Reducing the Algebraic Complexity of Geometric Algorithms

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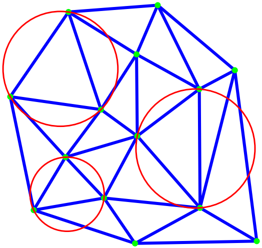
The example of Delaunay triangulations

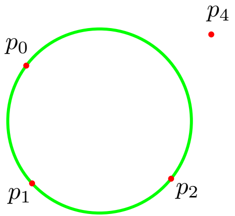
Finite set of points $P \in \mathbb{R}^d$



- $\sigma \in DT(P) \quad \Leftrightarrow \quad \exists c_\sigma : \quad \|c_\sigma - p\| \leq \|c_\sigma - q\| \quad \forall p \in \sigma \text{ and } \forall q \in P$
- It is embedded in $\mathbb{R}^d$ if P is generic wrt spheres [Delaunay 1934]

The curses of dimensionality

- The combinatorial complexity depends exponentially on the ambient dimension d
- The algebraic complexity depends on d



$$\begin{aligned}\texttt{insphere}(p_0,\ldots,p_{d+1}) &= \texttt{orient}(\hat{p}_0,\ldots,\hat{p}_{d+1}) \\ &= \operatorname{sign}\begin{vmatrix} 1 & \ldots & 1 \\ p_0 & \ldots & p_{d+1} \\ p_0^2 & \ldots & p_{d+1}^2 \end{vmatrix}\end{aligned}$$

Our results

- Low algebraic degree. We construct $\mathrm{Del}(P')$ for $P' \approx P$ using only degree 2 predicates (squared distance comparisons)
- Efficiency. The time complexity of the algorithm is $O\left(\frac{|P|}{\bar{\mu}^{d^2}}\right)$ where $\bar{\mu}$ is the sparsity ratio of P
- Simplex quality. We provide a lower bound on the thickness of the output simplices
- No need for coordinates. We simply need to know the interpoint (euclidean) distances

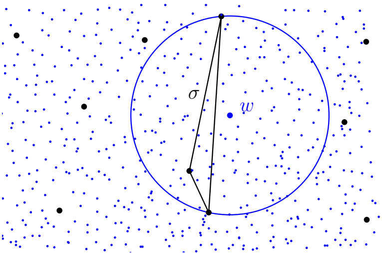
Our strategy

- Compute the witness complex, a weak form of DT that only needs to compare distances
- Identify conditions under which WC = DT
- Randomly perturb P around its initial position to satisfy the conditions above

Witness Complex [de Silva]

L a finite set of points (landmarks) vertices of the complex

W a dense sample (witnesses) pseudo circumcenters



Let σ be a (abstract) simplex with vertices in L, and let $w \in W$. We say that w is a witness of σ if

$$\|w - p\| \leq \|w - q\| \qquad \forall p \in \sigma \text{ and } \forall q \in L \setminus \sigma$$

The witness complex $\mathrm{Wit}(L, W)$ is the complex consisting of all simplexes σ such that for any simplex $\tau \subseteq \sigma$, τ has a witness in W

Construction of witness complexes

Time-complexity : $O\left((|WC| + |W|)\, d^2 \,\log |L|\right)$ [B., Maria]

Algebraic complexity : comparisons of (squared) distances : degree 2

Implementation and experimental results : ask Siargey !

Easy consequences of the definition

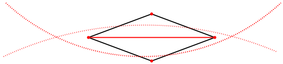
- The witness complex can be defined for any metric space and, in particular, for discrete metric spaces
- If $W' \subseteq W$, then $\mathrm{Wit}(L, W') \subseteq \mathrm{Wit}(L, W)$
- $\mathrm{Del}(L) \subseteq \mathrm{Wit}(L, \mathbb{R}^d)$

First identity theorem [de Silva 2008]

Theorem : $\mathrm{Wit}(L, W) \subseteq \mathrm{Wit}(L, \mathbb{R}^d) = \mathrm{Del}(L)$

Remarks

- Faces of all dimensions have to be witnessed

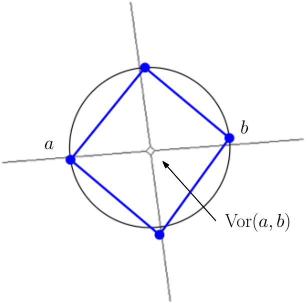


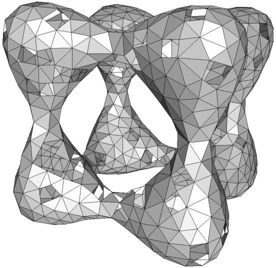
- $\mathrm{Wit}(L, W)$ is embedded in $\mathbb{R}^d$ if L is in general position wrt spheres

Case of sampled domains : $\mathrm{Wit}(L, W) \neq \mathrm{Del}(L)$

W a finite set of points $\subset \mathbb{T}^d$

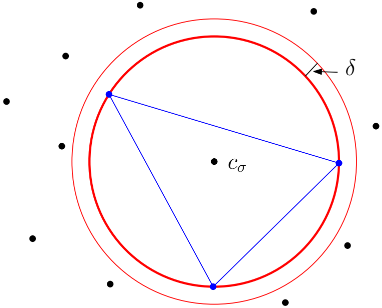
$\mathrm{Wit}(L, W) \neq \mathrm{Del}(L)$, even if W is a dense sample of $\mathbb{T}^d$





$$[ab] \in \mathrm{Wit}(L, W) \quad \Leftrightarrow \quad \exists p \in W, \quad \mathrm{Vor}_2(a, b) \cap W \neq \emptyset$$

Protection

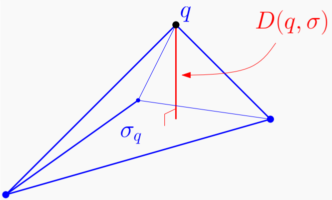


δ-protection We say that a Delaunay simplex $\sigma \subset L$ is δ-protected if

$$\|c_\sigma - q\| > \|c_\sigma - p\| + \delta \qquad \forall p \in \sigma \text{ and } \forall q \in L \setminus \sigma.$$

Simplex quality

Altitudes



If σ_q, the face opposite q in σ is protected, The *altitude* of q in σ is

$$D(q,\sigma) = d(q, \mathrm{aff}(\sigma_q)),$$

where σ_q is the face opposite q.

Definition (Thickness [Cairns, Whitney, Whitehead et al.])

The *thickness* of a j-simplex σ with diameter $\Delta(\sigma)$ is

$$\Theta(\sigma) = \begin{cases} 1 & \text{if } j = 0 \\ \min_{p\in\sigma} \frac{D(p,\sigma)}{j\Delta(\sigma)} & \text{otherwise.} \end{cases}$$

Protection implies thickness

Let L be a $(\lambda, \bar{\mu})$-net, i.e.

- $\forall x \in \mathbb{T}^d, \quad d(x, L) \leq \lambda$
- $\forall p, q \in P, \quad \|p - q\| \geq \bar{\mu}\,\lambda$

if any d-simplex $\sigma \in \mathrm{star}^2(p, \mathrm{Del}(L))$ is δ-protected, then we have for any simplex $\tau \in \mathrm{star}(p, \mathrm{Del}(L))$ (of any dimension)

$$\Theta(\sigma) > \Theta_0 = \frac{\bar{\mu}\,\delta}{4d}$$

Protection implies $\mathrm{Wit}(L, W) = \mathrm{Del}(L)$

Lemma If a d-simplex σ of $\mathrm{Del}(L)$ is δ-protected with $\delta \geq 2\varepsilon$, then $\sigma \in \mathrm{Wit}(L, W)$ (ε = sampling radius of W)

If true for all d-simplices of $\mathrm{Del}(L)$, then $\mathrm{Wit}(L, W) = \mathrm{Del}(L)$.

Proof

1. $\|c_\sigma - p_i\| = \|c_\sigma - p_j\| = r \quad \forall p_i, p_j \in \sigma$
2. $\|c_\sigma - p_l\| > r + \delta \quad \forall p_l \in L \setminus \sigma$
3. $\forall x \in B(c_\sigma, \delta/2)$,

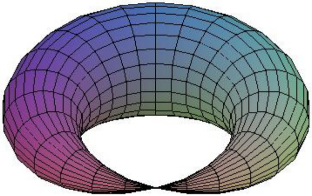
$$\begin{aligned} &\forall p_i \in \sigma, && |x - p_i| \leq |c_\sigma - p_i| + |c_\sigma - x| \leq r + \tfrac{\delta}{2} \\ &\forall p_l \in L \setminus \sigma && |x - p_l| \geq |c_\sigma - p_l| - |x - c_\sigma| > r + \delta - \tfrac{\delta}{2} = r + \tfrac{\delta}{2} \end{aligned}$$

Hence, x is a witness of σ. If $\varepsilon \leq \delta/2$, there must be a point $w \in W$ in $B(c, \delta/2)$ which witnesses σ.

Good links

A simplicial complex K is a k-pseudomanifold complex if

1. K is a pure k-complex
2. every $(k-1)$-simplex is the face of exactly two k-simplices



We say that a complex $K \subset \mathbb{T}^d$ with vertex set L has good links if

$$\forall p \in L, \text{ link } (p, K) \text{ is a } (d-1)\text{-pseudomanifold}$$

Good links implies $\mathrm{Wit}(L, W) = \mathrm{Del}(L)$

Lemma

If K is a triangulation of $\mathbb{T}^d$ and $K' \subseteq K$ a simplicial complex with the same vertex set

then $\quad K' = K \quad \Leftrightarrow \quad K'$ has good links

Corollary

If all vertices of $\mathrm{Wit}(L, W)$ have good links, $\mathrm{Wit}(L, W) = \mathrm{Del}(L)$

Turning witness complexes into Delaunay complexes

Input: L, W, ρ (perturbation radius)

Init : $L' := L$; compute $\mathrm{Wit}(L', W)$

while a vertex p' of $\mathrm{Wit}(L', W)$ has a bad link **do**

  perturb p' and the points of $I(p')$

  update $\mathrm{Wit}(L', W)$

Output: $\mathrm{Wit}(L', W) = \mathrm{Del}(L')$

The Lovász Local Lemma

Motivation

Given: A set of (bad) events $A_1, ..., A_N$,
each happens with $\mathrm{proba}(A_i) \leq \varpi < 1$

Question : what is the probability that none of the events occur?

The case of independent events

$$\mathrm{proba}(\neg A_1 \wedge ... \wedge \neg A_N) \geq (1-\varpi)^N > 0$$

What if we allow a limited amount of dependency among the events?

LLL : symmetric version [Lovász & Erdös 1975]

Under the assumptions

1. $\text{proba}(A_i) \leq \varpi$
2. A_i depends of $\leq \Gamma$ other events A_j
3. $\varpi \leq \frac{1}{e\,(\Gamma+1)}$ $\qquad e = 2.718...$

then

$$\text{proba}(\neg A_1 \wedge ... \wedge \neg A_N) > 0$$

Moser and Tardos' constructive proof of the LLL [2010]

$\mathcal{P}$ a finite set of mutually independent random variables

$\mathcal{A}$ a finite set of events that are determined by the values of $S \subseteq \mathcal{P}$

Two events are independent iff they share no variable

Algorithm

for all $P \in \mathcal{P}$ **do**
  $v_P \leftarrow$ a random evaluation of P;

while $\exists A \in \mathcal{A}$: A occurs **do**

  pick an arbitrary occuring event $A \in \mathcal{A}$;

  for all $P \in \mathrm{variables}(A)$ **do**
    $v_P \leftarrow$ a new random evaluation of P;

return $\{v_P, P \in \mathcal{P}\}$;

Moser and Tardos' theorem

if

1. $\text{proba}(A_i) \leq \varpi$
2. A_i depends of $\leq \Gamma$ other events A_j
3. $\varpi \leq \frac{1}{e\,(\Gamma+1)}$ $\qquad e = 2.718...$

then $\exists$ an assignment of values to the variables $\mathcal{P}$ such that no event in $\mathcal{A}$ happens

The randomized algorithm resamples an event $A \in \mathcal{A}$ at most $\boxed{\frac{1}{\Gamma}}$ expected times before it finds such an evaluation

The expected total number of resampling steps is at most $\boxed{\frac{N}{\Gamma}}$

Protecting Delaunay simplices via perturbation

Notations : L is a $(\lambda, \bar{\mu})$-net, W is a $(\varepsilon, \bar{\eta})$-net

Picking regions : pick p' in $B(p, \rho)$ Hyp. $\rho \leq \frac{\eta}{4} \ (\leq \frac{1}{2})$

Sampling parameters of a perturbed point set

If L is a $(\lambda, \bar{\mu})$-net, L' is a $(\lambda', \bar{\mu}')$-net, where

$$\lambda' = \lambda(1 + \bar{\rho}) \text{ and } \bar{\mu}' = \frac{\bar{\mu} - 2\bar{\rho}}{1 + \bar{\rho}} \geq \frac{\bar{\mu}}{3}$$

The LLL framework

Random variables : L' a set of random points $\{p',\ p' \in B(p,\rho), p \in L\}$

Event: an event happens at p' if Link(p') is not good

$I(p') :=$ the points of L' that

- can be in $\mathrm{star}^2(p')$
- can violate the δ-protection zone $Z_\delta(\sigma')$ of a d-simplex $\sigma' \in \mathrm{star}^2(p')$

Algorithm

Input: L, ρ, δ

while a vertex p' of $\mathrm{Wit}(L'W)$ has a bad link $L(p')$ **do**

perturb p' and the points in $I(p')$

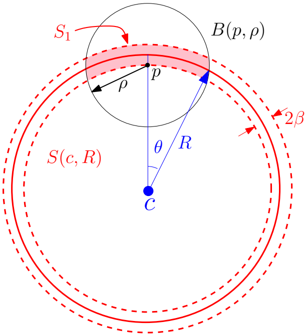
update $\mathrm{Wit}(L'W)$

Analysis

Bounding $|I(p')|$ and Γ : An event is independent of all but at most Γ other bad events where Γ depends on $\bar{\mu}$ and d

($L' = (\lambda', \bar{\mu}')$-net + a packing argument)

Bounding proba(link (p') is bad)



$$\text{proba}(\text{link}(\text{p}') \text{ is bad }) \leq \text{proba}(p \in Z_\delta(\sigma))$$
$$\sigma \in \text{star}^2(p')$$

$$\text{proba}(p \in Z(\sigma)) = \frac{\text{vol}_d(Z_\delta \cap B_\rho)}{\text{vol}_d(B_\rho)}$$
$$= O(\frac{\delta}{\rho})$$

Main result

Under the condition

$$\frac{\mu}{4} \geq \rho \geq \frac{24d\varepsilon}{\bar{\mu}J} \quad \text{where} \;\; J^{-1} = \left(\frac{2}{\bar{\mu}}\right)^{O(d^2)}$$

the algorithm terminates.

The expected complexity is linear in $|L|$

Sublinear algorithm

- The time to construct $\mathrm{Wit}(L, W)$ depends linearly on $|W|$
- Witnesses are (in general) redundant
- Challenge : Choose witnesses close to the CC of the simplices (without computing CCs)

Relaxed Delaunay complex

α-center for σ
$\|x-p\| \leq \|x-q\| + \alpha \ \forall p,q \in \sigma$

α-Delaunay center
$\|x-p\| \leq \|x-q\| + \alpha \ \forall p \in \sigma$ and $\forall q \in L$

Relaxed Delaunay complex $\mathrm{Del}^{\alpha}(L', W)$
The set of simplices that have an α-Delaunay centre in W

Full cells

Closeness to bisectors Let σ be a d-simplex and H_{pq} be the bisecting hyperplane of p and q. A point x that satisfies $d(x, H_{pq}) \leq \alpha$, for any $p, q \in \sigma$ is a 2α-center of σ.

Clustered α-Delaunay centers
If L is a $(\lambda, \bar{\mu})$-net and x is an α-Delaunay center for σ, then

$$\|c_\sigma - x\| < \frac{2\alpha}{\Theta_\sigma \bar{\mu}}$$

Full cells in a grid ε : cells that are intersected by all bisectors of σ

The number of full cells is $O(\frac{1}{(\Theta_\sigma \bar{\mu}')^d} \log \frac{\lambda}{\varepsilon})$

$\mathrm{Del}_0^{2\varepsilon}(L', W) = \{\sigma \in \mathrm{Del}^{2\varepsilon}(L', W) \quad \text{s.t.} \quad \Theta_\sigma \geq \Theta_0\}$

- If
  - the d-simplices in $\mathrm{Del}(L')$ are δ-protected
  - $\Theta_0 = \frac{\bar{\delta}\bar{\mu}'}{8d}$

  then $\mathrm{Del}(L') \subseteq \mathrm{Del}_0^{2\varepsilon}(L', W)$

- if, in addition, every d-simplex of $\mathrm{Del}_0^{2\varepsilon}(L', W)$ is protected

  then $\mathrm{Del}_0^{2\varepsilon}(L', W) = \mathrm{Del}(L')$

Turning relaxed-Delaunay to Delaunay complexes

Protected Delaunay triangulation from $\mathrm{Del}_0^{2\varepsilon}(L', W)$

input: L, W, ρ, ε, λ, μ
$L' \leftarrow L$
compute: $\mathrm{Del}_0^{2\varepsilon}(L', W)$
while a vertex p' of $\mathrm{Del}_0^{2\varepsilon}(L', W)$ has a bad link or $\texttt{check}(p') = \mathrm{FALSE}$ **do**
  perturb p' and the points in $I(p')$
  update $\mathrm{Del}_0^{2\varepsilon}(L', W)$
output: $\mathrm{Del}_0^{2\varepsilon}(L', W) = \mathrm{Del}(L')$ with δ^* protection (See Lemma **??**)

procedure $\texttt{check}(p')$

if all d-simplices $\sigma \in \mathrm{star}(p'; \mathrm{Del}_0^{2\varepsilon}(L', W))$ satisfy

1. The diameter of the full leaves is at most $\frac{16\sqrt{d}\varepsilon}{\Theta_0 \bar{\mu}'}$.
2. There is a $(\delta - 2\varepsilon)$-protected full-leaf-point **then**
   check$(p') = \mathrm{TRUE}$

Extensions

- Weighted points and power distance
- Delaunay triangulation of non-flat manifolds
- Other geometric constructions