From binary search through games to graphs

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11^{th} Workshop on GRAph Searching,
Theory and Applications
Bertinoro October 26, 2023

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Outline

1. (A little dubious) motivation via multi-criteria optimisation
2. Multidimensional binary search as a game
3. Switch to graphs: edge and pair queries

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2. Multidimensional binary search as a game
3. Switch to graphs: edge and pair queries

Let us start with a definition!

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Definition (Renyi)

Mathematician is a device that turns coffee into theorems.

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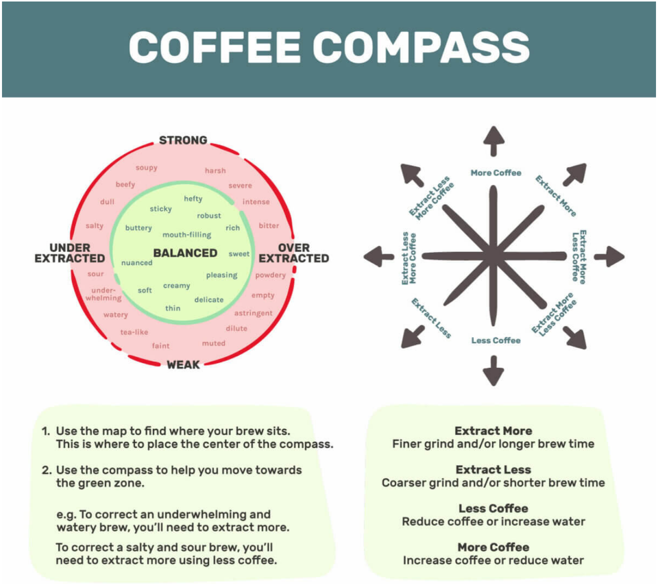
Mathematician is a device that turns coffee into theorems.

Hence, we need a good coffee.
(Computer Scientists too).

Recipe for good coffee



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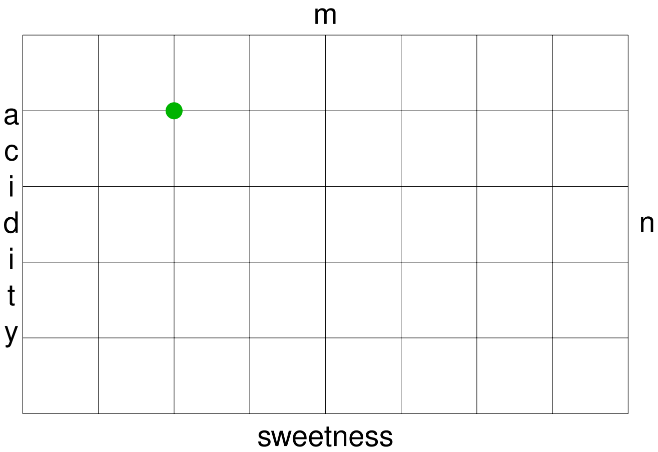

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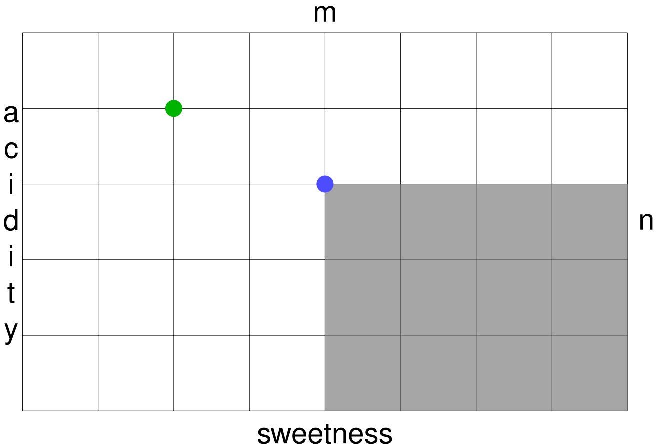

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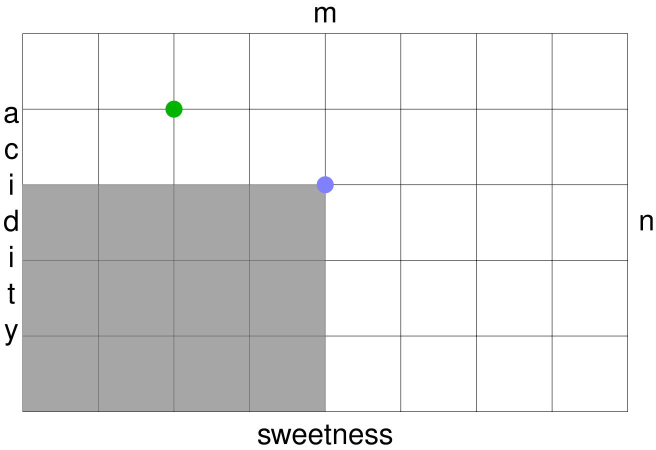
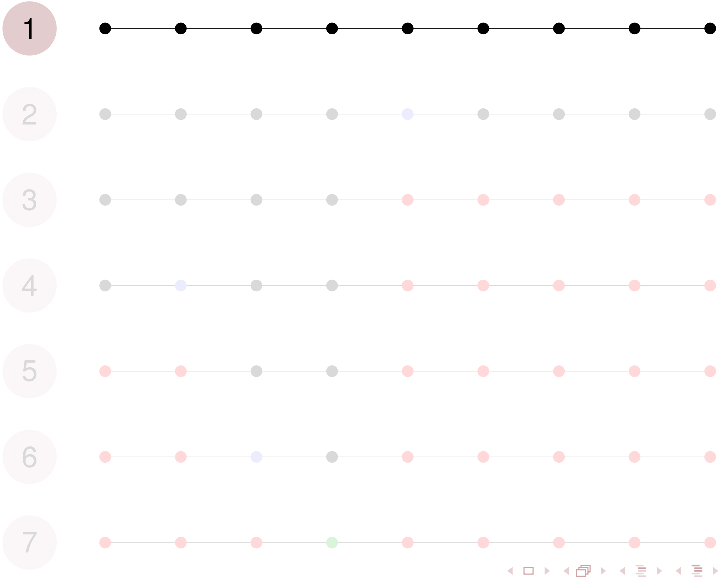


One dimensional solution — a binary search



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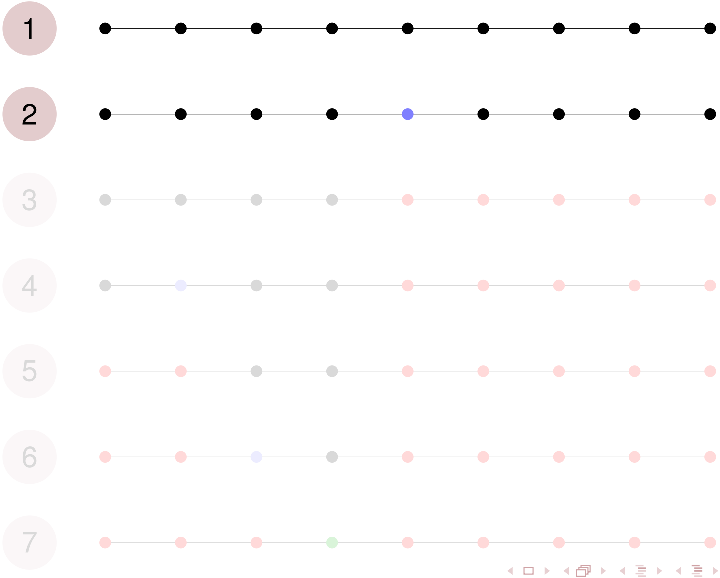

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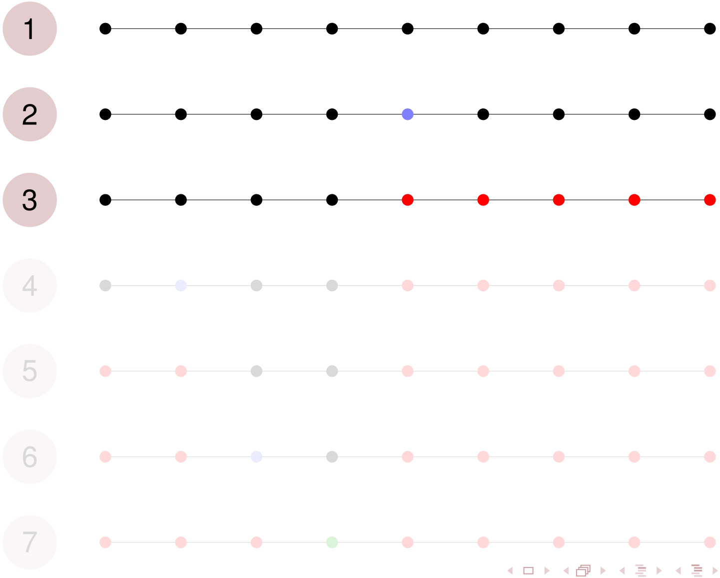

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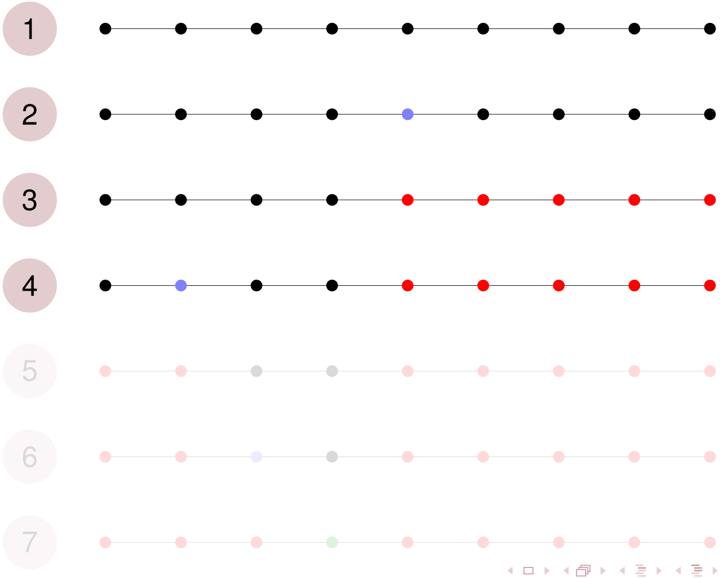

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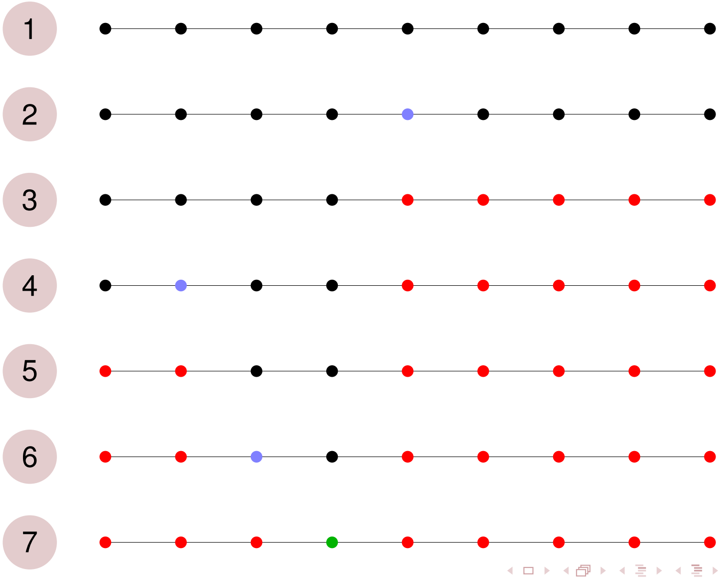
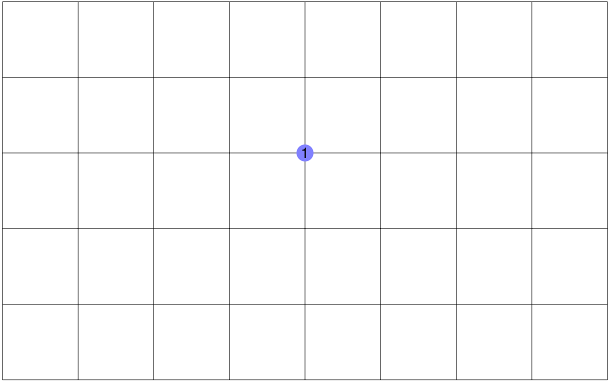


Outline

Multidimensional binary search as a game

1. A game between Algorithm and Adversary
2. A board: d-dimensional grid $n_1 \times n_2 \times \dots n_d$
3. Adversary hides a target in some point of the grid
4. At each step Algorithm picks a point $(x_1, x_2, \dots x_d)$
5. Adversary answers with (d-dimensional) interval with one end at $(x_1, x_2, \dots x_d)$ and another at some corner, where there is no target.
6. The Algorithm' goal is to find the target as soon as possible
7. Adversary wants to play long
8. It is good to think that there is no target point, but the area of possible target places shrinks at each step
9. Let $T(n_1, n_2 \dots n_d)$ be the number of steps provided both player play optimally. Surely $T(n) = \Theta(\log n)$.

2D example



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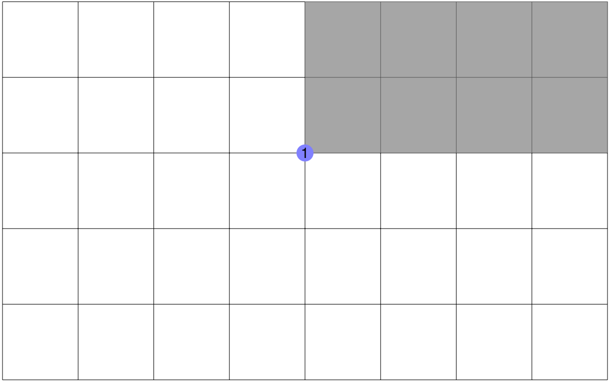

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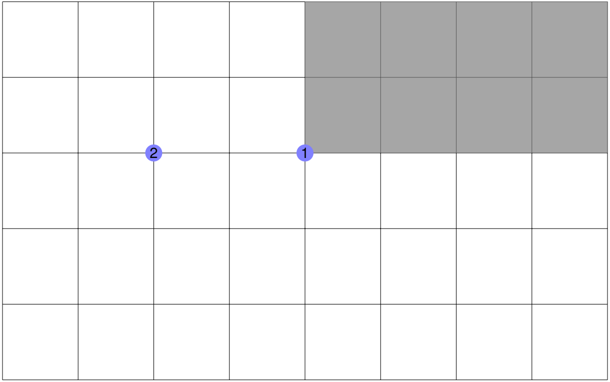

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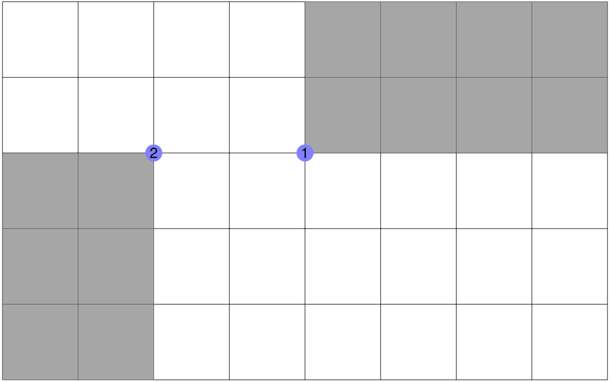

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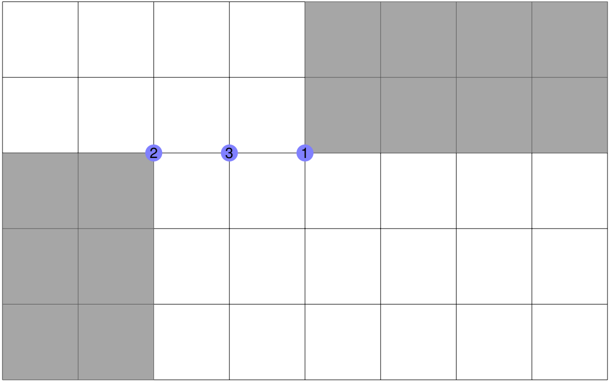

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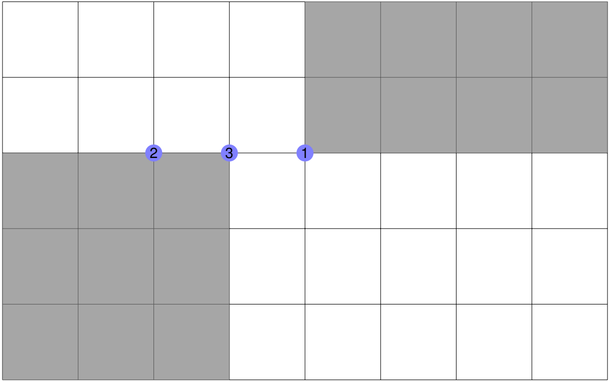

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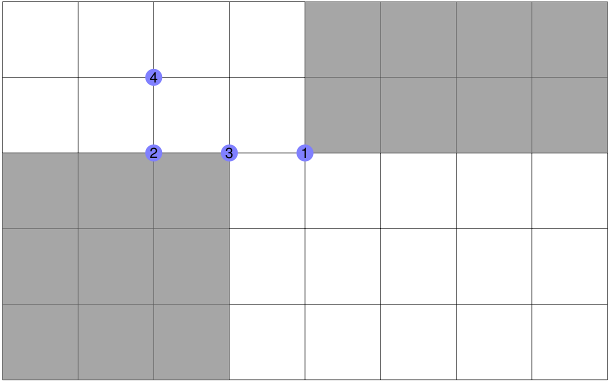

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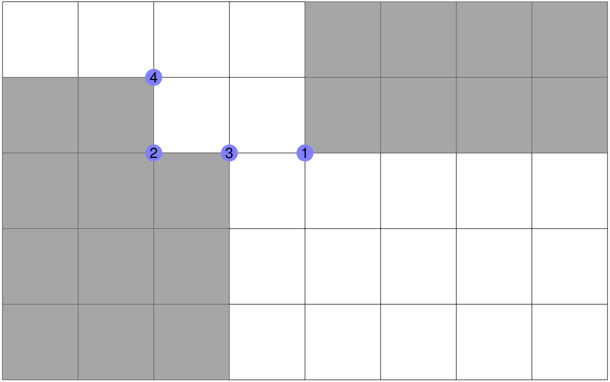

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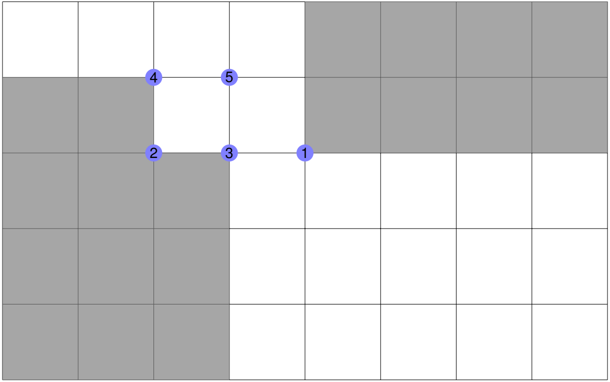

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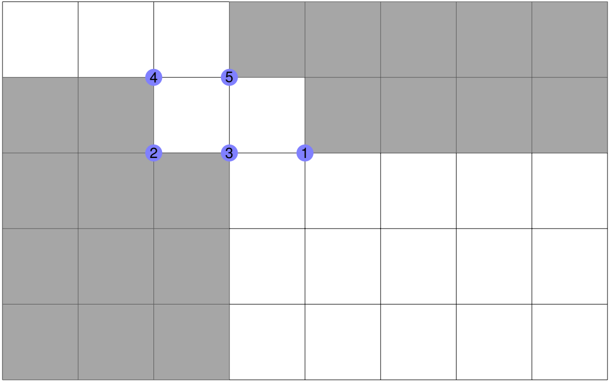

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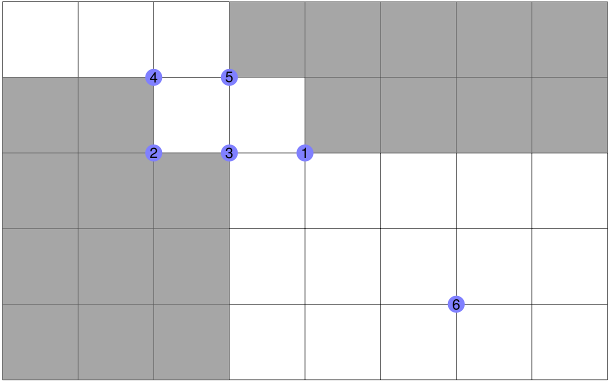

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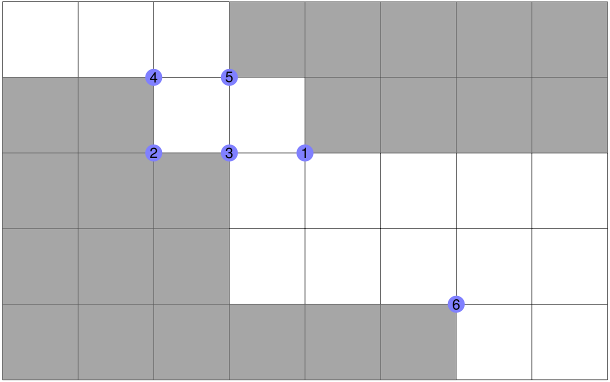

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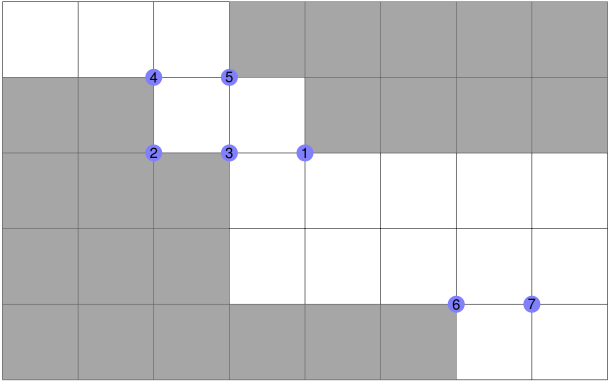

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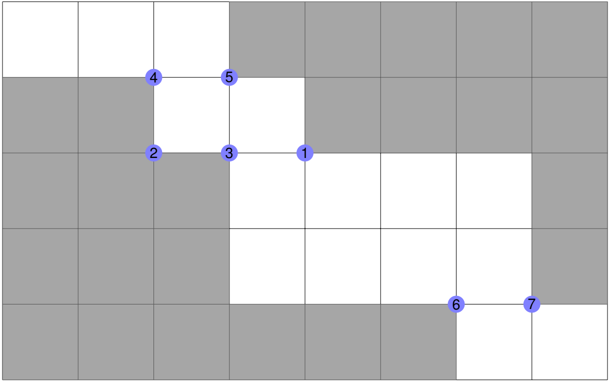

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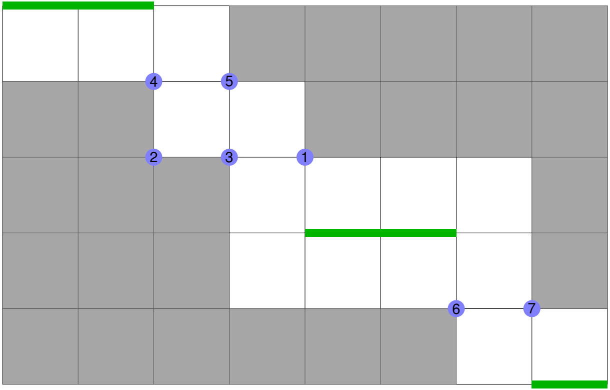

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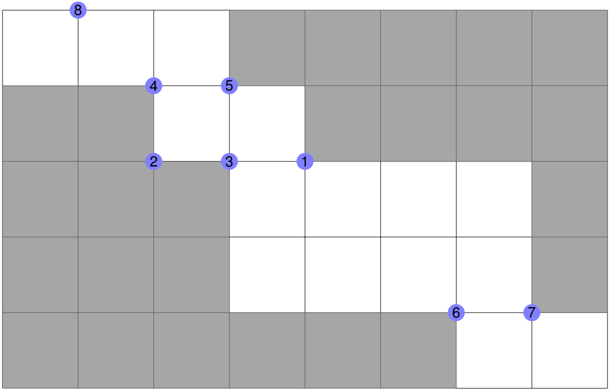

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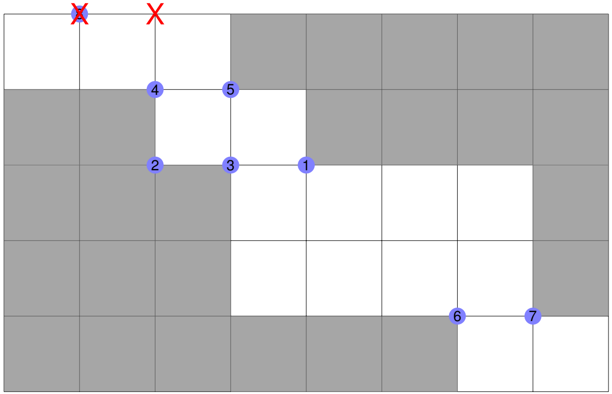

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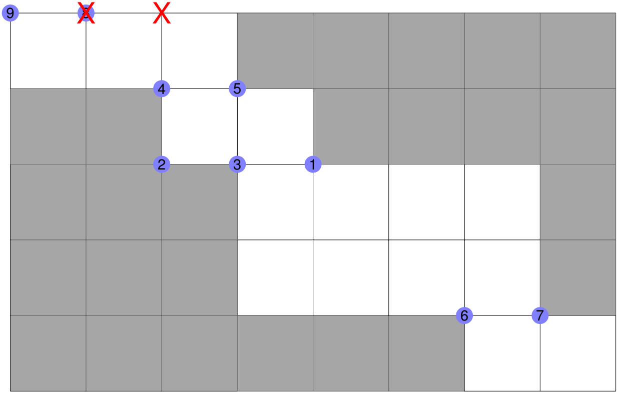

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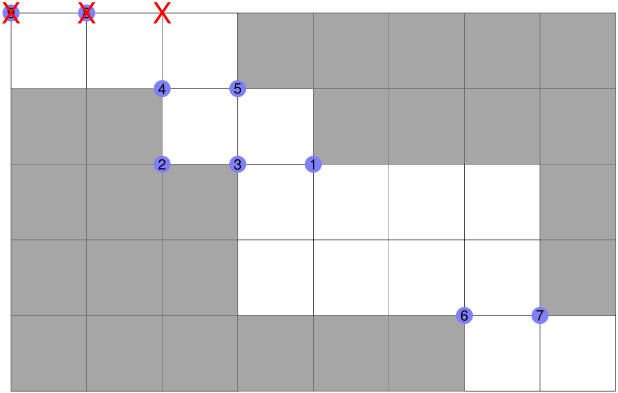

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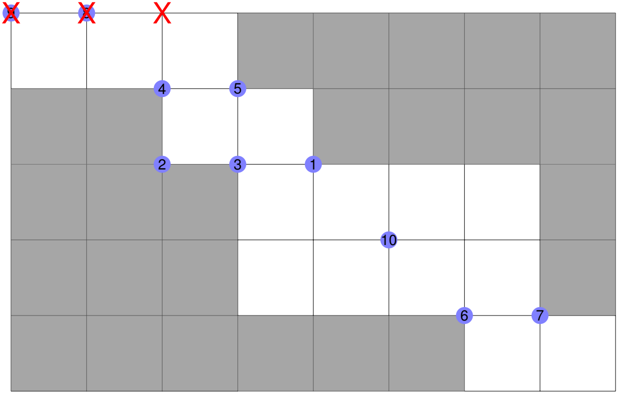

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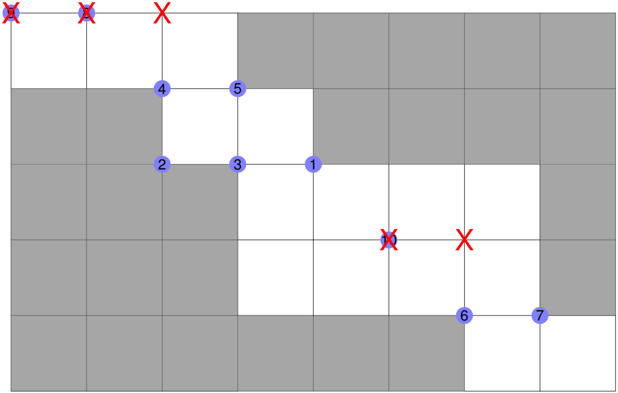

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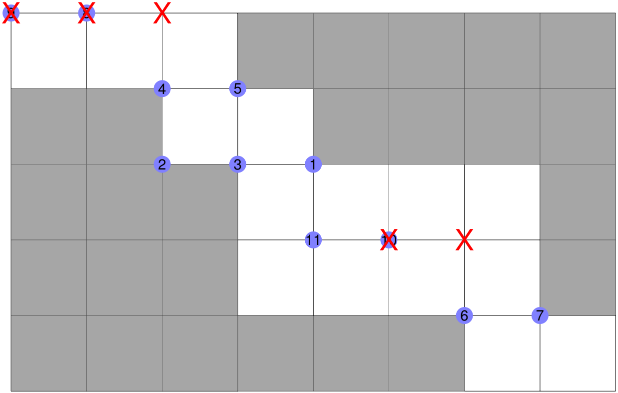

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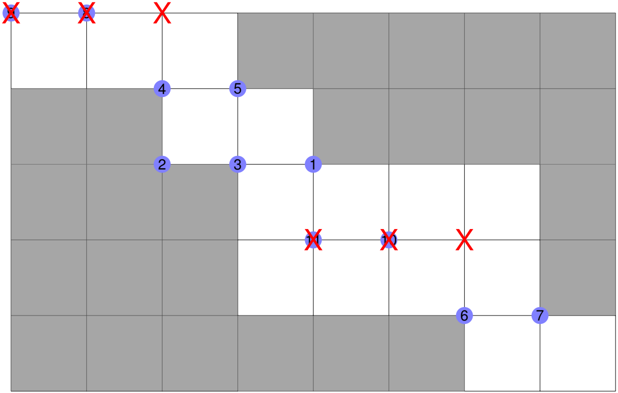

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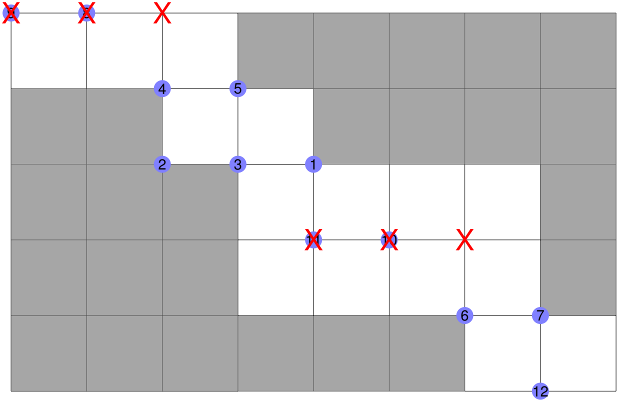

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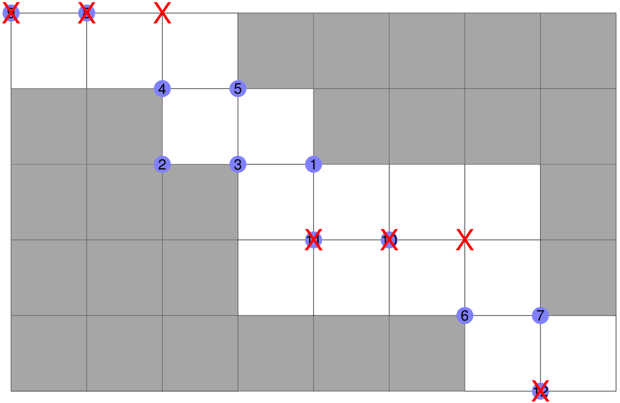

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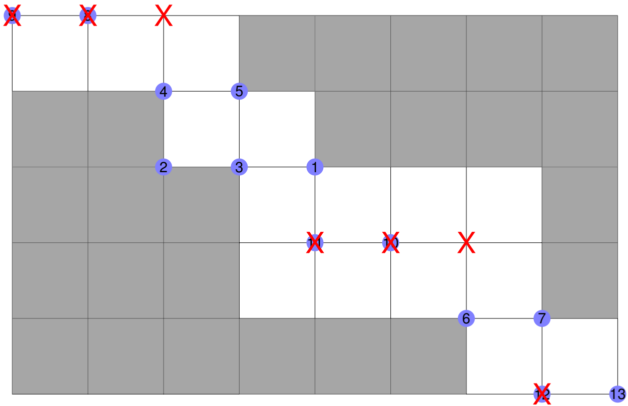

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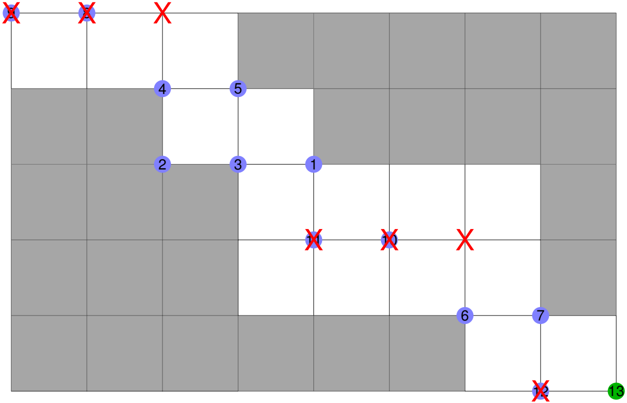

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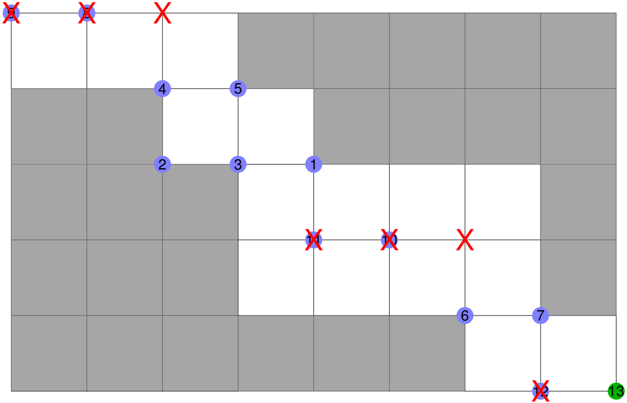
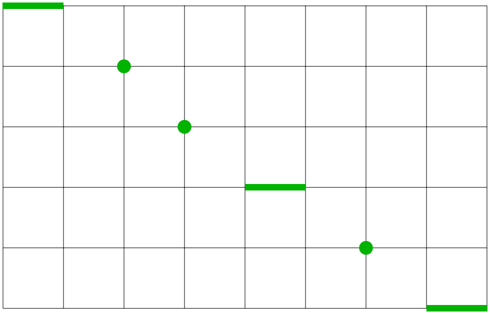


Theorem (Dereniowski, G., Wróbel)

Given $n \leq m$, $T(n,m) = \Theta(n \log(\frac{m}{n} + 1))$.

Informal proof (2D) $T(n,m) = \Theta(n \log(\frac{m}{n} + 1))$

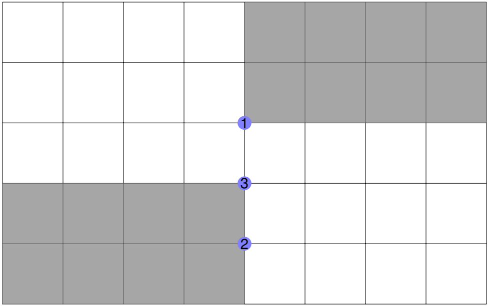
From below $T(n,m) = \Omega(n \log(\frac{m}{n} + 1))$:



- n (independent) segments with roughly $\frac{m}{n}$ elements,
- for each perform binary search with $\log_2 \frac{m}{n} + 1$ queries,

Informal proof (2D) $T(n, m) = \Theta(n \log(\frac{m}{n} + 1))$

From above $T(n, m) = \mathcal{O}(n \log(\frac{m}{n} + 1))$:



- $\log n$ queries to split the board into 2 (independent) pieces,
- for Adversary it is optimal to have them of the same size,
- hence $T(n, m) \leq \log n + 2T(\frac{n}{2}, \frac{m}{2})$
- after $\mathcal{O}(n)$ steps we are left with diagonal from the lower bound

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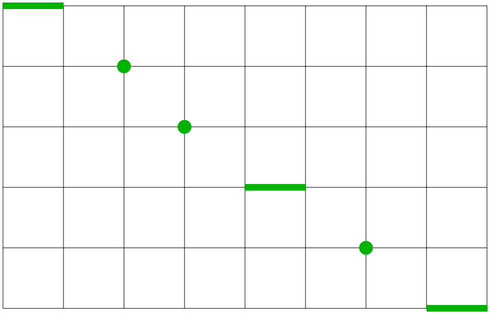

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d-dimensional case

Theorem (Dereniowski, G., Wróbel)

Given d and $n_1, \ldots n_d$, such that $n_i \leq n_{i+1}$ there is

$$T(n_1, \ldots, n_d) = \mathcal{O}\left(\left(\prod_{i=1}^{d-1} n_i\right) \log\left(\frac{n_d}{n_{d-1}} + 1\right)\right).$$

For proof: Algorithm consider n_1 $(d-1)$-dimensional cubes.

Theorem (Dereniowski, G., Wróbel)

For fixed d there is $T(n, \ldots, n) \geq \frac{2}{d-1} n^{d-1}$,
i.e $T(n, \ldots, n) = \Omega\left(\frac{n^{d-1}}{d}\right)$.

For proof: Adversary hides target at the hyperplane given by

$$x_d = n - \left\lceil \frac{x_1 + \cdots + x_{d-1}}{d-1} \right\rceil.$$

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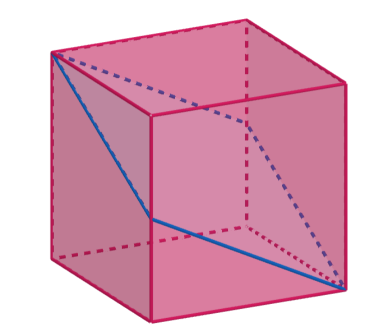

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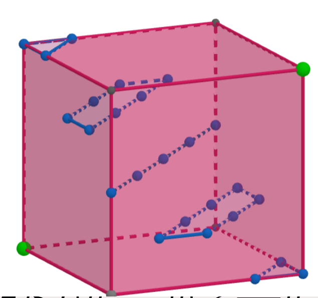

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For proof: Adversary hides target at the hyperplane given by

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d-dimensional case — further work

Theorem (Dereniowski, G., Wróbel)

For fixed d there is $T(n,\ldots,n) = \Omega\left(\frac{n^{d-1}}{d}\right)$ and $T(n,\ldots,n) = \mathcal{O}\left(n^{d-1}\right)$.

To do:

- Fill the gap.
- Lower bound when $(n_1,\ldots,n_d) \neq (n,\ldots,n)$ (we did it for $d \leq 3$).

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Why graphs? Graphical embeddings of data

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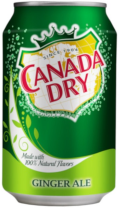

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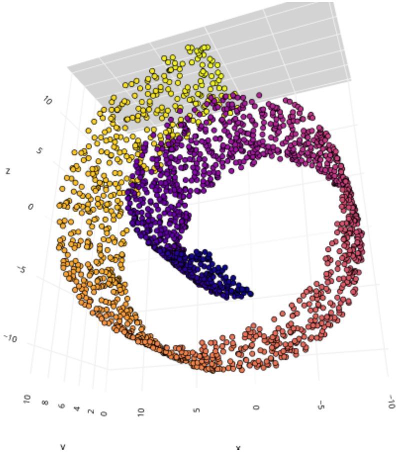

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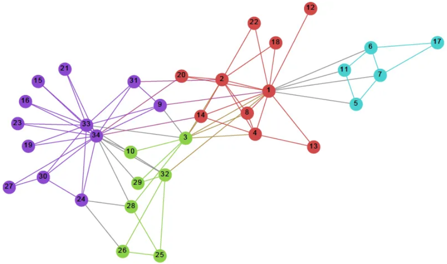

Edge and pair queries (Dereniowski, G., Prałat [2023])

1. A game between Algorithm and Adversary
2. A board: graph G
3. Adversary hides a target at some vertex $x \in V(G)$
4. At each step Algorithm picks two (adjacent) vertices u, v
5. Adversary answers which vertex is closer to x breaking ties arbitrary
6. The Algorithm' goal is to find the target as soon as possible
7. Adversary wants to play long
8. It is good to think that there is no target vertex, but the area of possible target places shrinks at each step
9. Let pq(G) (eq(G)) be the number of rounds provided both player play optimally. Surely pq(G) $\leq$ eq(G).

Edge and pair queries — simple properties

Observation

For any connected graph G on n vertices,

$$\log_2 n \leq \mathrm{pq}(G) \leq \mathrm{eq}(G) \leq n - 1.$$

In fact, there exists a strategy of the algorithm that in each round eliminates at least one vertex from the search space.

Lower bound is achieved by P_n (binary search), upper — by K_n and $K_{1,n-1}$.

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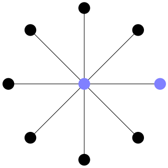

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For any $n \in \mathbb{N} \setminus \{1\}$, there exists a graph G on $\Theta(n)$ vertices such that $\mathrm{eq}(G) = \Omega(n)$ and $\mathrm{pq}(G) = \mathcal{O}(\log n)$.

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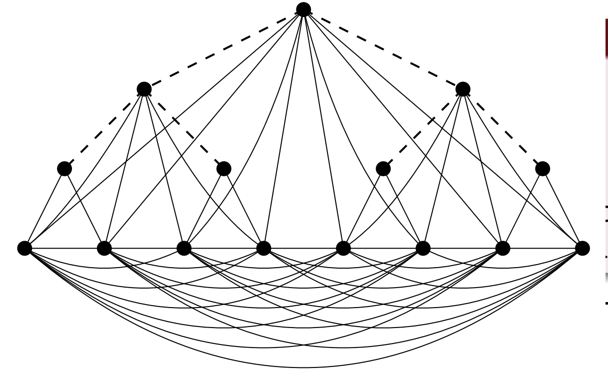

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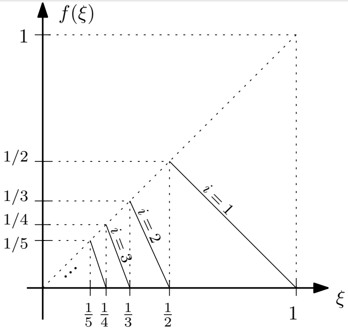
Edge and pair queries — results

Theorem (Dereniowski, G., Prałat [2023])

Suppose that $pn = n^{\xi+o(1)}$, where $\xi \in (\frac{1}{i+1}, \frac{1}{i})$ for some $i \in \mathbb{N}$. Then, a.a.s. for $G \in \mathcal{G}(n,p)$.

$$\mathrm{pq}(G) = \Theta\Big(\mathrm{eq}(G)\Big) = \Theta\left(n^{f(\xi)+o(1)}\right), \textit{ for } f$$



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For fixed $p \in (0,1)$ a.a.s. for $G \in \mathcal{G}(n,p)$.

$$\mathrm{pq}(G) = \Theta\Big(\mathrm{eq}(G)\Big) = \Theta\left(\log n\right).$$

Theorem (Dereniowski, G., Prałat [2023])

Calculating $\mathrm{pq}(G)$ or $\mathrm{eq}(G)$ is $\mathcal{NP}$-hard, even for graphs of diameter at most 3.

Edge and pair queries — questions

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What happen for particular graph classes?
(eg. grids, intersection graphs)

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Thank you for the attention!

Source of pictures:
www.homegrounds.co (Coffee Compass)
some commercials (cold drink samples)
towardsdatascience.com (graph embeddings)