Kinematic analysis of a steering mechanism: a follow-up with interval analysis

J-P. Merlet

INRIA Sophia-Antipolis

BP 93, 06902, Sophia-Antipolis, France

Jean-Pierre.Merlet@sophia.inria.fr

Abstract: In a recent article Pramanik has presented the design analysis of a steering mechanism where 3 precision points were used to determine the 3 design parameters of the mechanism by solving a system of non-linear equations. Pramanik was able to find one design solution using the Newton scheme. Using interval analysis we will show that there may be more than one solution and we exhibit a case with 2 design solutions.

Assuming uncertainties in the design parameters we are able to compare the different design solutions in term of the maximal deviation at the precision points.

Finally we show that we are able to extend the number of design parameters until up to 5 precision points may be defined.

1 Introduction

The steering mechanism proposed by Pramanik [3] is presented in figure 1. The purpose of this mechanism is to enable to orient differently the inner and outer

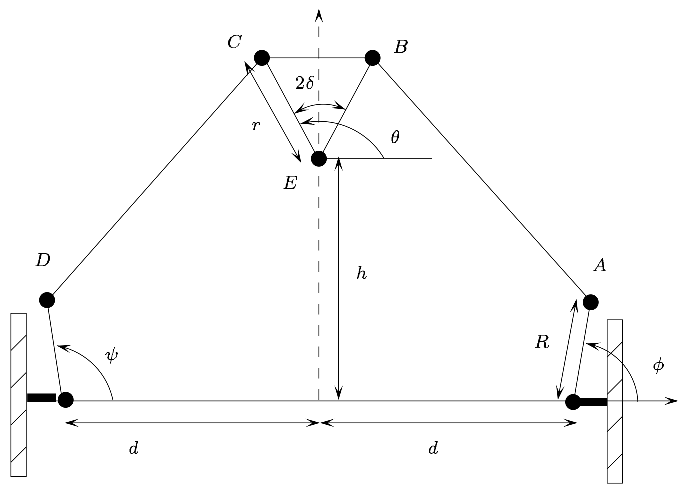

Figure 1: The steering mechanism proposed by Pramanik

wheel of a car while it is turning, the wheels being parallel when running in straight line. It is constituted of 5 rigid bodies O_1D, DC, CAE, AB, O_2A connected by revolute joints. A gear system (not represented on the drawing) transmit the rotation of the links O_1D, O_2A to the wheel. When the wheels are parallel the system is designed so that the angle ψ, ϕ between O_1C, O_2A and the x axis have a value:

$$\psi_0 = 79.951505^\circ \qquad \phi_0 = 100.048495^\circ$$

Ideally these two angles should be such that:

$$cot(\psi) - cot(\phi) = w \tag{1}$$

where w is the ration of track width to wheelbase (supposed to be 0.6 in this paper).

The kinematic equation of the system may be obtained by writing that the lengths of links AB, CD remain constant for any configuration of the mechanism. This leads to 2 linear equations in term of the sine and cosine of the rotation angle θ of the triangular body:

$$
\begin{aligned}
&((1+k_2\ \cos(\phi))k_1\ \cos(2\,\delta) - (k_2\ \sin(\phi) - k_3)k_1\ \sin(2\,\delta))\cos(\theta) + \\
&((1+k_2\ \cos(\phi))k_1\ \sin(2\,\delta) + (k_2\ \sin(\phi) - k_3)k_1\ \cos(2\,\delta))\sin(\theta) - \\
&k_2\,(\cos(\phi) - \cos(\phi_0)) + k_2\,k_3\,(\sin(\phi) - \sin(\phi_0)) - \\
&(1+k_2\ \cos(\phi_0))k_1\ \sin(\delta) - (k_2\ \sin(\phi_0) - k_3)k_1\ \cos(\delta) = 0 \qquad (2) \\
&(k_2\ \cos(\psi) - 1)k_1\ \cos(\theta) + (k_2\ \sin(\psi) - k_3)k_1\ \sin(\theta) + k_2\ \cos(\psi) + \\
&k_2\,k_3\ \sin(\psi) - k_1\ \sin(\delta) - (1 - k_1\ \sin(\delta))k_2\ \cos(\psi_0) + k_1\,k_3\ \cos(\delta) - \\
&(k_3 + k_1\ \cos(\delta))k_2\ \sin(\psi_0) = 0
\end{aligned}
$$

Solving this system and expanding the relation $\sin^2(\theta) + \cos^2(\theta) = 1$ enable to obtain the kinematic relationship between ψ and ϕ as a function of the design parameters r, R, d, h, δ but Pramanik assumes that δ has the fixed value π. Under this assumption we may use normalized design parameters $k_1 = r/d, k_2 = R/d, k3 = h/d$. Introducing E, F as:

$$
\begin{aligned}
E &= k_2\ (\cos(\phi) - \cos(\phi_0)) - k_2\,k_3\ (\sin(\phi) - \sin(\phi_0)) + \\
&\quad (1 + k_2\ \cos(\phi_0))\,k_1\ \sin(\delta) + (k_2\ \sin(\phi_0) - k_3)\,k_1\ \cos(\delta) \\
F &= -k_2\ \cos(\psi) - k_3\,k_2\ \sin(\psi) + k_1\ \sin(\delta) + (1 - k_1\ \sin(\delta))\,k_2\ \cos(\psi_0) - \\
&\quad k_1\,k_3\ \cos(\delta) + (k_3 + k_1\ \cos(\delta))\,k_2\ \sin(\psi_0)
\end{aligned}
$$

The input-output relation between the angle ϕ and ψ is:

$$
\begin{aligned}
&(E(k_2\ \sin(\psi) - k_3) - F((1 + k_2\ \cos(\phi)) \sin(2\,\delta) + \\
&(k_2\ \sin(\phi) - k_3) \cos(2\,\delta)))^2 + (F((1 + k_2\ \cos(\phi)) \cos(2\,\delta) - \\
&(k_2\ \sin(\psi) - k_3) \sin(2\,\delta)) - E(k_2\ \cos(\psi) - 1))^2 - (((1 + k_2\ \cos(\phi)) \cos(2\,\delta) \\
&-(k_2\ \sin(\phi) - k_3) \sin(2\,\delta))(k_2\ \sin(\psi) - k_3)k_1 - ((1 + k_2\ \cos(\phi)) \sin(2\,\delta) + \\
&(k_2\ \sin(\phi) - k_3) \cos(2\,\delta))(k_2\ \cos(\psi) - 1)k_1)^2 = 0
\end{aligned} \tag{3}
$$

Note that this equation is algebraic of degree 2 in k_1, 4 in k_2, k_3 and is of total degree 6 in the design parameters. To determine the values of the k_i's Pramanik proposes to define 3 precision points i.e. a triplet of (ψ, ϕ) that define 3 configurations of the mechanism. For each of this precision equation (3) can be used to define a constraint relation between the design parameters. Hence we obtain a set of 3 equations in the variable k_1, k_2, k_3. Pramanik obtains one solution of this system using the classical Newton scheme. This system will usually admit a finite number of solution and an upper bound on the number of possible solutions is obtained with Bezout's theorem as 216 $(6 \times 6 \times 6)$. But in practice all the solutions of this system will not leads to an acceptable design solution. Indeed very small or very large values for the k_i's cannot be accepted as this will lead to a mechanism in which the link will have very dissimilar length. Hence it is reasonable to assume that these ratio have to lie within some acceptable range. Our purpose is to show that even under this assumption there is more than one solution to the system. To solve this system we will use a numerical method, *interval analysis*, that is appropriate for determining all the solutions of a system of equations within some ranges for the unknowns.

2 Solving with interval analysis

A basic tool of interval analysis [1, 2], is interval arithmetics that allows to determine very simply lower and upper bounds for the value of a function when the variables lie within some given ranges. The simplest way to determine these bounds is to use the *natural evaluation* of the function which consists in replacing all the mathematical operators of the function by their interval equivalent. For example the sum of two intervals $X1_1 = [\underline{x_1}, \overline{x_1}]$, $X2_2 = [\underline{x_2}, \overline{x_2}]$ is defined as:

$$X_1 + X_2 = [\underline{x_1} + \underline{x_2}, \overline{x_1} + \overline{x_2}]$$

which means that for any value of x_1, x_2 in the range X_1, X_2 the sum $x_1 + x_2$ will always lie in $X_1 + X_2$. Such interval equivalent can be defined for all the classical mathematical operators. These equivalent can be implemented to take into account round-off errors: hence the interval evaluation of a function is guaranteed to include the value of the function for any instance of the variables in their ranges. Consequently a nice property of interval arithmetics is that if an interval evaluation of a function $f(\mathbf{X})$ does not include 0 for some ranges for the variables, then there is no solution of $F(\mathbf{X}) = 0$ within the ranges. The main drawback is that for a given function f the interval evaluation $f(\mathbf{X})$ will provide a range $[a, b]$ that is usually overestimated (i.e. there is no value of x

in X such that $f(x) = a$ or $f(x) = b$ or equivalently for all x in X we have $a < f(x) < b$). This overestimation is due to the fact that each occurrence of a variable is considered as a new example. For example the interval evaluation of $x - x$ when x lie in the range [-1,1] is not 0 but [-2,2]. The amplitude of this overestimation decrease with the width of the intervals but may be large for complex expressions even for small intervals. However some methods allows to decrease the amplitude of the overestimation.

For a problem with multiple unknowns a *box* denotes a set of intervals, one for each of the unknowns.

Methods known as *filters* can be used to reduce the search space for the solutions. For example consider the equation $x^2 + x - 6 = 0$ for which we are looking for a solution in the range [1,3]. This equation may be written as $x^2 = 6 - x$ and we may compute the interval evaluation U_l, U_r of the left and right term of this equation. Clearly if a solution exists for the equation in this range, then the solution must lie in he intersection of U_l and U_r. Here we have $U_l = [1, 9]$, $U_r = [3, 5]$: hence x^2 for a solution must lie in [3,5] and consequently x must lie in $[\sqrt{3}, \sqrt{5}]$. A simple operation has enable to decrease the size of the search space from 2 to 0.5. Note that such a filter may determine that there is no solution within the given ranges.

Another methods are the *existence operators* that allow to show that there is a unique solution within a given box, solution that can be computed exactly with an iterative scheme. For example Kantorovitch theorem [4] may be used for each box to determine a ball in which there is a unique solution which is guaranteed to be found by the Newton scheme with as initial guess the center of the ball.

The filter and existence operators may not give any result on a box: in that case the box is *bisected*. We select an interval $X_j = [\underline{x_j}, \overline{x_j}]$ of the box and we create 2 new boxes by duplicating the initial box except for the variable j: for this variable one of the box will have the range $[\underline{x_j}, (\underline{x_j} + \overline{x_j})/2.]$ while the other box will have as j-th range $[(\underline{x_j} + \overline{x_j})/2., \overline{x_j}]$.

A solver based on interval analysis uses a list $\mathcal{L}$ of boxes which is initialized with the box which defines the search space. The i-th box of the list will be denoted $\mathcal{B}_i$ and n will be total number of boxes in the list. At some step of the algorithm a specific box $\mathcal{B}_k$ will be processed. The solver proceed along the following steps:

1. if $k > n$, then return the solution set
2. apply the filter operators on $\mathcal{B}_k$. If the filters show that there is no solution in $\mathcal{B}_k$, then $k = k + 1$, go to step 1
3. apply the existence operators on $\mathcal{B}_k$: if a solution is found add it to the solution set
4. calculation of the interval evaluation of all the equations of the system. If one of these interval evaluation does not include 0, then $k = k + 1$, go to step 1

5. if the width of all the ranges in the box is lower than a given threshold, then store the box in the solution set

6. bisect $\mathcal{B}_k$ and store the 2 new boxes in $\mathcal{L}$, $k = k + 1$, go to step 1

It may be seen that two different types of solution is found with this algorithm:

- solution found at step 3: those are the guaranteed solutions.
- solution found at step 5: here a small box is returned as a solution but we cannot guarantee that there is a solution in the box. Such case may occur either is the system is ill-conditioned or is singular i.e. does not admit a finite number of solution

Such solver is implemented in our `ALIAS` library, a C++ library using the package `BIAS/Profil` for interval arithmetics. One specificity of this library is that it is interfaced with the symbolic computation package Maple: being given a set of equation defined in Maple the user may call a specific Maple procedure that will automatically create the necessary C++ code for solving the system, compile and execute it and then return the result to Maple.

3 Finding the design parameters

One of the solver proposed in the `ALIAS` library has been used to solve the system of equation (3) obtained for the following triplet of precision point (ψ, ϕ)

in radian:

$$\phi_1 = 2.036469113 \quad \psi_1 = 1.744482854$$

$$\phi_2 = 2.239097787 \quad \psi_2 = 2.065623437$$

$$\phi_3 = 2.304774965 \quad \psi_3 = 2.180815168$$

The ranges for the parameters k_i's has been defined as [0.06,2] as the k_i's cannot be 0. The following solutions has been found:

$$k_1 = 0.2895756 \quad k_2 = 0.3285485 \quad k_3 = 0.27163277$$

$$k_1 = 0.339353 \quad k_2 = 0.34453138 \quad k_3 = 0.3215609$$

An extension of the ranges to [0.06,1000] does not provide any additional solution.

4 Verifying the solutions

For given k_i's equation (3) defines a variety $\mathcal{S}$ in the ϕ, ψ space that may have different non-connected components: in other words the mechanism may follow different branches. If there is no singularity, then these branches do not intersect and the mechanism will follow the branch to which belong the initial assembly point. A problem with the precision points approach is that the precision points may lie on different branches and hence the mechanism may go through only some of them. It is therefore necessary to verify that all the precision points lie on the same branch. In our case for a given ψ angle there are two possible locations for the triangle and for each of this location there are 2 possible values of the angle ϕ and consequently the system has 4 branches.

For the purpose of following the branches we use a specific module of the ALIAS library that allows to draw the branches of a 1-dimensional variety. Here we will consider the system (2) in the unknown ψ, θ and we will draw the branches as functions of the parameter ϕ, supposed to lie in the range $[\phi_0, \phi_3]$. Our algorithm starts by solving the system (2) in the unknown θ, ψ for $\phi = \phi_0$ to get the initial starting point of the branches. Then the algorithm will calculate a change $\Delta\phi$ that allows to determine for $\phi = \phi_0 + \Delta\phi$ a new point on each branch with a certified Newton scheme. This process will be repeated for the new value of ϕ until the value ϕ_3 is reached.

The 4 branches for the 2 design solutions are presented in figure 2 and it may be seen that for the two design solutions all the precision points lie on the same branch: hence we have got correct design solutions.

5 Error analysis

It may be interesting to compare the design solutions to determine if one of them is the most appropriate as a steering mechanism. A first approach will be to compare the branch on which lie the precision points with the desired path defined by equation (1). The error between the two path for a given $\psi = \psi_d$

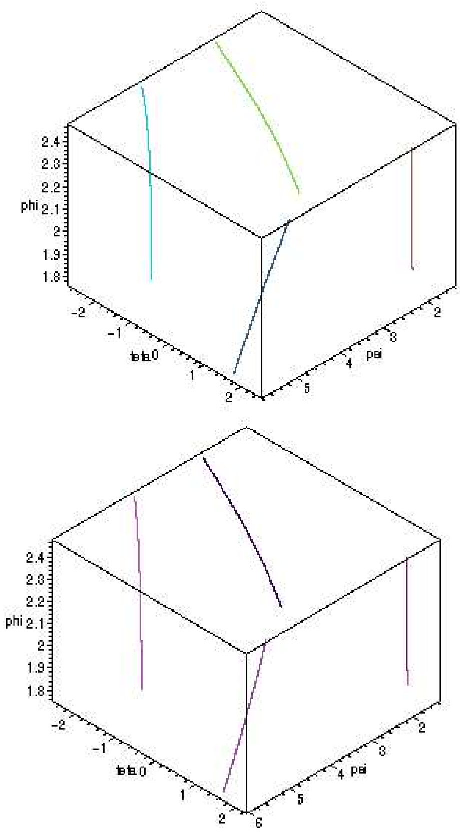

Figure 2: The four branches of the two solution mechanisms

may be defined as the absolute value of the difference between the values of ϕ found on the branch and the desired value computed as $arcot(cot(\psi_d) - 0.6)$ As we have computed the branches for each solution in the previous section we may calculate the mean and maximal values of this error. The results are presented in table 1. It appears here that solution 2 presents the best solution in term of

Solution	Mean value	Maximal value
1	$0.144379\ e^{-2}$	$0.47267\ e^{-2}$
2	$0.66249e^{-3}$	$0.16221e^{-2}$

Table 1: Mean and maximal value of the error between the desired steering path and the mechanism path (in radian)

deviation with respect to the reference path.

Another approach for choosing the design solution is to consider that the calculated solutions are only theoretical as in practice manufacturing errors will introduce uncertainties in the design parameters i.e. the k_i will in fact lie in range $[\underline{k_i}, \overline{k_i}]$. A possible way to choose the best solution among the 2 design solutions is to analyze what will be the minimum and maximum values of the angle ψ for the angle ϕ at the precision points, being given the ranges for the k_i's. Let F denote the right term of equation (3): for a given ϕ these extremum will be obtained either when the k_i's values are at their lower or upper bound or when the following system of equations in the 4 unknowns k_1, k_2, k_3, ψ is satisfied:

$$\frac{\partial F}{\partial k_1} = 0 \quad \frac{\partial F}{\partial k_2} = 0 \quad \frac{\partial F}{\partial k_2} = 0 \quad F = 0 \tag{4}$$

An initial range $[\psi_m, \psi^M]$ for the minimum and maximum values of ψ will be obtained by fixing the values of the k_i's either to $\underline{k_i}$ or $\overline{k_i}$ and solving numerically equation (3) in ψ (hence 8 such equations have to be solved). Then we solve the system (4) using a special filter method that allows to eliminate the boxes for which the range for ψ is included in $[\psi_m, \psi^M]$ and updating the boxes that have an intersection with this range. Each solution of this system is used to update eventually the value of ψ_m, ψ^M.

Let k_i^{nom} be the nominal value for the k_i's and assume that the effective k_i will lie in the range $[k_i^{nom} - 1/100, k_i^{nom} + 1/100]$ we get for each solution and for each precision point the error bounds presented in figure 3. It can be seen that solutions 1 and 2 exhibit similar errors for the precision points 1 and 2 while solution 2 has a lower error for the precision point 3. Hence solution 2 is definitely the best design solution.

6 Conclusion

Interval analysis is clearly an appropriate method to solve the problem presented in Pramanik's paper, especially as the unknowns may easily be bounded. But we have shown that it allows also to analyze the different design solutions in realistic terms.

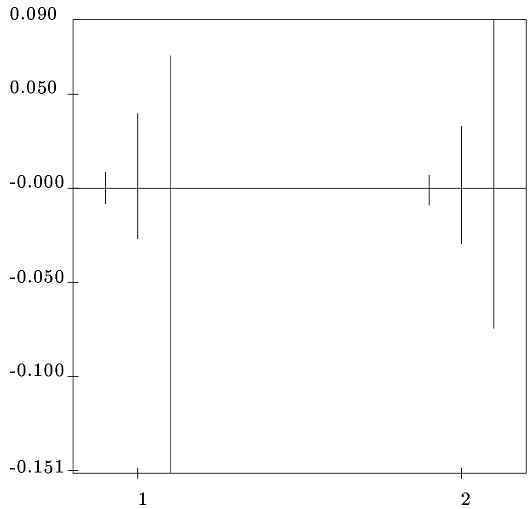

Figure 3: Extremal errors on ψ for the 3 precision points and for the 2 solutions

We are currently investigating a much more complex problem: in Pramanik analysis it has been assumed that the angle δ, ψ_0 were known but may also assume that there are design parameters. This allows to specify two additional precision point to get a system of five equations in the unknowns $k_1, k_2, k_3, \delta, \psi_0$. This system is much more complex than the system obtained by fixing δ, ψ_0. Preliminary test have shown that solutions may be obtained but that the computation time for getting all the solutions will be very large. This may be explained as the equations that have to be solved are highly complex with numerous occurrence of the variables: this leads to large overestimation of the bounds for the equations. As for any method based on interval analysis a compromise has to be made between a set of complex equations and a small number of unknowns and simpler equations with more unknowns. We are currently investigating various models for the equations for finding the best compromise that will allow to determine all the possible design solutions.

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