Open problems of GRASTA 2023

Grasta Community

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1 Open Problems

1.1 Hunters and Rabbit in Interval graphs. Nicolas Nisse

In the Hunters and Rabbit game, an invisible rabbit is initially located in some vertex of a graph. At each turn, k vertices are shot. If the Rabbit is on a shot vertex, then the Hunters win. Otherwise, the Rabbit must move to an adjacent vertex. Given a graph G, $h(G)$ denotes the smallest k such that k Hunters are sufficient to win in G, whatever the Rabbit does. The computational complexity of $h(G)$ is not known in general graphs G (not even in trees).

For interval graphs, it is known that $pw(G) \leq h(G) \leq pw(G) + 1$ where $pw(G)$ denotes the pathwidth of G [1]. Moreover, if every maximum clique of G has a simplicial vertex, then $h(G) = pw(G)$. Otherwise (if some maximum clique has no simplicial vertex), $h(G)$ may be equal to $\mathsf{pw}(G)+1$ (an example is given in [2]) or to $pw(G)$ (consider any clique K and add a path of length ≥ 3 whose each vertex is adjacent to all vertices of K).

Can the interval graphs for which $pw(G) = h(G)$ be characterized?

More generally, what is the time-complexity of deciding whether $h(G) \leq k$ in general graphs? in trees?

[1] Thomas Dissaux, Foivos Fioravantes, Harmender Gahlawat, Nicolas Nisse: Recontamination Helps a Lot to Hunt a Rabbit. MFCS 2023: 42:1-42:14

[2] Tatjana V. Abramovskaya, Fedor V. Fomin, Petr A. Golovach, Michal Pilipczuk: How to hunt an invisible rabbit on a graph. Eur. J. Comb. 52: 12-26 (2016)

1.2 Broadcast "Search" (Pierre Fraigniaud)

Let $G = (V, E)$ be a simple connected graph. Initially, a vertex $v \in V$ has some information, and the objective is to broadcast this information to all the other vertices of G. Broadcast proceeds as a sequence of rounds. At each round, each vertex that has the information may send it to one of its neighbours. The minimum number of rounds required to broadcast the information from v in G is denoted by $b(G, v)$. Since the number of informed vertices can at most double at each round, we have $\lceil \log_2 n \rceil \leq b(G, v) \leq n - 1$ in any (connected) n-vertex graph. Let $b(G) = \max_{v \in V} b(G, v)$ denotes the *broadcast time* of G.

Open Problem A. Design a polynomial-time algorithm for approximating the broadcast time of a graph; Design FPT parametrized algorithms for computing the broadcast time of a graph. See [FFG23] for the known results on these matters.

A graph G for which $b(G) = \lceil \log_2 n \rceil$ is called *broadcast graph*. For instance, complete graphs and hypercubes are broadcast graphs. The minimum number of edges of a broadcast graph with n vertices is denoted by $B(n)$. It is known [GP91] that $B(n) = \Theta(n \cdot L(n))$ where $L(n)$ denotes the number of leading 1's in the binary representation of $n-1$. (In fact, hypercubes are minimum broadcast graphs.) Nevertheless, the behavior of the function $B(n)$ is not fully understood. For instance, it is not known whether B is non-decreasing between two powers of 2. Also, the complexity of computing $B(n)$ is not known.

Open Problem B. What is the complexity of the following problem:
Input: integers $n \geq 1$ and $k \geq 0$;
Question: $B(n) \leq k$?

Bibliography:

[FFG23] Fedor V. Fomin, Pierre Fraigniaud, Petr A. Golovach: Parameterized Complexity of Broadcasting in Graphs. In proc. WG 2023: 334-347.

[GP91] Michelangelo Grigni, David Peleg: Tight Bounds on Minimum Broadcast Networks. SIAM J. Discret. Math. 4(2): 207-222 (1991)

1.3 Graph Searching in rigs. Sebastian Wiederrecht

A graph H is a *rig* of a graph G if there exists a collection $\mathcal{I} = (G_v)_{v \in V(H)}$ of connected subgraph of G such that H is the intersection graph of $\mathcal{I}$. Given a minor closed graph class $\mathcal{C}$, let $rig(\mathcal{C})$ be the class of all rigs of graphs in $\mathcal{C}$. Note that $rig(\mathcal{C})$ is induced minor closed.

The *clique game* is defined as the classical node search game where searchers may guard any clique instead of a single vertex (this is a variant of Marshall game [Gottlob, Leone, Scarcello] in hypergraphs). It is known that $rig(\mathcal{C})$ has bounded search-clique number if and only if $\mathcal{C}$ has bounded treewidth.

Recall that there are relationship between classical variants of graph searching and graph parameters: visible/lazy $\Leftrightarrow$ admissability, visible/agile $\Leftrightarrow$ treewidth $\Leftrightarrow$ inisible/lazy, inisible/agile $\Leftrightarrow$ pathwidth, and LIFO $\Leftrightarrow$ treedepth.

1. Fill the previous table for Marshall games?
2. Extend these relation to rigs?
3. Given an induced minor closed class $\mathcal{I}$, decide if $I = rig(\mathcal{C})$ for some minor closed class $\mathcal{C}$.

More information here

1.4 H-extensions of search games. Dimitrios M. Thilikos

Given a graph G, we denote by $\mathsf{cc}(G)$ the set of all connected components of G. For a graph G and a set $X \subseteq V(G)$, the *stellation* of X in G is the graph $\mathsf{stell}(G, X)$ obtained from G if, for every $C \in \mathsf{cc}(G \setminus X)$, we contract all the edges of C to a single vertex v_C. The *torso* of X in G is the graph $torso(G, X)$ obtained from $\mathsf{stell}(G, X)$ if, for every v_C where $C \in \mathsf{cc}(G \setminus X)$, we add all edges between neighbors of v_C and finally remove all v_C's from the resulting graph.

Let $\mathcal{H}$ be a graph class and let p be a graph parameter, mapping graphs to non-negative integers. The graph parameter $\mathcal{H}$-p is defined so that

$$\mathcal{H}\text{-}\mathsf{p}(G) = \min\{k \mid \exists X : \mathsf{p}(\mathsf{torso}(G, X)) \leq k \wedge G \setminus X \in \mathcal{H}\}$$

In the case p is the parameter of treewidth tw, then $\mathcal{H}$-tw corresponds to the monotone search game against an active and visible fugitive of infinite speed where the fugitive is captured if his/her free space is inducing a graph in $\mathcal{H}$. But what about the monotonicity of this game? As proved by Seymour and Thomas, the game is monotone and corresponds to treewidth when $\mathcal{H}$ is the class $\mathcal{H}_\emptyset$ containing only the empty graph.

The emerging question is whether there is some condition for the defining class $\mathcal{H}$ that implies the monotonicity of the general game.

For proving monotonicity one may use the approach of Seymour and Thomas, and define the notion of a *$\mathcal{H}$-bramble* that is a collection $\mathcal{B} = \{B_1, \ldots, B_r\}$ of connected induced subgraphs of G that do not belong to $\mathcal{H}$ and where for every two of them B_i, B_j either they have some common vertex or there is an edge with one endpoint in B_i and the other in B_j. The *order* of $\mathcal{B}$ is the minimum number of vertices that intersects all elements of $\mathcal{B}$. The *$\mathcal{H}$-bramble number* of a graph G, $\mathcal{H}$-$\mathsf{bn}(G)$ is the maximum order of a $\mathcal{H}$-bramble of G. According to the result of Seymour and Thomas, for every graph G, $\mathcal{H}_\emptyset$-$\mathsf{tw}(G) = \mathsf{tw}(G)$ is one less than the $\mathcal{H}_\emptyset$-bramble number of G. So the question is whether a graph class $\mathcal{H}$ has the following property:

[tw/bn duality property] for every graph G in $\mathcal{H}$, $\mathcal{H}$-$\mathsf{tw}(G) + 1 = \mathcal{H}$-$\mathsf{bn}(G)$.

It is easy to see that if $\mathcal{H}$ has the tw/bn duality property, then this implies the monotonicity of the $\mathcal{H}$-extension of the search game against an active and visible fugitive of infinite speed where the fugitive is captured if his/her free space is inducing a graph in $\mathcal{H}$.

There are several instantiations of $\mathcal{H}$ that are known (or have been conjectured) to have the tw/bn duality property. For instance, there is a proof for the class of bipartite graphs. The emerging open problem is to find a general criterion for a class $\mathcal{H}$ to have the tw/bn duality property.

All the above discussion can be extended to other parameter dualities. For instance, one may consider pathwidth pw instead of treewidth and blockages instead of brambles. What is the correct notion of an $\mathcal{H}$-blockage that can imply the monotonicity of the search game against an active and invisible fugitive of infinite speed? What are the classes $\mathcal{H}$ for which the pathwidth/blockage duality can be extended? To what extend the above notion of $\mathcal{H}$-extension of a search game is applicable to other games and/or parameter dualities?

1.5 Searching with Predictions. Spyros Angelopoulos

A very active trend on the design and analysis of algorithms with incomplete information focuses on settings in which there is some *prediction* concerning the (unknown) input. This prediction is given to the algorithm by means of some imperfect, i.e., erroneous, oracle. The objective is then to design algorithms that simultaneously perform well if the prediction is perfect (or near-perfect) and are not too far from worst-case guarantees when the prediction is very unreliable.

This setting naturally applies to search problems, since they are inherently problems in which we do not know key elements of the input, notably the position of the hider. In [Spyros Angelopoulos. Online Search with a Hint. *Information and Computation 2023*] we introduced a setting for search games with predictions, where the objective is to find optimal tradeoffs between the *consistency* (i.e., performance when the prediction is perfect) and the *robustness* (i.e., performance when the prediction is adversarially incorrect). In particular, the above work focuses on *Pareyo-optimal* trade-offs between consistency and robustness for the well-known problem of *linear search*, informally known as the *cow-path* problem. Here, the prediction may take several forms: it can indicate the right direction to reach the hider, or the exact position of the hider, or it may be a potentially erroneous k-bit string, in the spirit of advice complexity of online algorithms. In follow-up work [Spyros Angelopoulos. Competitive Search on the Line and the Star with Predictions, MFCS 2023] we further studied searching in a star-like environment, and in on-going work with Thomas Lidbnetter we focus on randomized strategies (i.e., the full power of search games) for some fundamental search games, including linear search.

This framework should be applicable to other search games and search problems. Here are two broad directions:

(a) Can predictions help improve cops and robbers games? Here, consistency/robustness can be related, for instance, to the cop number in the two extreme situations.

(b) Can we extend the study of search problems under the (vanilla) advice complexity model, to the untrusted advice complexity model, in which the advice is allowed to be erroneous? What are the Pareto-tradeoffs in this case?

1.6 Harmender Galhawat

Cops and Robber in some interesection graphs Consider the classical Cops and Robber game. First, the Cop-Player places k cops at (not necessarily distinct) vertices of a graph. Then one Robber is placed at some vertex. Turn-by-turn, the Cop-Player may move each of its cops to an adjacent vertex, and then the Robber-Player can move its token to a neighbour of its current position. The Cop-Player wins if one cop eventually occupies the same vertex as the Robber, and the Robber-Player wins otherwise. The *cop-number* $cn(G)$ of a graph G is the minimum k such that k cops have a winning strategy in G.

String graphs are intersection graphs of strings (continuous curves) in the plane. It is knwon that, for any string graph G, then $cn(G) \leq 13$ [1]. Can this be improved? If, moreover, any two strings intersects at most once, then $cn(G) \leq 6$ [?], can this be improved?

What about the cop-number of intersection graphs of a set of rectangles? of a set of (unit) squares? Case of large girth?
Remark of Sebastian: String graphs are rigs of planar graphs. If a class of graphs has bounded cop-number, what about the cop-number of its rigs?

[1] Sandip Das, Harmender Gahlawat: On the Cop Number of String Graphs. ISAAC 2022: 45:1-45:18

Pushing cops. Let D be an oriented graph and let $v \in V(D)$. *Pushing* v consists in inverting the orientation of all arcs incident to v. Consider the Cops and Robber game in oriented graphs (where both cops and robber must follow the orientations of the arcs). Moreover, at its turn, the Cop-Player may either move its cops or push some vertex.

Are all graphs cop-win (i.e., one cop always wins) in this variant?

1.7 Jan Kratochvil

OuterString Graphs. IFA (Interval Filaments) [Gavril]: intersection graph of continuous curves leaving above a border "horizontal" line (and with both ends in the border line). More general class than interval graphs. 2 cops are enough in IFA (in any variant: normal, lazy...).

OuterString graphs [Fellows?]: intersection graph of strings starting at the border line. Cop number 3 or 4?

Firefighter in Hexagonal grid. Fire game. Fire and firemen take turns. The fire starts in one vertex. Each turn, one vertex is protected by firemen, then fire extends to its unprotected neighbours. Who wins in Hexagonal grid?

If the firemen have (in total) 2 extra protected vertex (at any moment), then they win. [Tomas Gavenciak, Jan Kratochvíl, Pawel Pralat: Firefighting on square, hexagonal, and triangular grids. Discret. Math. 337: 142-155 (2014)]

Known with one 1 extra protected vertex.

Plane cops and robber game [Irsic, Mohar, Wesmek?]

1.8 Multidimentional binary search problem. Przemysław Gordinowicz

Consider the following searching model. Consider the game between Algorithm and Adversary on a d-dimensional grid $n \times n \times \dots n$. Adversary hides a target in some point of the grid. At each step Algorithm picks a point $(x_1, x_2, \dots x_d)$ and then Adversary answers with (d-dimensional) interval with one end at $(x_1, x_2, \dots x_d)$ and another at some corner of the grid, where there is no target. The Algorithm' goal is to find the target as soon as possible while Adversary wants to play long. It is good to think that there is no target point, but the area of possible target places shrinks at each step. Let $T(n, d)$ be the number of steps provided both player play optimally. Surely $T(n, 1) = \Theta(\log n)$.

Because of the geometry of the target area the bound is no longer logarithmic for higher dimensions. For $d \geq 2$ we can prove $T(n, d) = \Omega\left(\frac{n^{d-1}}{d}\right)$ and $T(n, d) = O\left(n^{d-1}\right)$.

Question: Which bound is closer? Fill the gap.

1.9 Cops and Robber in temporal graphs. Frédéric Simard

Is there a periodic temporal sequence of graphs based on a graph G such that the temporal cop-number of the sequence is higher than a function of $tw(G)$? Is the class of periodic temporal graphs with planar footprint bounded for the temporal cop number?

1.10 Fionn Mc Inerney

Maker Breaker complexity POS CNF (Schaeffer 1978). Maker wants to satisfy the formula, Breaker wants to falsify it. In turns, starting with Maker, they assign a value to one variable. Alternative definition in terms of hypergraphs: Maker and Breaker pick vertices, Breaker wants to "fill" an hyperedge, Maker wants a transversal of the hyperedges.

Schaeffer showed POS-CNF-11 (each clause has 11 litterals) is PSPACE-complete. Then, Rahman, Watson (STACS 2021) showed POS-CNF-6 (each clause has 6 litterals) is PSPACE-complete. Recently, Gaillot, Gravier... (2023+) showed that POS-CNF-3 is in P.

Eternal Domination Given a graph, first player initially places k agents on vertices (several agents may occupy a vertex). Each turn, the second player attacks one vertex. Then, each agent may move to an adjacent vertex, ensuring that the attacked vertex is occupied at the end of the turn. Deciding the smallest k such that First Player has a winning strategy is NP-hard [].

Is it PSPACE-complete? EXPTIME-complete? It seems that when at most one guard can move each turn, this is PSPACE-hard (or EXPTIME-hard?) if the initial positions of the guards is given? [McGillivray, Mynhardt,Virgile 2023?]