

Parameterized Analysis of the Cops and Robber Game

Harmender Gahlawat and Meirav Zehavi

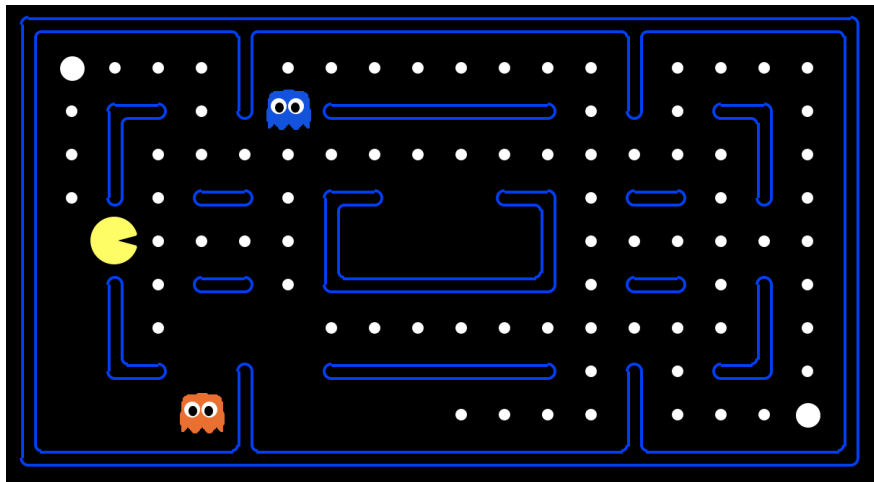
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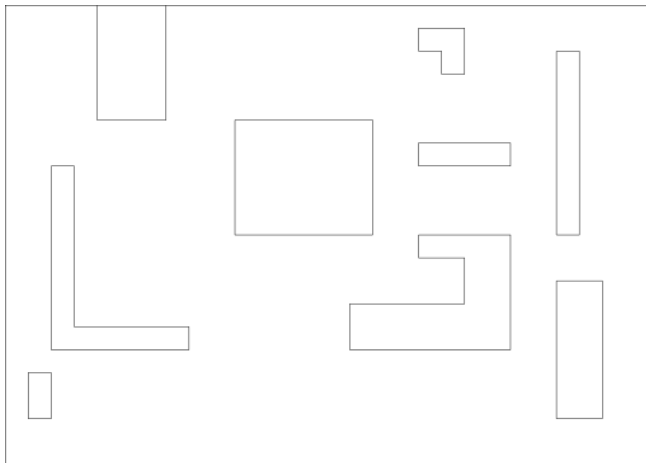


October 24, 2023

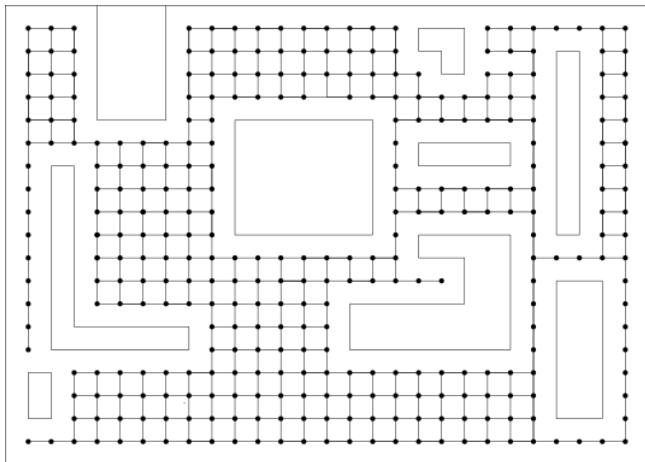
PacMan



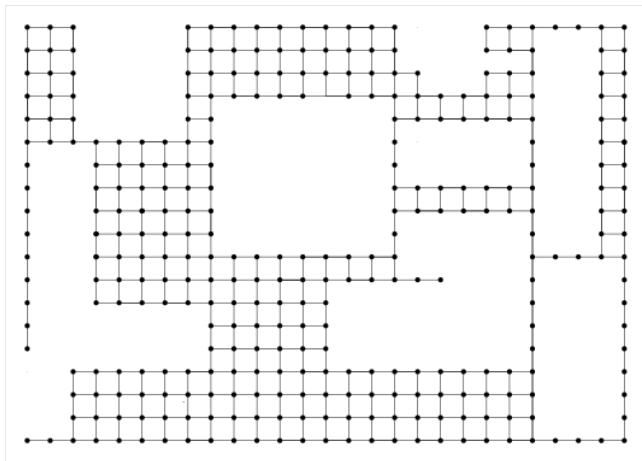
PacMan



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Cops and Robber

Playground

A simple, connected and finite graph.

Two teams

k cops and a single robber.

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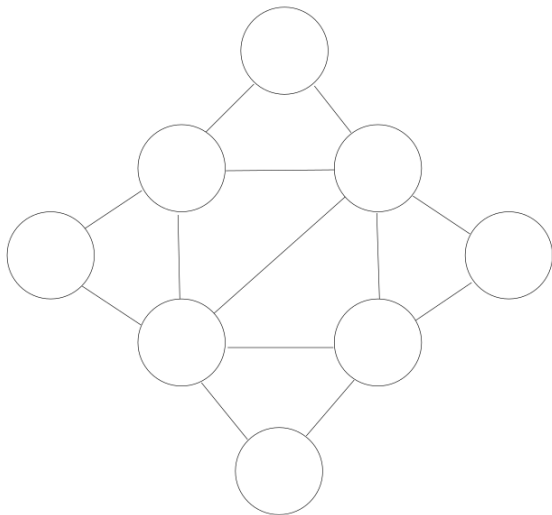
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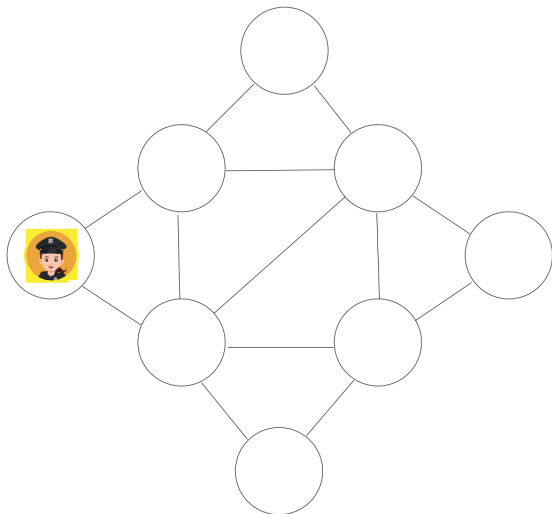
Winning

- Cops win if some cop and robber occupy the same vertex. (*Capture*)
- Robber wins otherwise.

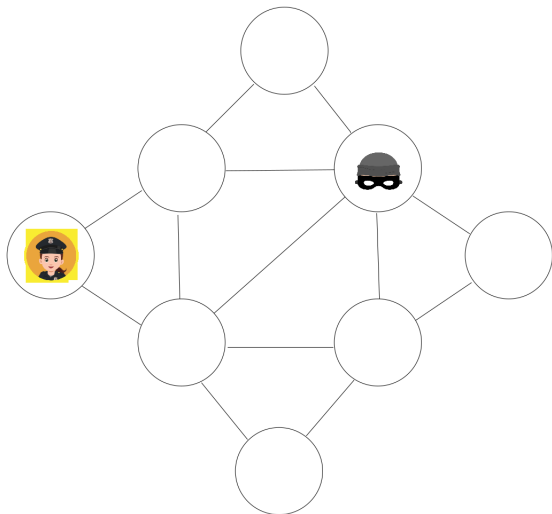
Demo(Playground)

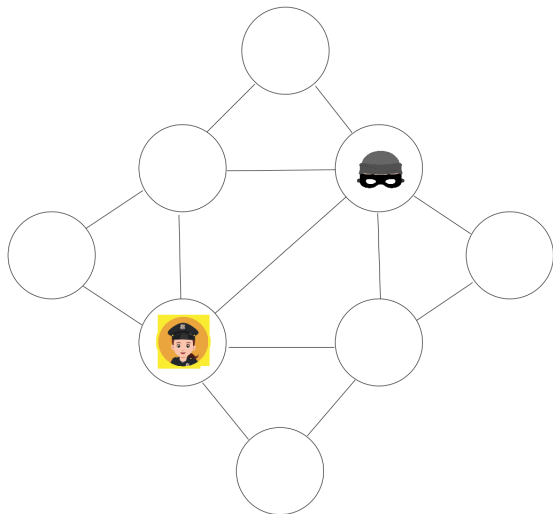


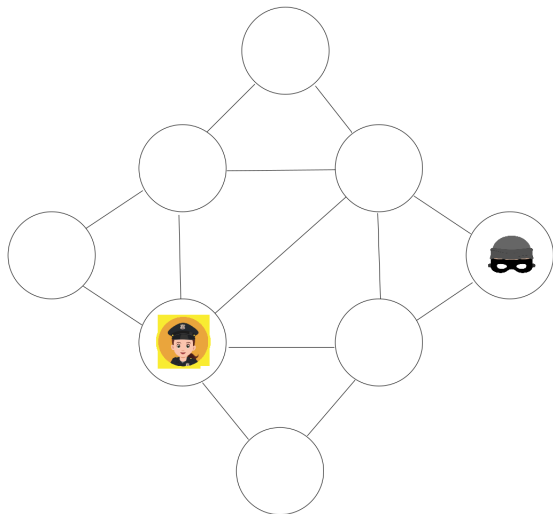
Demo(Initialization)

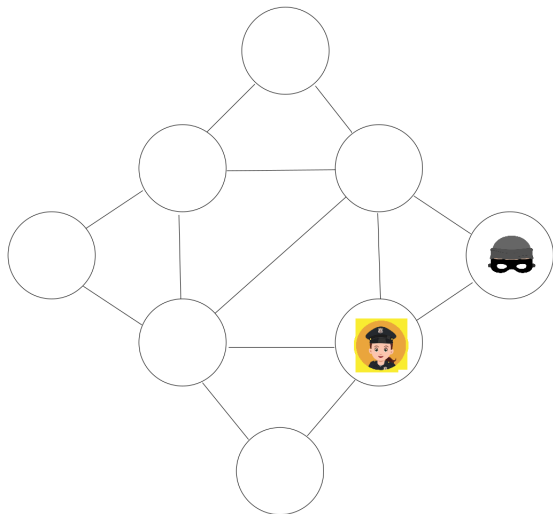


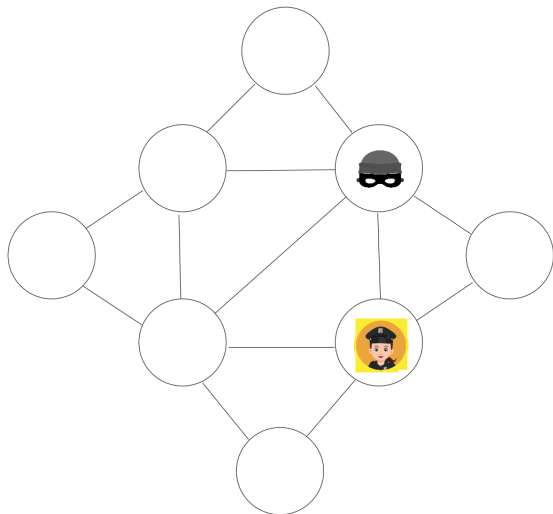
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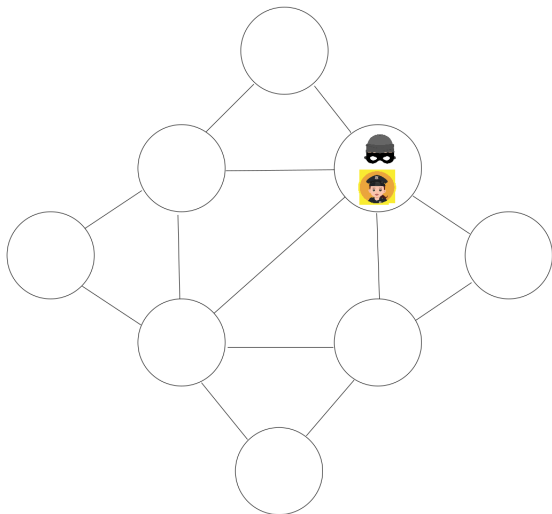








Demo(Capture)



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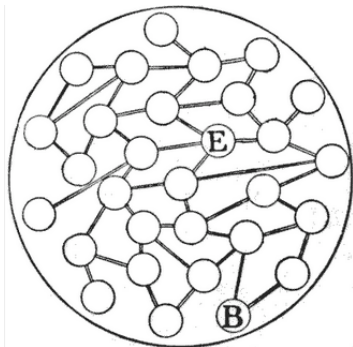
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Prehistory

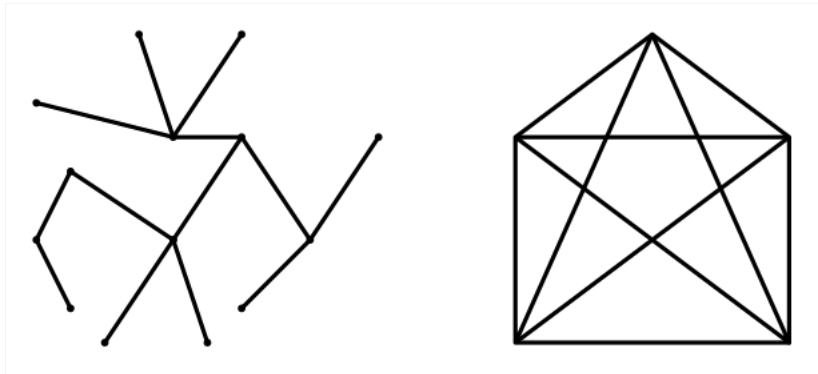
In the book *Amusements in Mathematics*, published in 1917, Henry Ernest Dudeney asked the following question.



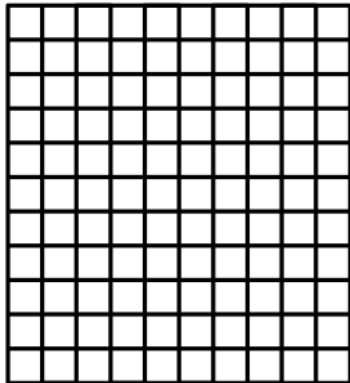
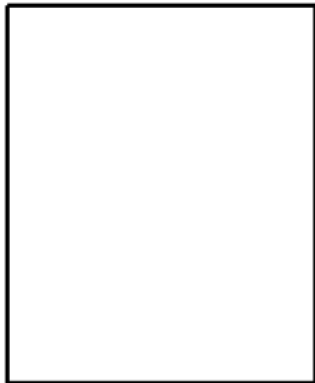
One Cop vs Robber

- First considered by A. Quilliot in his doctoral thesis in 1978.
- Considered independently by Nowakowski and Winkler in 1983.
- Both characterized the *cop-win* graphs, where one cop can win.

Some Copwin Graphs



Some Not Copwin Graphs



More cops to come...

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Cop number

- is the minimum number of cops required to capture a robber in the graph.
- is denoted by $c(G)$.
- is upper bounded by *domination number*.

Complexity

- EXPTIME-Complete [Kinnersley, 2015]

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- XP-time algorithm for k [Berarducci and Intrigila, 1993]

Our Results: Kernelization

Our Main Lemma

If $N(u) \subseteq N(v)$ and $k \geq 2$. Then, G is k -copwin $\iff G - u$ is k -copwin.

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Kernelization Results

Let t denote the structural parameter in consideration. Then, we have the following kernels:

- Vertex Cover: $t + \frac{2^t}{t}$.
- Cluster Vertex Deletion: $2^{2^t + P_t}$.
- Deletion to stars: $2^{2^t + t^{1.5}}$.
- Neighbourhood Diversity: t .

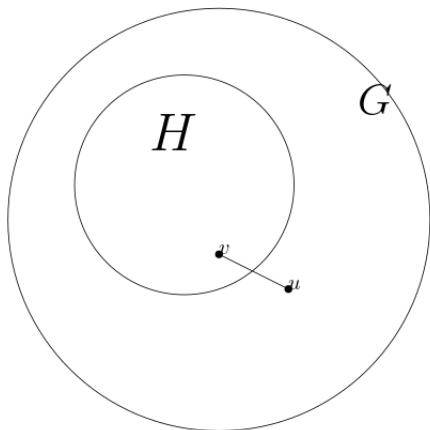
Vertex Cover

Cops and Robber parameterized by vertex cover number is unlikely to admit a polynomial kernel.

Definitions

Guarding a subgraph

Let H be a subgraph of G . Cops *guard* H if R cannot enter H without being captured.



Two Important Results [Aigner and Fromme, 1984]

Path Guarding Lemma

Let P be an isometric path in G . Then one cop can guard P .

High Girth High Min-degree Lemma

Let G be a graph with girth at least 5. Then, $c(G) = \lfloor \frac{n}{5} \rfloor$.

One Application of Guarding Paths Lemma

Lemma

Let $U \subseteq V(G)$ such that each connected component of $G - U$ has cop number at most k . Then, $c(G) \leq \frac{|U|}{2} + k$.

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Proof Sketch

Fix an ordering $u_1, \dots, u_{|U|}$. Now find a shortest path between u_i, u_{i+1} for each odd i and guard it. This restricts R to a component having cop number at most k .

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Corollary

$c(G)$ is bounded by

- feedback vertex set,
- distance to chordal graphs,
- distance to planarity,
- and so on...

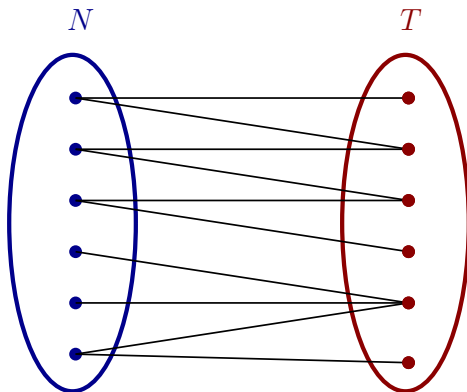
Polynomial Parameter Transformation

- A polynomial-time algorithm that, given an instance $(I; k)$ of Π_1 , generates an equivalent instance $(I^0; k^0)$ of Π_2 such that $k^0 = p(k)$, for some polynomial $p(\cdot)$.

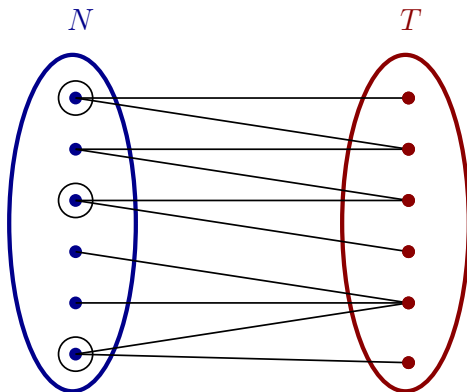
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- A polynomial-time algorithm that, given an instance $(I; k)$ of Π_1 , generates an equivalent instance $(I^0; k^0)$ of Π_2 such that $k^0 = p(k)$, for some polynomial $p(\cdot)$.
- If Π_1 does not admit a polynomial compression, then Π_2 does not admit a polynomial compression.

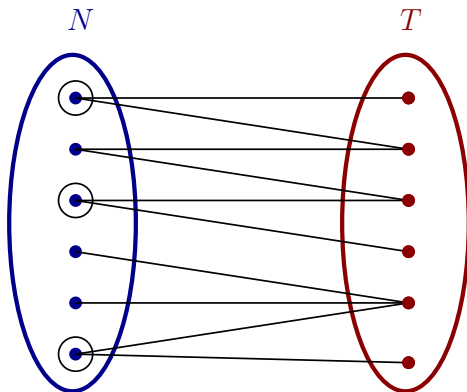
Red Blue Dominating Set



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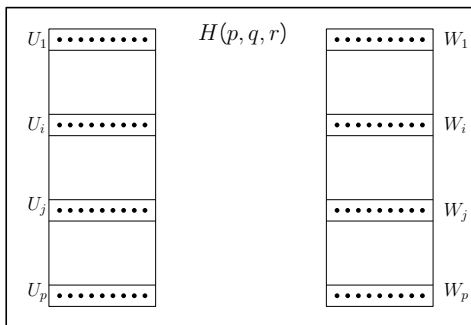
Red Blue Dominating Set



RBDS Incompressibility [Dom et al., 2011]

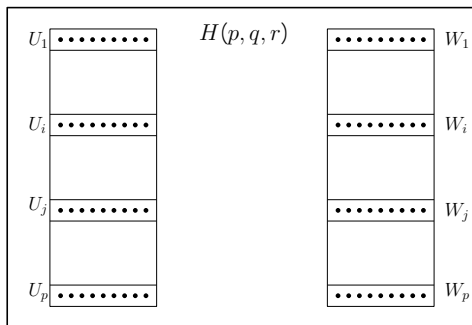
RBDS parameterized by $|T| + k$ is unlikely to admit a polynomial kernel.

Incompressibility: A construction with large degree and girth [Fomin et al., 2013]



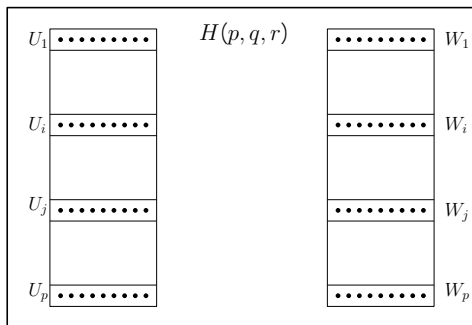
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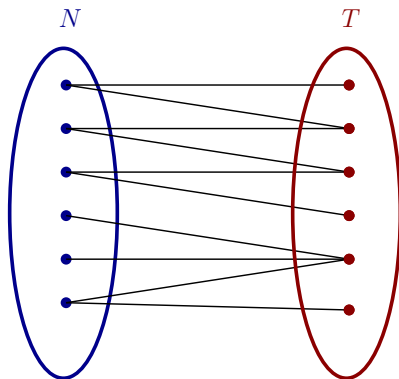
- H is a bipartite graph with partitions U and W , and girth 6.
- Both U and W has exactly p blocks.
- Each block contains exactly q vertices.
- Each vertex in U_i has at least $r - 1$ neighbours in each W_j , and vice-versa.

A construction with large degree and girth [Fomin et al., 2013]

Let $q = 2p(r+1) \frac{(p(r+1)-1)^6}{(p(r+1)-1)^2 - 1}$. Then, we can construct $H(p; q; r)$ in time $O(r \cdot q \cdot p^2)$ with the following properties.

- The girth of $H(p; q; r)$ is at least 6.
- For every vertex $z \in V(H_{i,j})$ and every $i, j \in [p]$, we have $r+1 \leq \deg_{i,j}(z) \leq r+1$.

Incompressibility: Reduction



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Set $p = jTj$ and $r = k + 2$.

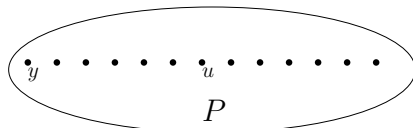
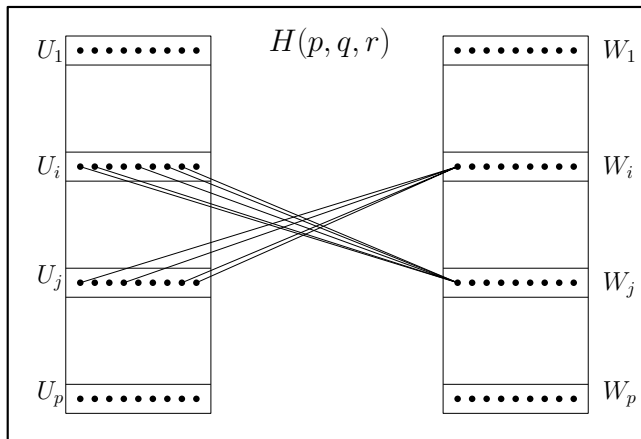
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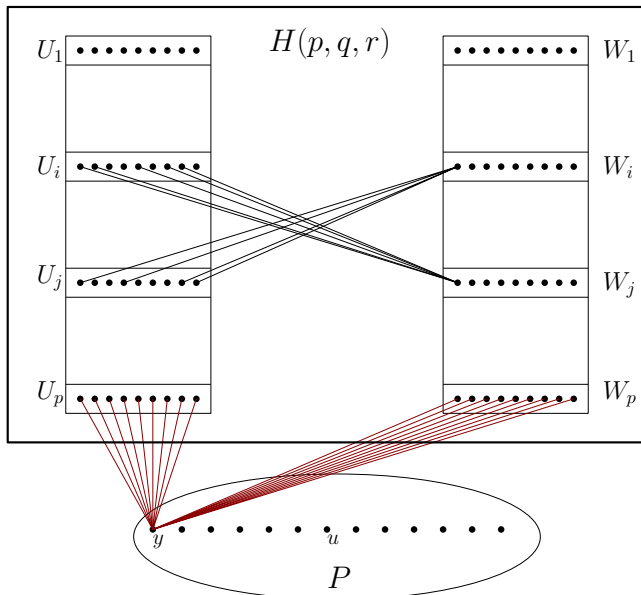
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$p = jTj$ and $r = k + 2$. For $1 \leq i, j \leq p$, each vertex in U_i is connected to at least $k + 1$ vertices in W_j , and vice-versa.

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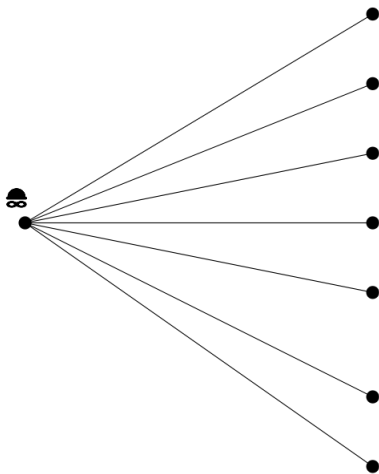
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Finally, connect y to each vertex in P .

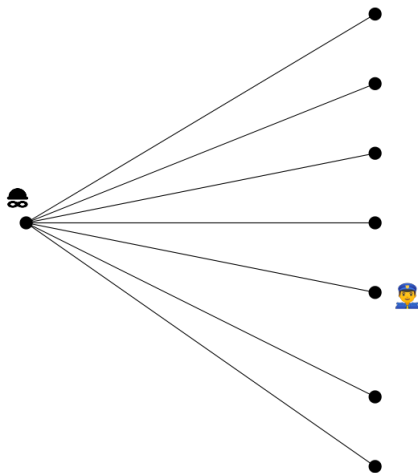
Reduction: RBDS of size $k \Rightarrow k$ -copwin

Reduction: No RBDS of size $k \Rightarrow$ Not k -copwin

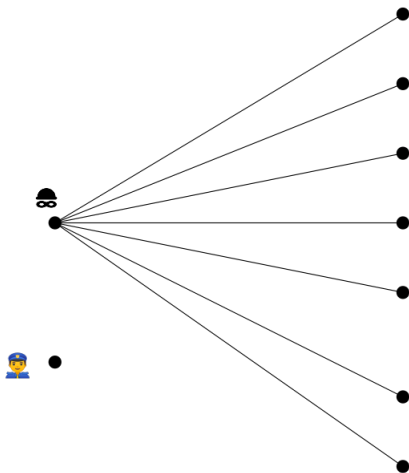
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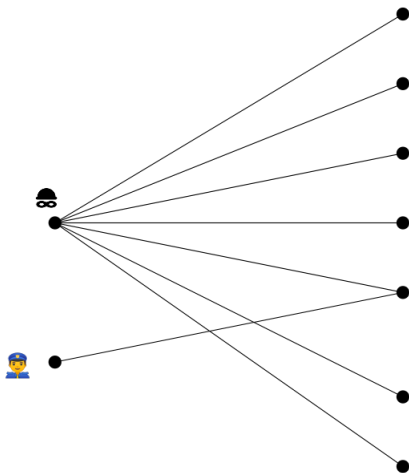
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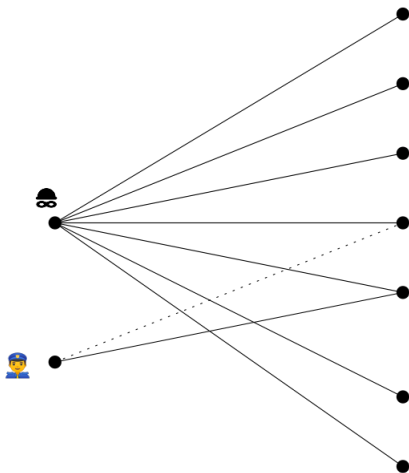
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Variations

- Cops and attacking Robber.
- Lazy Cops and Robber.
- Fully active Cops and Robber.
- Cops and fast Robber.
- Cops and Surrounding Robber
- Cops and Robber on oriented graphs.
- Generalized Cops and Robber.

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- k cops and a single robber.

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- Some of the cops can have a gun with a predefined (possibly different) range.

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Kernelization

Generalized Cops and Robber respects keeping $k + 1$ twins rule. Hence, it admits a kernel with at most $2^t : t + t$ vertices.

Open Problems

- Is Cops and Robber FPT parameterized by treewidth or by feedback vertex set?

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- Is $c(G) = o(vc)$?
- Does Cops and Robber admit a single exponential FPT algorithm parameterized by vertex cover number?

Thank You