Inverting Ramsey: from class properties to graph parameters

GRASTA 2023
11th Workshop on GRAph Searching, Theory and Applications
Bertinoro, October 22–27, 2023

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October 24, 2023

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A **graph parameter** or **graph invariant** is a mapping from the class of all graphs to some set (typically to $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$) that does not distinguish between isomorphic graphs.

A graph parameter ρ is **friendly** if it is defined as the **maximum** or **minimum** cardinality of a set of vertices of the graph satisfying a certain property π_ρ associated with the parameter.

Many graph parameters are friendly.

A trivial example:

- order, $n(G)$ $\pi_G(S)$: $S = V(G)$

Maximization parameters:

- clique number, $\omega(G)$ $\pi_G(S)$: S is a clique in G
- independence number, $\alpha(G)$ $\pi_G(S)$: S is an independent set in G

Minimization parameters:

- vertex cover number, $\tau(G)$ $\pi_G(S)$: S is a vertex cover in G
- dominating set number, $\gamma(G)$ $\pi_G(S)$: S is a dominating set in G

In some cases, the description of the property π_G may be less obvious.

chromatic number: $\chi(G) = \min\{|S| \colon \pi_G(S)\}$ where

$\pi_G(S)$: there exists a proper vertex coloring of G such that S is a set of maximum cardinality among all sets of vertices that contain at most one vertex from each color class

degeneracy: $\mathrm{dgn}(G) = \min\{|S| \colon \pi_G(S)\}$ where

$\pi_G(S)$: there exists a linear ordering $(v_1, \ldots, v_n)$ of $V(G)$ such that S is a set of maximum cardinality among all sets of the form $N_G(v_j) \cap \{v_i \colon i > j\}$, for $j \in \{1, \ldots, n\}$

We also allow that $\rho(G)$ is a **strictly increasing integer-valued function** of either the maximum or the minimum cardinality of a set S of vertices of G satisfying property π_G.

This way the definition also captures parameters such as treewidth and pathwidth.

treewidth: $\mathrm{tw}(G) = \min\{|S| : \pi_G(S)\} - 1$ where

$\pi_G(S)$: S is a maximum cardinality bag of some tree decomposition of G

pathwidth: $\mathrm{pw}(G) = \min\{|S| : \pi_G(S)\} - 1$ where

$\pi_G(S)$: S is a maximum cardinality bag of some path decomposition of G

treedepth: $\mathrm{td}(G) = \min\{|S| : \pi_G(S)\}$ where

$\pi_G(S)$: there exists a rooted forest F such that G is a subgraph of the transitive closure of F and S is a set of maximum cardinality among all the vertex sets of root-to-leaf paths in F

Replacing in the above definition all cardinalities $|S|$ of sets of vertices with $\alpha(G[S])$, the independence number of the corresponding induced subgraph, we obtain the **independence variant of ρ**, denoted by α-ρ.

This framework captures, for example:

- the independence variant of **degeneracy**, called inductive independence number [Ye and Borodin, '12]
- the independence variant of **treewidth**, called tree-independence number and introduced independently by [Yolov, '18] and [Dallard, M., Štorgel, '21+]

$\mathcal{G}$: a hereditary graph class (closed under vertex deletion)

ρ: a graph parameter

We say that $\mathcal{G}$ has **ω-bounded ρ** or **clique-bounded ρ** if there exists a function f such that $\rho(G) \leq f(\omega(G))$ for all graphs $G \in \mathcal{G}$.

That is, the only reason for large ρ is the presence of a large clique.

Examples:

- ρ is the chromatic number: χ-bounded graph classes
  [Gyarfás, '87; many authors in the last decade]
- ρ is treewidth: (tw, ω)-bounded graph classes
  [Chaplick, Töpfer, Voborník, Zeman '21; Dallard, M., Štorgel, '21]

Ramsey's theorem: for any two nonnegative integers a and b, there exists a least nonnegative integer $R(a, b)$ such that any graph with at least $R(a, b)$ vertices has either a clique of size a or an independent set of size b.

Ramsey's theorem implies:

Proposition

Let ρ be a friendly graph parameter and let $\mathcal{G}$ be a hereditary graph class with bounded α-ρ. Then, $\mathcal{G}$ has clique-bounded ρ.

$$(\exists k)(\forall G \in \mathcal{G})(\alpha\text{-}\rho(G) \leq k) \quad \Longrightarrow \quad (\exists f)(\forall G \in \mathcal{G})(\rho(G) \leq f(\omega(G)))$$

For example:

- $\rho = \mathrm{tw}$: every graph class with bounded tree-independence number is (tw, ω)-bounded
- it was conjectured that the converse holds [Dallard, M., Štorgel, '22]

$$\text{bounded } \alpha\text{-}\rho \quad \Longrightarrow \quad \omega\text{-bounded } \rho$$

When does this implication reverse?

For which graph parameters ρ is boundedness of α-ρ not only a sufficient but also a necessary condition for ω-boundedness of ρ?

We call such graph parameters **$\alpha\omega$esome** (pronounced "awesome").

If ρ is an $\alpha\omega$esome graph parameter, then

$$\text{bounded } \alpha\text{-}\rho \quad \Longleftarrow \quad \omega\text{-bounded } \rho$$

$$\text{bounded } \alpha\text{-}\rho \quad \overset{Ramsey}{\Longrightarrow} \quad \omega\text{-bounded } \rho$$

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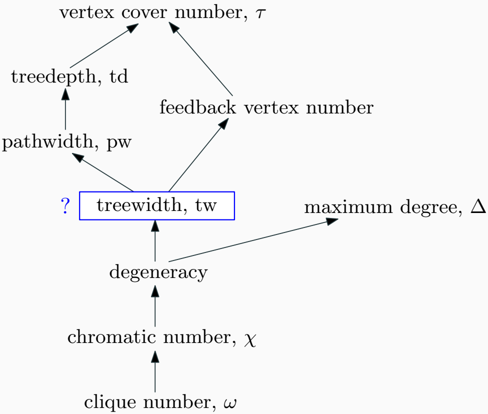
This explains the title,
"Inverting Ramsey: from class properties to graph parameters":

- ω-bounded ρ: a class property involving **two graph parameters**
- bounded α-ρ: a simpler class property involving **a single graph parameter**

Since the treewidth is conjectured to be $\alpha\omega$esome, it is natural to consider a number of parameters related to treewidth.

Which of the following graph parameters are $\alpha\omega$esome?

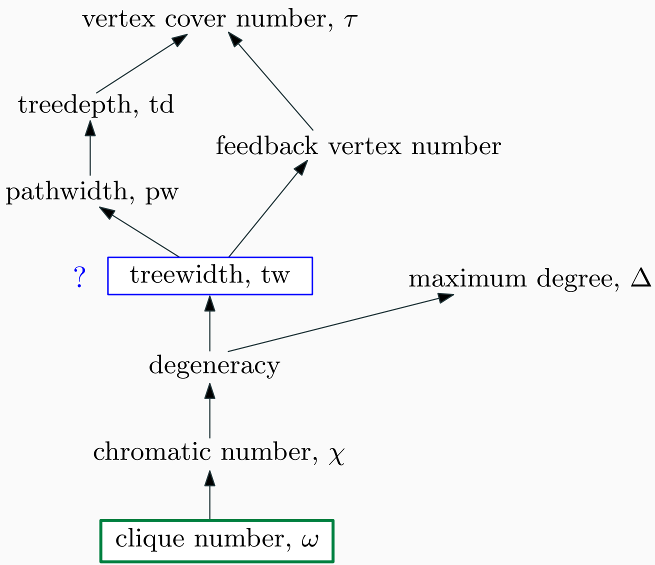


Observation

Clique number is an $\alpha\omega$esome parameter.

The independence number of the subgraph induced by any feasible solution is bounded (by 1).

More generally, any friendly graph parameter for which all feasible solutions induce subgraphs with bounded independence number is $\alpha\omega$esome.


vertex cover number, τ
treedepth, td
feedback vertex number
pathwidth, pw
?
treewidth, tw
maximum degree, Δ
degeneracy
chromatic number, χ
clique number, ω

Observation

Maximum degree is an $\alpha\omega$esome parameter.

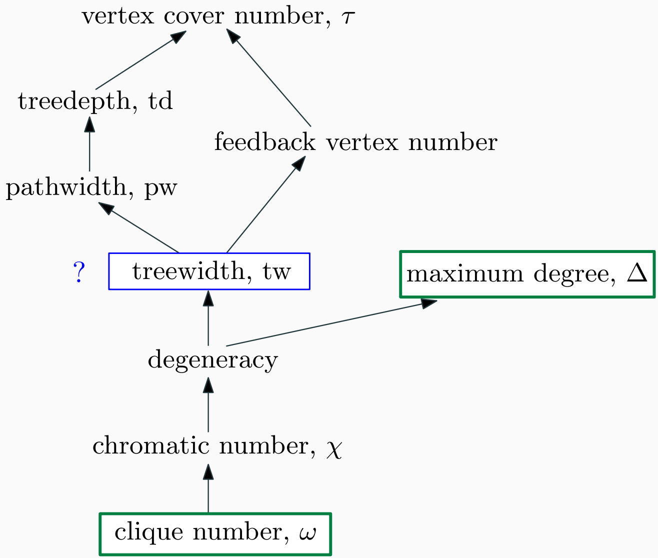
Let $\mathcal{G}$ be a hereditary graph class with clique-bounded Δ, that is, $\Delta(G) \leq f(\omega(G))$ for all $G \in \mathcal{G}$.

α-Δ, the independence variant of the maximum degree $=$ maximum size of an independent set contained in the neighborhood of some vertex

Fix a graph $G \in \mathcal{G}$ and a vertex $v \in V(G)$, and let S be an independent set such that $S \subseteq N(v)$.

Let G' be the subgraph of G induced by $S \cup \{v\}$. Since $G' \in \mathcal{G}$ and $\omega(G') \leq 2$, we have $|S| \leq \Delta(G') \leq f(2)$.

Thus, $\mathcal{G}$ has bounded α-Δ.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Theorem

Vertex cover number and feedback vertex number are $\alpha\omega$esome parameters.

- vertex cover number of G = minimum cardinality of a set $S \subseteq V(G)$ such that $G - S$ is edgeless
- feedback vertex number of G = minimum cardinality of a set $S \subseteq V(G)$ such that $G - S$ is acyclic

In other words:

- vertex cover number of G = minimum cardinality of a set $S \subseteq V(G)$ such that $\mathrm{tw}(G - S) \leq 0$
- feedback vertex number of G = minimum cardinality of a set $S \subseteq V(G)$ such that $\mathrm{tw}(G - S) \leq 1$

Theorem

For any $k \geq 0$, treewidth-k modulator number is an $\alpha\omega$esome parameter.

- treewidth-k modulator number of G = minimum cardinality of a set $S \subseteq V(G)$ such that $\mathrm{tw}(G - S) \leq k$

More generally:

Theorem

If large clique number implies large ρ, then for any $k \geq 0$, the ρ-k modulator number is an $\alpha\omega$esome parameter.

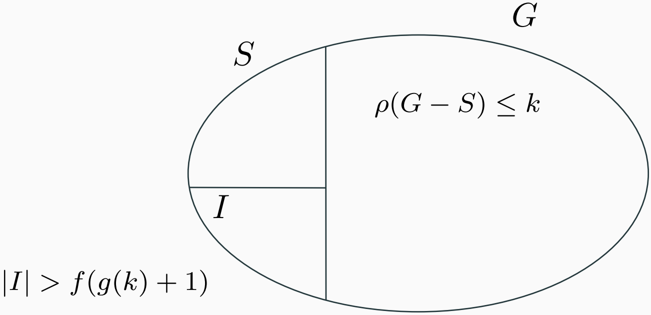
- $\tau_k^\rho(G)$ = ρ-k modulator number of G = minimum cardinality of a set $S \subseteq V(G)$ such that $\rho(G - S) \leq k$

Let $\mathcal{G}$ be a hereditary graph class with clique-bounded τ_k^ρ, that is, $\tau_k^\rho(G) \leq f(\omega(G))$ for all $G \in \mathcal{G}$.

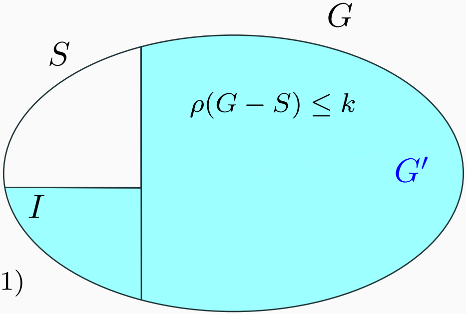
Then, for every graph $G \in \mathcal{G}$, every minimum ρ-k modulator S satisfies $\alpha(G[S]) \leq f(g(k)+1)$, where $\omega(H) \leq g(\rho(H))$ for all graphs H.

Proof: Suppose $\alpha(G[S]) > f(g(k)+1)$.

Let $I \subseteq S$ be independent with $|I| > f(g(k)+1)$.



Let $G' = G[(V(G) \setminus S) \cup I]$.



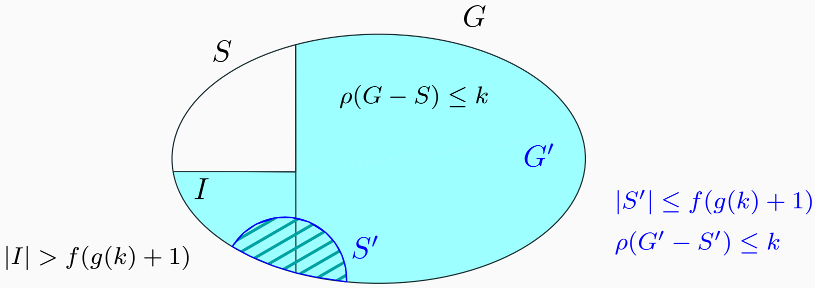
$\rho(G - S) \leq k$ implies $\omega(G - S) \leq g(k)$, which implies

$$\omega(G') \leq \omega(G - S) + \omega(G[I]) \leq g(k) + 1\,.$$

Since the class $\mathcal{G}$ is hereditary, we have

$$\tau_k^{\rho}(G') \leq f(\omega(G')) \leq f(g(k) + 1)\,.$$

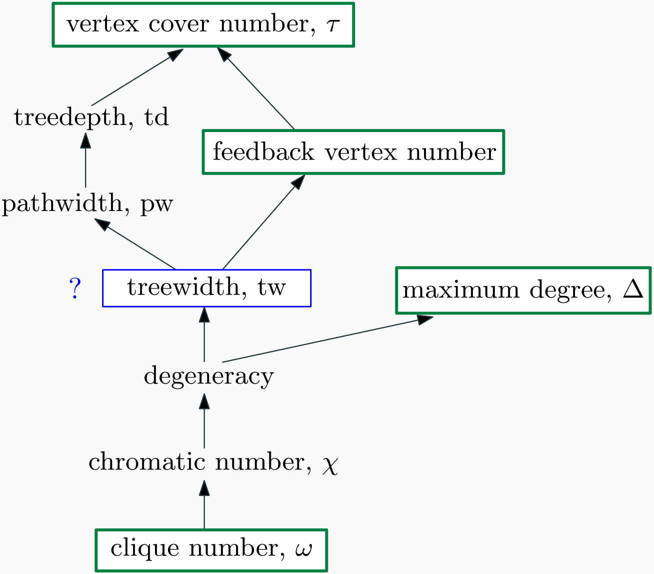
Let S' be a ρ-k modulator in G' such that $|S'| \leq f(g(k)+1)$.



Let $\widehat{S} = (S \setminus I) \cup S'$.

Note that $G - \widehat{S} = G' - S'$, thus $\widehat{S}$ is a ρ-k modulator in G.

Since $|S'| < |I|$, we have $|\widehat{S}| < |S|$, contradicting the fact that S is a minimum ρ-k modulator in G.


vertex cover number, τ
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degeneracy
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clique number, ω

Theorem

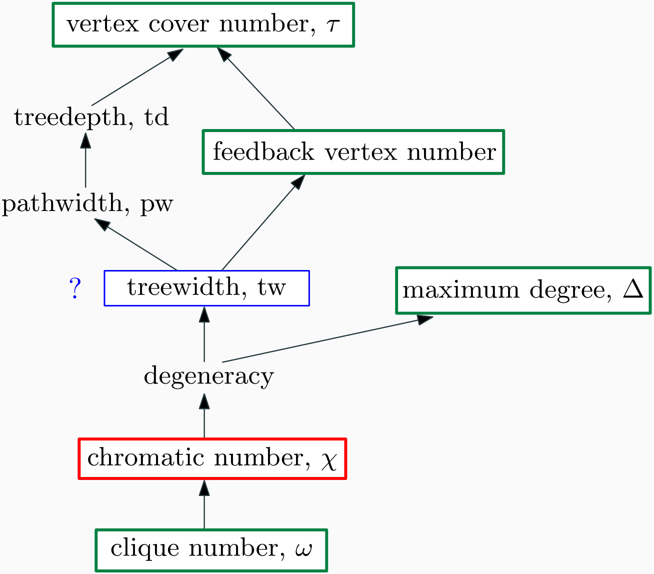
Chromatic number is not an $\alpha\omega$esome parameter.

The proof based on Ramsey's theorem showing that for any hereditary graph class $\mathcal{G}$ with bounded α-ρ, there exists a function f such that $\rho(G) \leq f(\omega(G))$ for all $G \in \mathcal{G}$, in fact establishes the existence of a **polynomial** function f.

Hence, if ρ is an $\alpha\omega$esome graph parameter, then for any hereditary graph class $\mathcal{G}$, clique-boundedness of ρ is equivalent to **polynomial clique-boundedness** of ρ.

But for chromatic number, this equivalence was recently disproved.

Brianński, Davies, Walczak, Combinatorica, August 2023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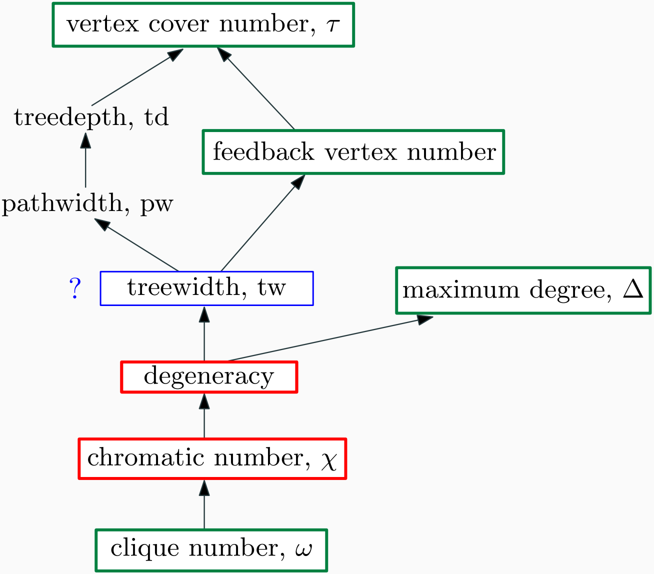
clique number, ω

Using a random construction, Girão, McCarty, Steiner, and Wiederrecht recently proved:

Theorem

Degeneracy is not an $\alpha\omega$esome parameter.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Theorem

Pathwidth and treedepth are not $\alpha\omega$esome parameters.

To prove this, we show that

$$\omega\text{-bounded treedepth} \centernot\implies \text{bounded } \alpha\text{-pathwidth}$$

This suffices since

$$\omega\text{-bounded treedepth} \implies \omega\text{-bounded pathwidth}$$

and

$$\text{bounded } \alpha\text{-treedepth} \implies \text{bounded } \alpha\text{-pathwidth}$$

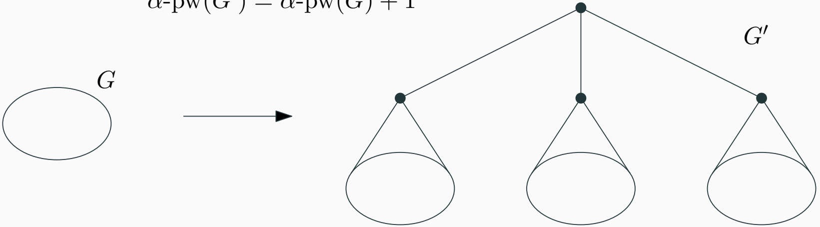
To show that ω-bounded treedepth $\not\Longrightarrow$ bounded α-pathwidth, we construct a hereditary graph class Γ such that

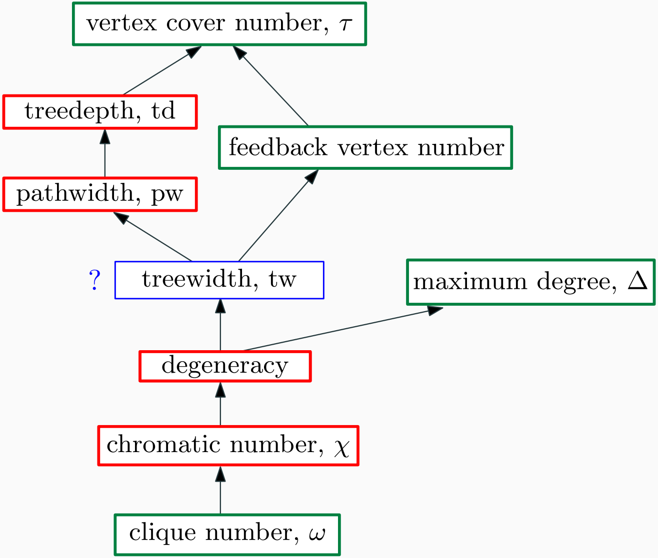
- each graph $G \in \Gamma$ satisfies $\mathrm{td}(G) \leq 2\omega(G) - 1$;
- the class Γ has unbounded α-pathwidth.

The second property is based on the following transformation that **increases α-pathwidth by one**:

given a graph G, substitute a copy of G into each leaf of the subdivided claw

$$\alpha\text{-pw}(G') = \alpha\text{-pw}(G) + 1$$

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Conclusion:

We introduced the concept of an $\alpha\omega$esome graph parameters, which allow to simplify two-parametric properties of graph classes – involving the clique number and the parameter ρ – into properties involving a single parameter, the independence variant of ρ.

We identified an infinite family of $\alpha\omega$esome graph parameters and several non-$\alpha\omega$esome ones.

Open questions:

- Is treewidth an $\alpha\omega$esome graph parameter?
- Is there an efficiently computable graph transformation that increases α-treewidth by some fixed positive integer?

Thank you!