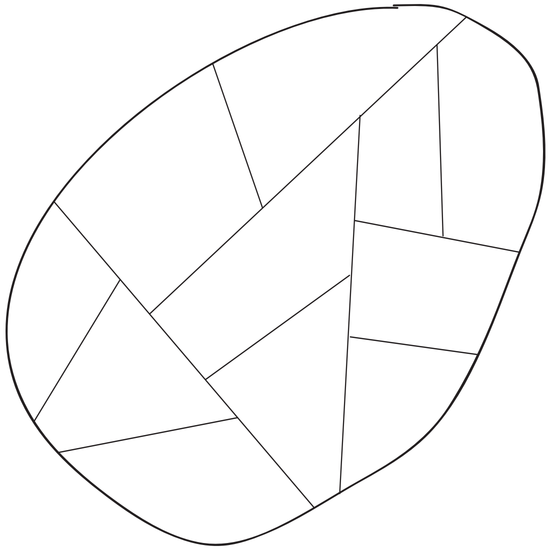
Theorem 4. *For any continuous map* $f\colon \mathbb{S}^d \to \mathbb{R}^k$ *there is a point* $z \in \mathbb{R}^k$ *such that for every* $\varepsilon > 0$, $vol(f^{-1}(z) + \varepsilon) \geq vol(\mathbb{S}^{d-k} + \varepsilon)$

Can you cut a convex polygon
into convex regions with
the same area and the same perimeter?

Conjecture 1 (Nandakumar-Ramana Rao)**.** *For any number n and any planar convex body K. There is a partiton of K into n convex pieces, such that all pieces have the same area and the same perimeter.*

Theorem 6. *Given a prime power* $n = p^m$ *convex body* $K \subset \mathbb{R}^d$*, a measure* μ *and* $d-1$ *continuous functionals* $F_1, F_2, \dots F_k \colon \mathcal{K}(\mathbb{R}^d) \to \mathbb{R}$ *there exists a partition* $K = \cup K_i$*, where* K_i *is convex* $\mu(K_i) = \frac{\mu(K)}{n}$ *and* $F_l(K_i) = F_l(K_j)$*, for all* i, j, l*.*

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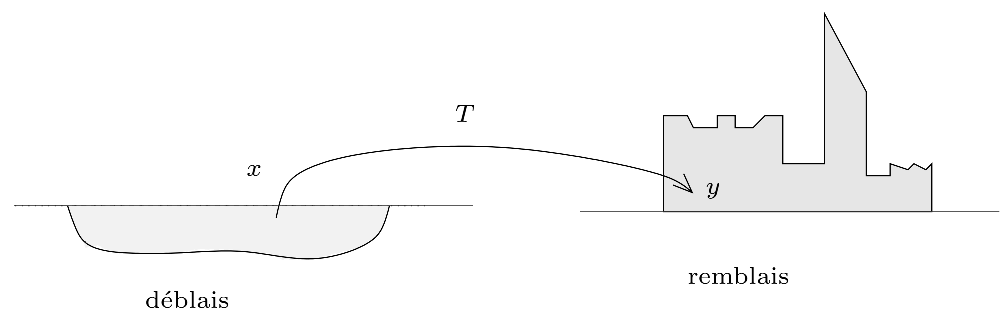
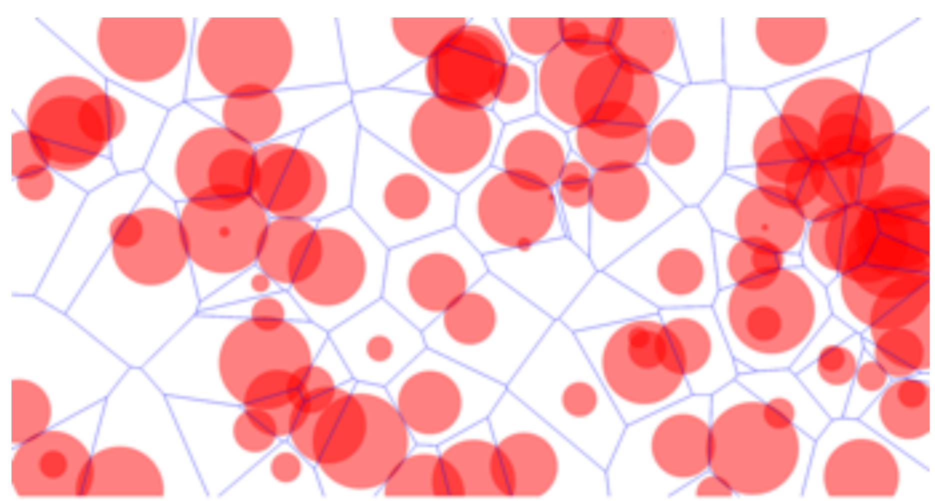
Same theorem is true for Sphere and Hyperbolic space. In the case of sphere can take convex body to be the whole sphere.



$$\inf \left\{ \int_X c(x, T(x))\, \mathrm{d}\mu(x) \,\middle|\, T_*(\mu) = \nu \right\},$$



Theorem 7. *Given $n = 2^m$, a continuous center map $c : \mathcal{K}(\mathbb{S}^d) \to \mathbb{S}^d$, a measure μ and a map $f \colon \mathbb{S}^d \to \mathbb{R}^{d-1}$ there exists a partition $\mathbb{S}^d = \cup K_i$, where K_i is convex $vol(K_i) = \frac{vol(\mathbb{S}^d)}{n}$ and $f(c(K_i)) = f(c(K_j))$, for all i, j, l.*

For m large one can take the sets to be close to something k-dimensional.

$$A + B = \{x + y : x \in A,\, y \in B\}.$$

The Brunn-Minkowski theorem says that if A, B and $A+B$ are measurable, then

$$\mathrm{vol}(A + B)^{1/n} \geq \mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n}. \tag{2–1}$$

Definition 4.1 *A convexely derived measure on $\mathbb{S}^n$ (resp. $\mathbb{R}^n$) is a limit of a vaguely converging sequence of probability measures of the form $\mu_i = \frac{vol|S_i}{vol(S_i)}$, where S_i are open convex sets.*

Lemma 4.1 *Let S be a geodesically convex set of dimension k of the sphere $\mathbb{S}^n$ with $k \leq n$. Let μ be a convexely derived measure defined on S (with respect to the normalized Riemannian measure on the sphere). Then μ is a probability measure having a continuous density f with respect of the canonical Riemannian measure on $\mathbb{S}^k$ restricted to S. Furthermore the function f is $\sin^{n-k}$-concave on every geodesic arc contained in S.*