Parameterized Analysis of the Cops and Robber Game

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Ben-Gurion University of the Negev, Beer-Sheva, Israel

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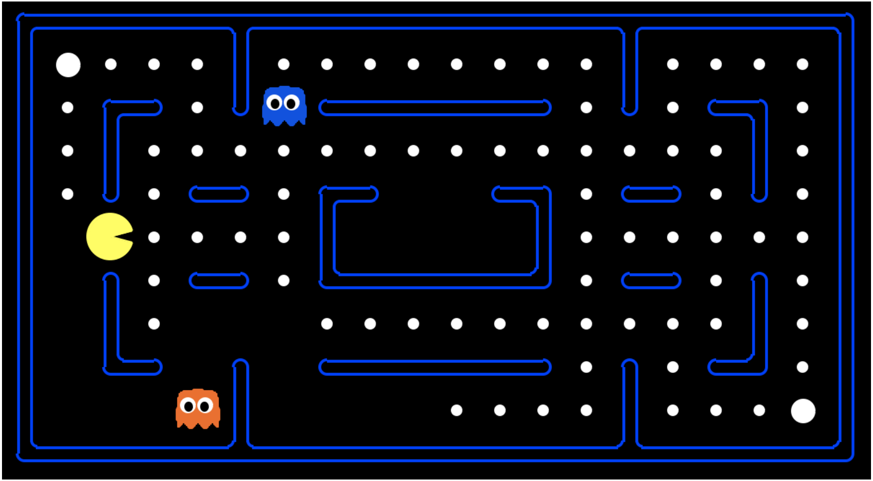
PacMan



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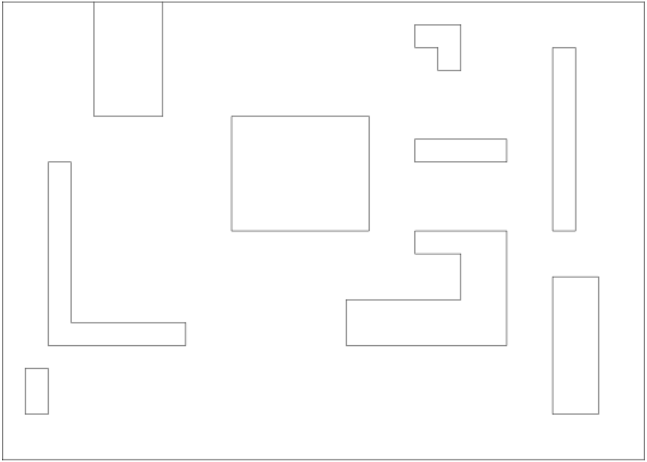

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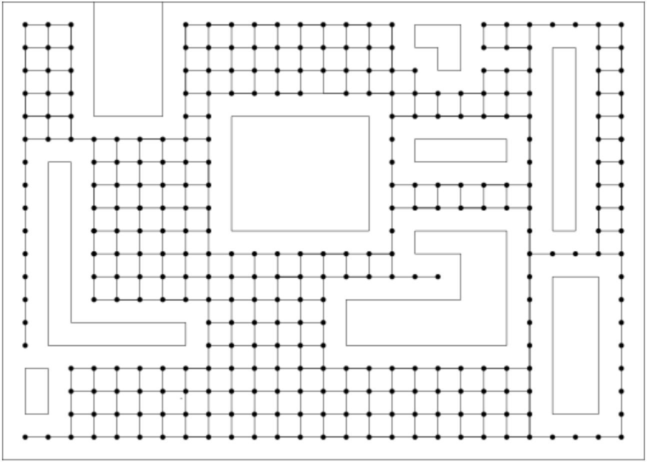

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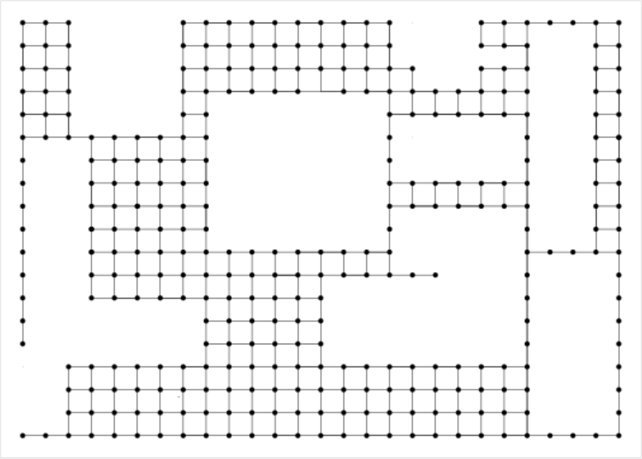

Cops and Robber

Playground

A simple, connected and finite graph.

Two teams

k cops and a single robber.

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Winning

- Cops win if some cop and robber occupy the same vertex. (*Capture*)
- Robber wins otherwise.

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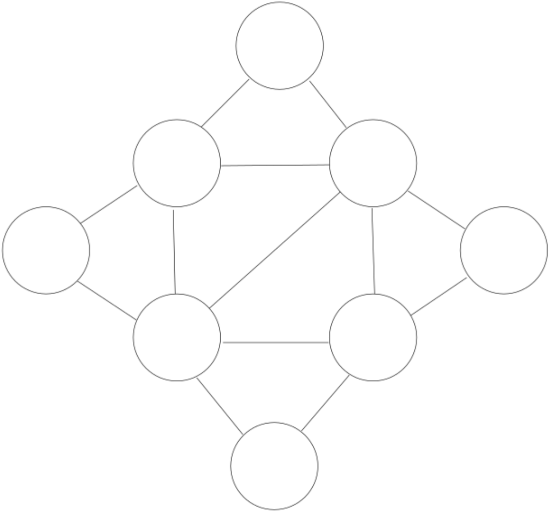
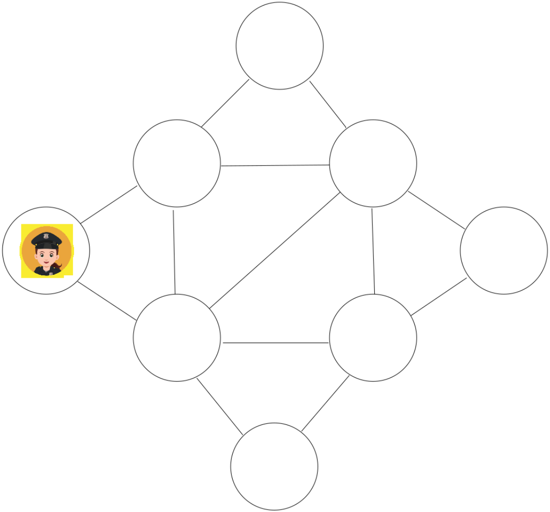


Demo(Initialization)



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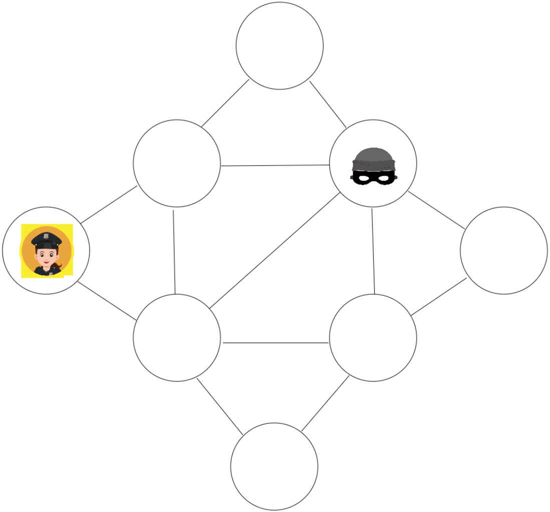

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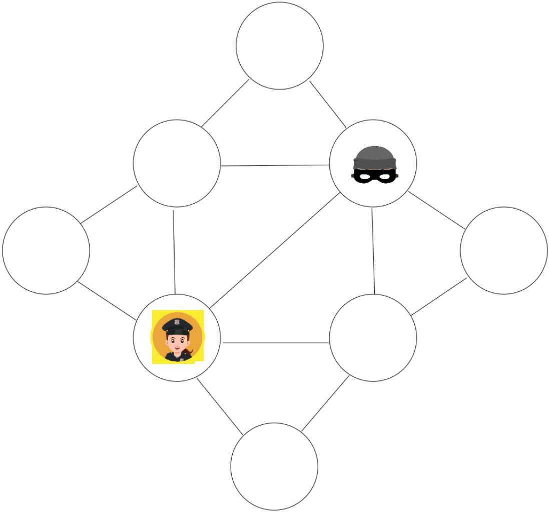

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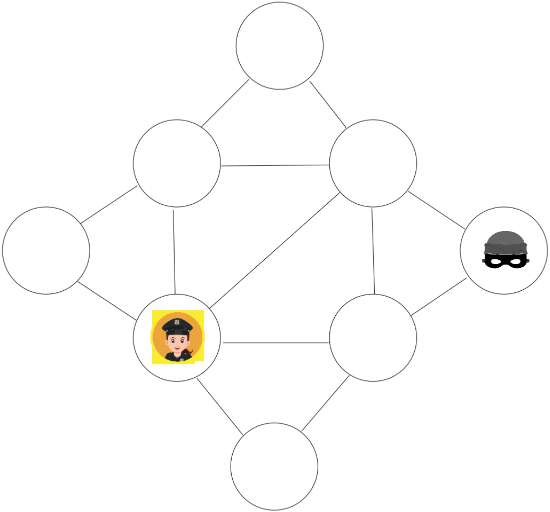

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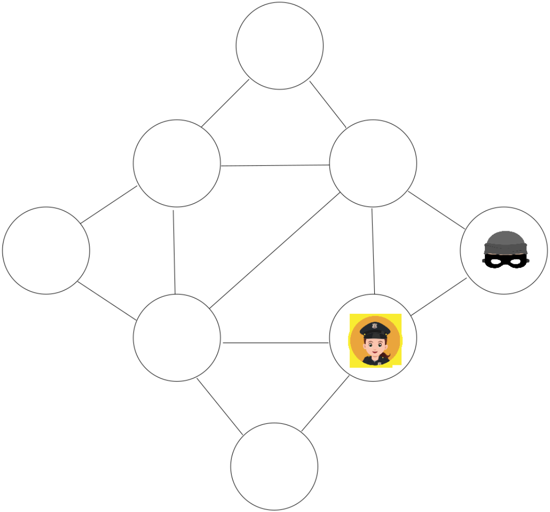

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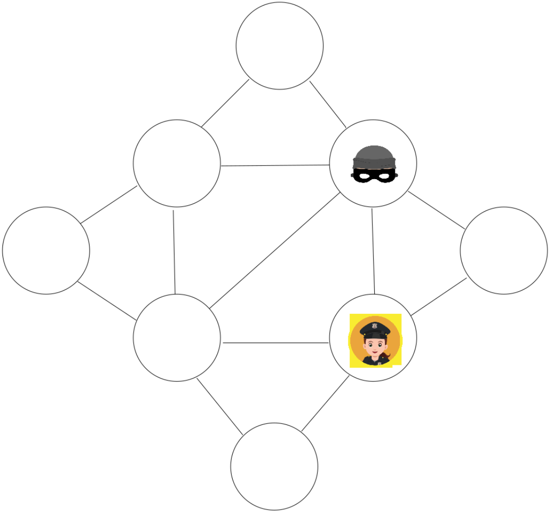

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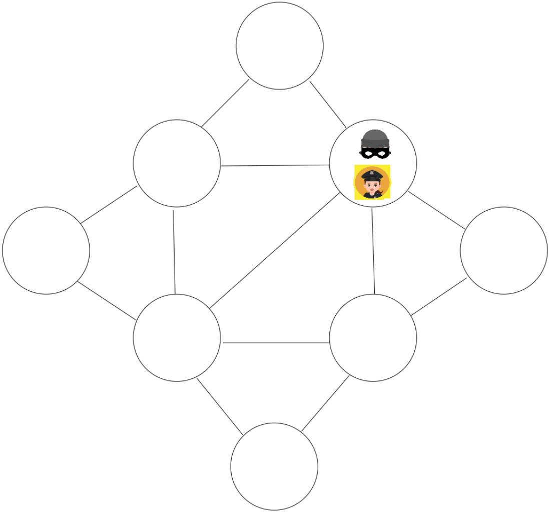

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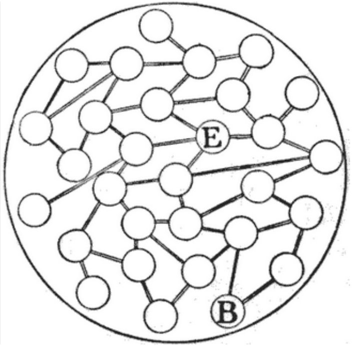
Prehistory

In the book *Amusements in Mathematics*, published in 1917, Henry Ernest Dudeney asked the following question.

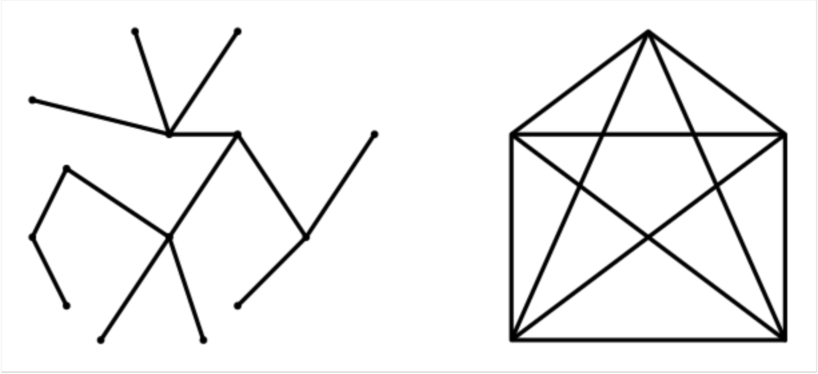


Origin

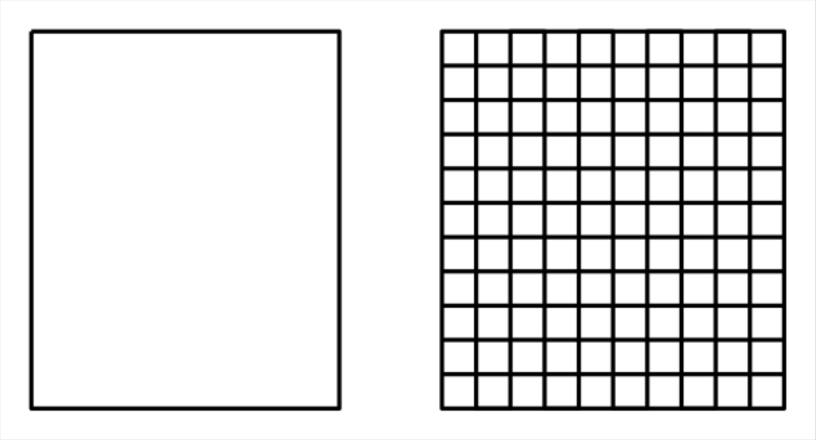
One Cop vs Robber

- First considered by A. Quilliot in his doctoral thesis in 1978.
- Considered independently by Nowakowski and Winkler in 1983.
- Both characterized the *cop-win* graphs, where one cop can win.

Some Copwin Graphs



Some Not Copwin Graphs



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Cop number

- is the minimum number of cops required to capture a robber in the graph.
- is denoted by $c(G)$.
- is upper bounded by *domination number*.

Computational complexity

Complexity

- EXPTIME-Complete [Kinnersley, 2015]

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- XP-time algorithm for k [Berarducci and Intrigila, 1993]

Our Results: Kernelization

Our Main Lemma

If $N(u) \subseteq N(v)$ and $k \geq 2$. Then, G is k-copwin $\iff$ $G - u$ is k-copwin.

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Kernelization Results

Let t denote the structural parameter in consideration. Then, we have the following kernels:

- Vertex Cover: $t + \frac{2^t}{\sqrt{t}}$.
- Cluster Vertex Deletion: $2^{2^t + \sqrt{t}}$.
- Deletion to stars: $2^{2^t + t^{1.5}}$.
- Neighbourhood Diversity: t.

Incompressibility

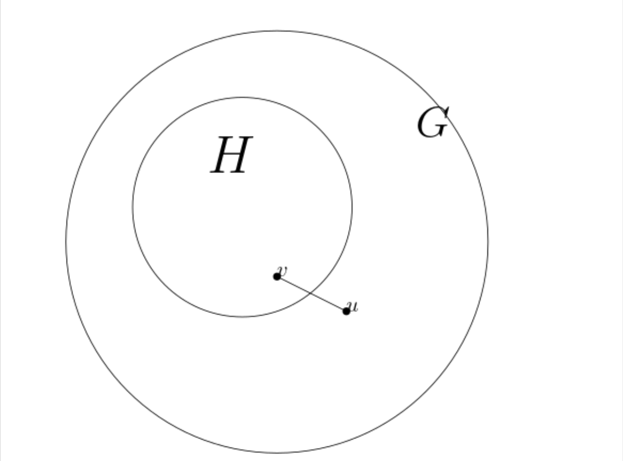
Vertex Cover

Cops and Robber parameterized by vertex cover number is unlikely to admit a polynomial kernel.

Definitions

Guarding a subgraph

Let H be a subgraph of G. Cops *guard* H if $\mathcal{R}$ cannot enter H without being captured.



Two Important Results [Aigner and Fromme, 1984]

Path Guarding Lemma

Let P be an isometric path in G. Then one cop can guard P.

High Girth High Min-degree Lemma

Let G be a graph with girth at least 5. Then, $\mathrm{c}(G) \geq \delta(G)$.

One Application of Guarding Paths Lemma

Lemma

Let $U \subseteq V(G)$ such that each connected component of $G - U$ has cop number at most α. Then, $c(G) \leq \frac{|U|}{2} + \alpha$.

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Proof Sketch

Fix an ordering $u_1, \ldots, u_{|U|}$. Now find a shortest path between u_i, u_{i+1} for each odd i and guard it. This restricts $\mathcal{R}$ to a component having cop number at most α.

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Corollary

$c(G)$ is bounded by

- feedback vertex set,
- distance to chordal graphs,
- distance to planarity,
- and so on...

Incompressibility

Polynomial Parameter Transformation

- A polynomial-time algorithm that, given an instance (I, k) of Π_1, generates an equivalent instance (I', k') of Π_2 such that $k' \leq p(k)$, for some polynomial $p(\cdot)$.

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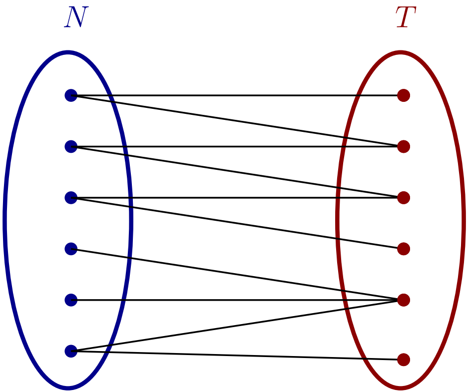
Polynomial Parameter Transformation

- A polynomial-time algorithm that, given an instance (I, k) of Π_1, generates an equivalent instance (I', k') of Π_2 such that $k' \leq p(k)$, for some polynomial $p(\cdot)$.
- If Π_1 does not admit a polynomial compression, then Π_2 does not admit a polynomial compression.

Red Blue Dominating Set



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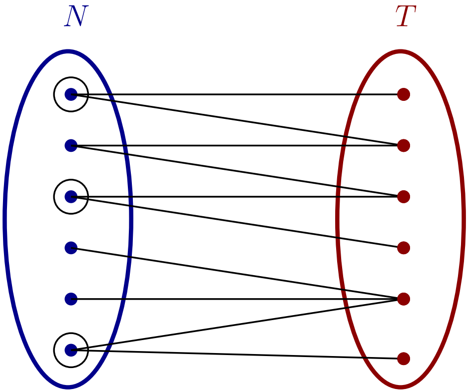

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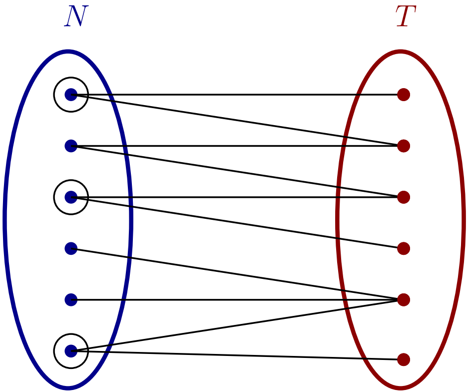
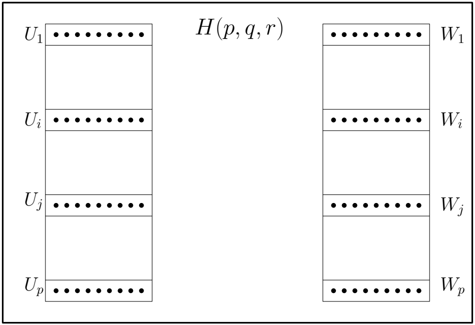


RBDS Incompressibility [Dom et al., 2011]

RBDS parameterized by $|T| + k$ is unlikely to admit a polynomial kernel.

Incompressibility: A construction with large degree and girth [Fomin et al., 2013]



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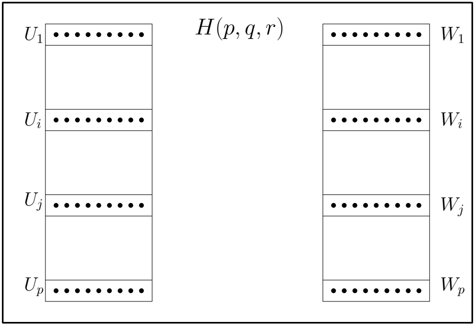

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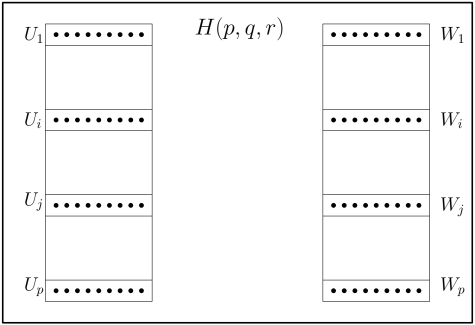

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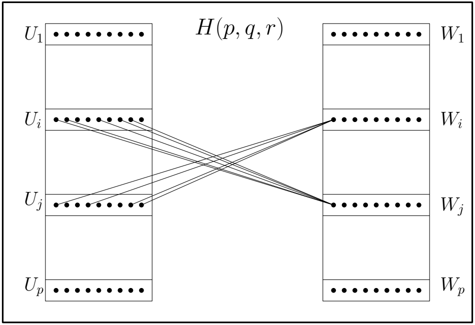

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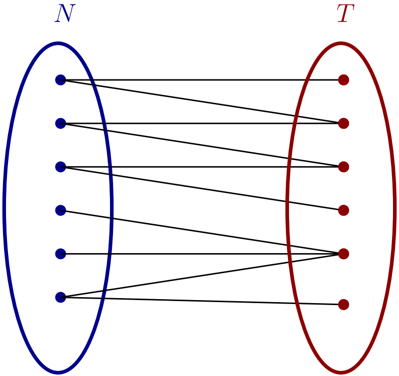
- Both U and W has exactly p blocks.
- Each block contains exactly q vertices.
- Each vertex in U_i has at least $r-1$ neighbours in each W_j, and vice-versa.

A construction with large degree and girth [Fomin et al., 2013]

Let $q \geq 2p(r+1)\frac{(p(r+1)-1)^6-1}{(p(r+1)-1)^2-1}$. Then, we can construct $H(p,q,r)$ in time $\mathcal{O}(r \cdot q \cdot p^2)$ with the following properties.

- The girth of $H(p,q,r)$ is at least 6.
- For every vertex $z \in V(H_{i,j})$ and every $i,j \in [p]$, we have $r-1 \leq \deg_{i,j}(z) \leq r+1$.

Incompressibility: Reduction



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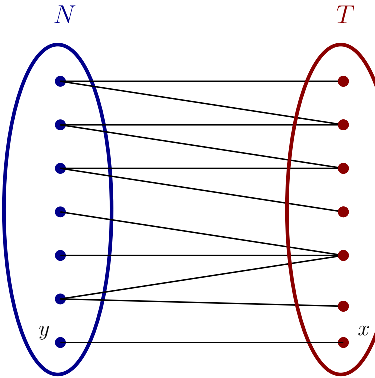

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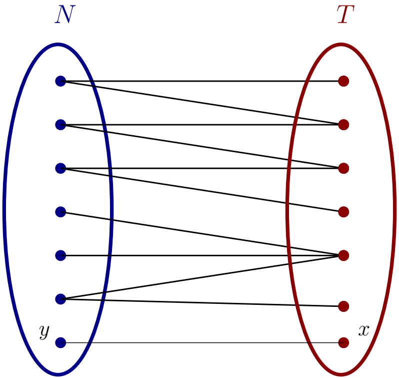

Set $p = |T|$ and $r = k + 2$.

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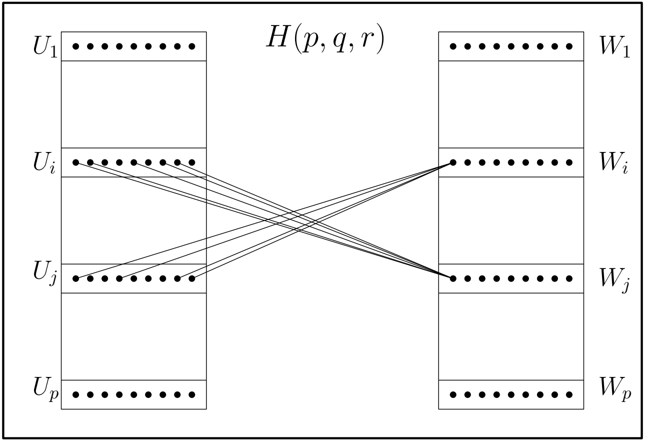

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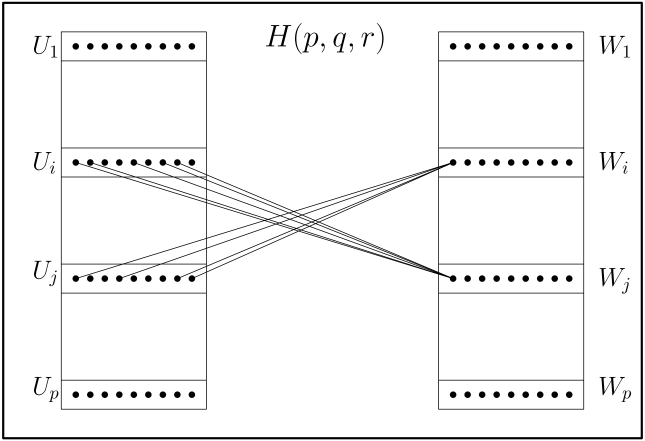

$p = |T|$ and $r = k + 2$. For $1 \leq i, j \leq p$, each vertex in U_i is connected to at least $k + 1$ vertices in W_j, and vice-versa.

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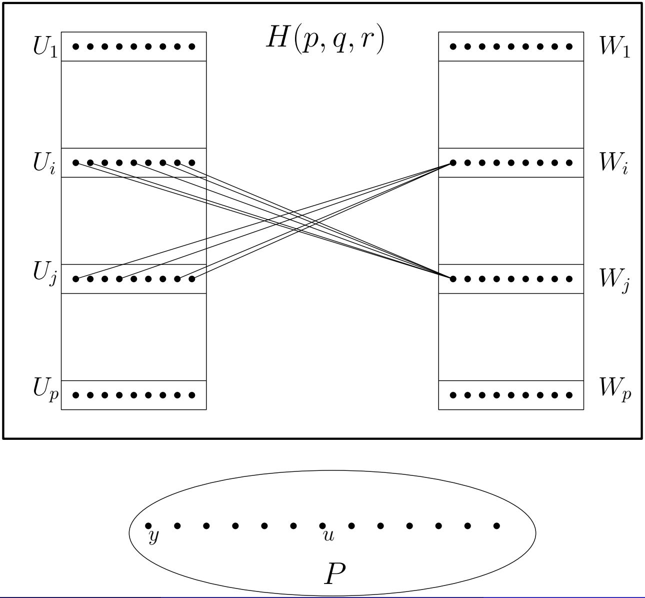

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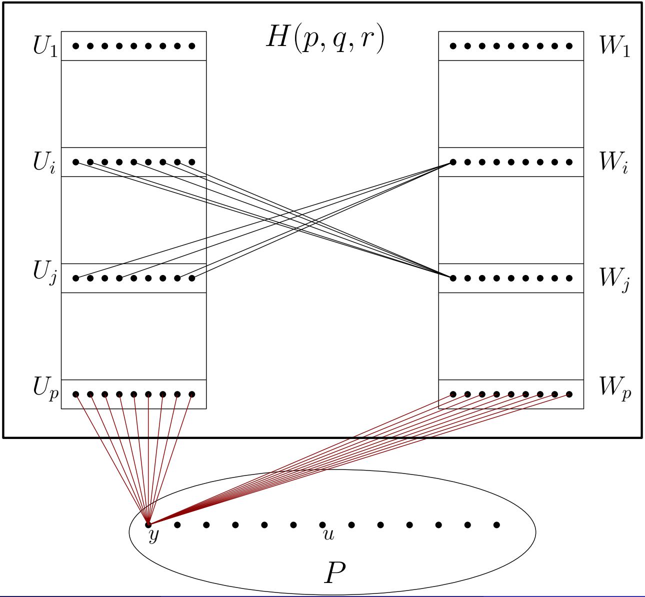

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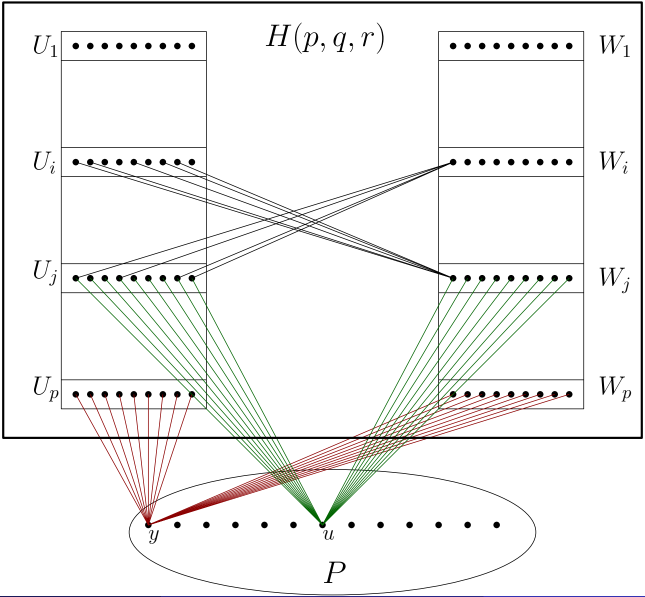

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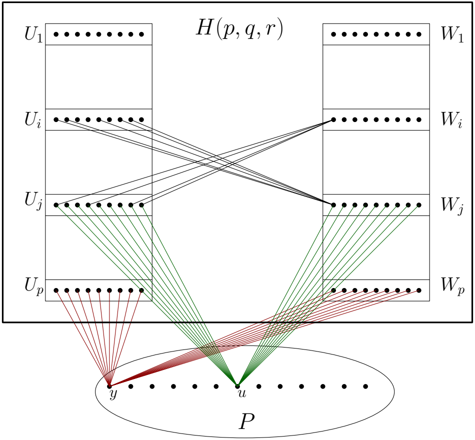
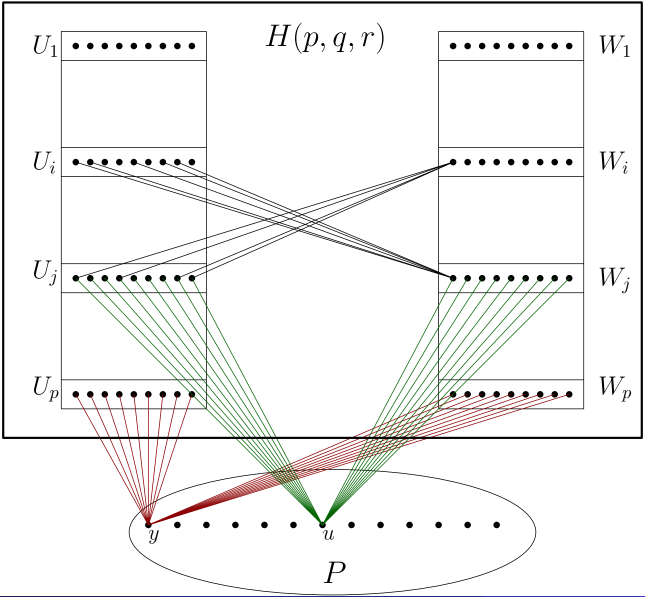
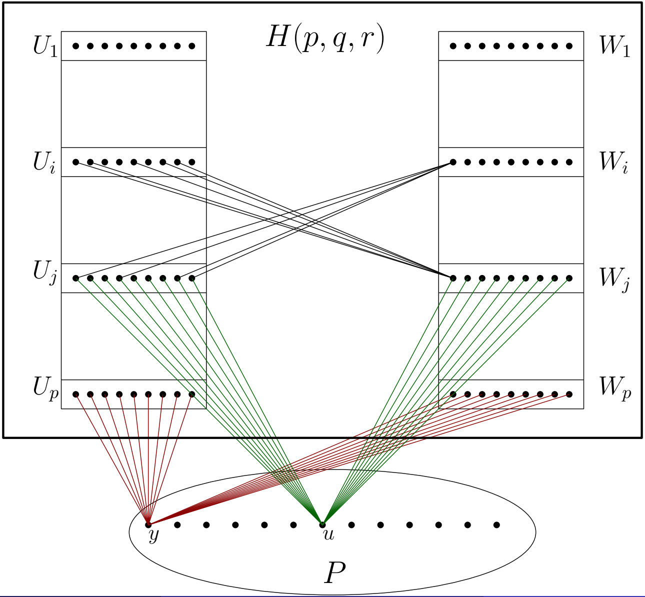


Finally, connect y to each vertex in P.

Reduction: RBDS of size $k \implies k$-copwin



Reduction: No RBDS of size $k \implies$ Not k-copwin



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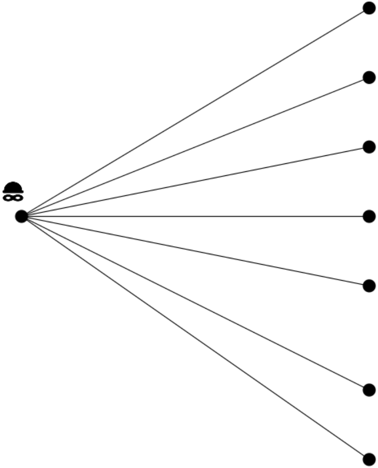

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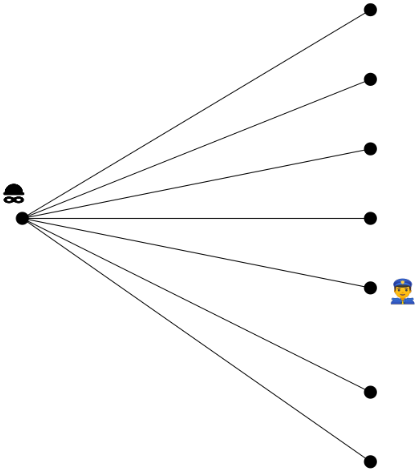

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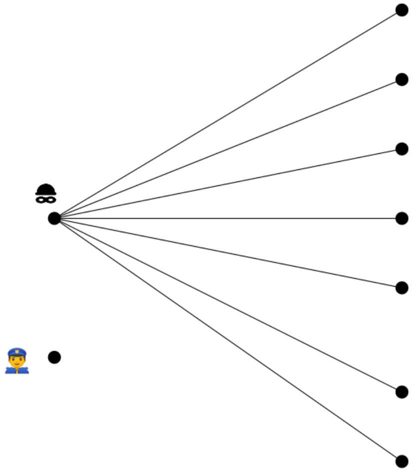

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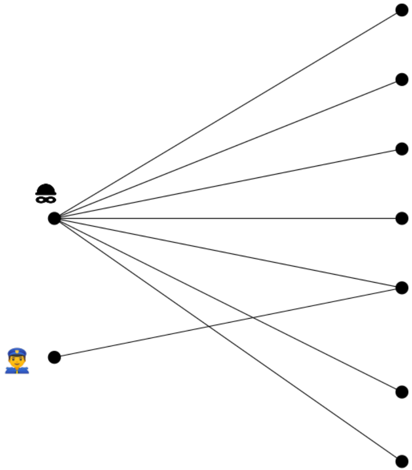

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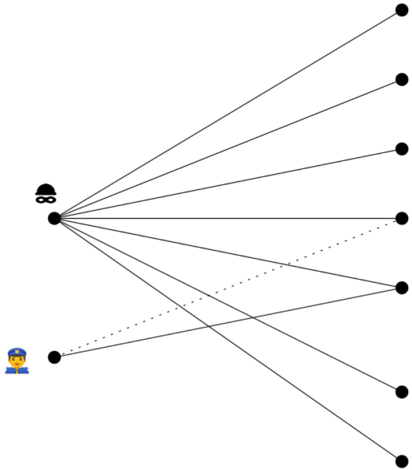

Variations Considered

Variations

- Cops and attacking Robber.
- Lazy Cops and Robber.
- Fully active Cops and Robber.
- Cops and fast Robber.
- Cops and Surrounding Robber
- Cops and Robber on oriented graphs.
- Generalized Cops and Robber.

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Kernelization

Generalized Cops and Robber respects keeping $k+1$ twins rule. Hence, it amits a kernel with at most $2^t.t + t$ vertices.

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- Is $c(G) = o(vc)$?
- Does Cops and Robber admit a single exponential FPT algorithm paramterized by vertex cover number?

Thank You