Edge Periodic Temporal Graphs

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[◇] Universität Trier - Germany

Time-Varying Graphs

- Graphs that vary over time, e.g., edge availability or edge transition time changes over time.
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- Model among others dynamic networks e.g. mobile ad hoc networks and vehicular networks.
- Different settings in literature: discrete vs. continuous time; explicitly vs. implicitly given.
- Special case edge periodic graphs (EPG) where availability of each edge e is given by a periodic discrete function $\tau(e)$.

Formal Definition

Edge Periodic Graph (EPG)

Time domain $\mathbb{N}$.

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Edge periodic graph $\mathcal{G} = (V, E, \tau)$ with underlying graph $G = (V, E)$ and function $\tau \colon E \to \{0,1\}^*$ where $e \in E$ is present in time step $t \geq 0$ iff $\tau(e)[t]^\circ = 1$ where $\tau(e)[t]^\circ := \tau(e)[t \mod |\tau(e)|]$.

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Least common multiple of all edge periods $\mathrm{lcm}(L_{\mathcal{G}})$.

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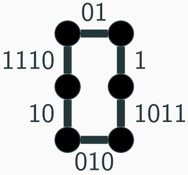
$\mathcal{G}$ is called edge periodic cycle (EPC) iff underlying graph G is a cycle.

Edge Periodic Temporal Graphs



EPG $\mathcal{G}$ together with snapshots $\mathcal{G}(t)$ for $0 \leq t \leq 4$.
In general sequence repeats only after $\text{lcm}(L_{\mathcal{G}})$ (exponential).

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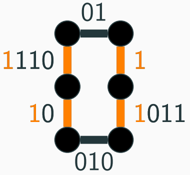

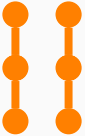

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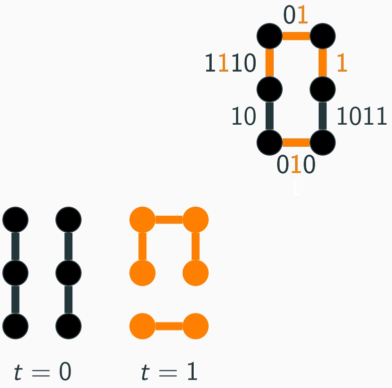

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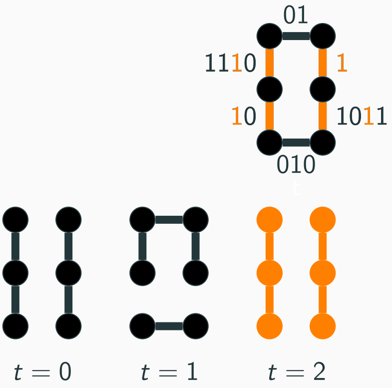

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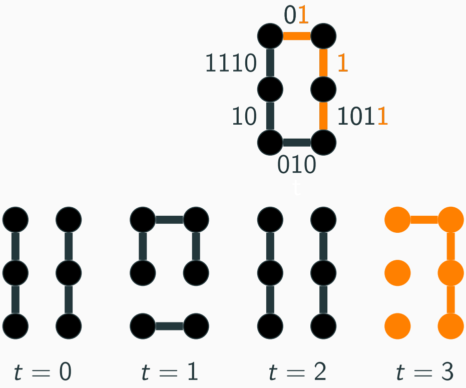

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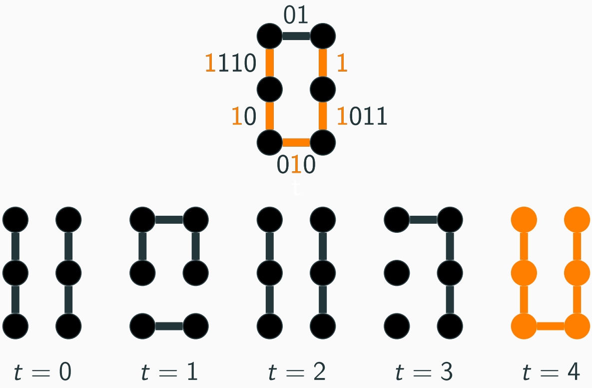

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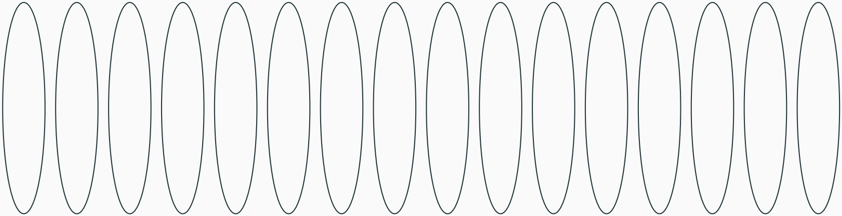
Motivation

Normal temporal graphs



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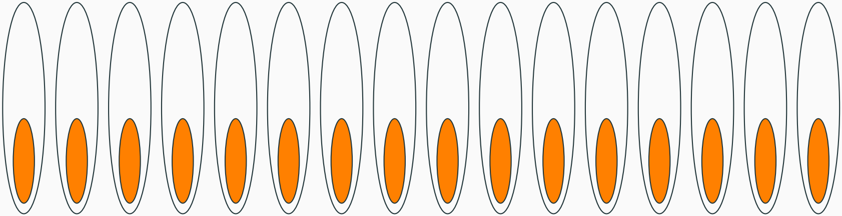

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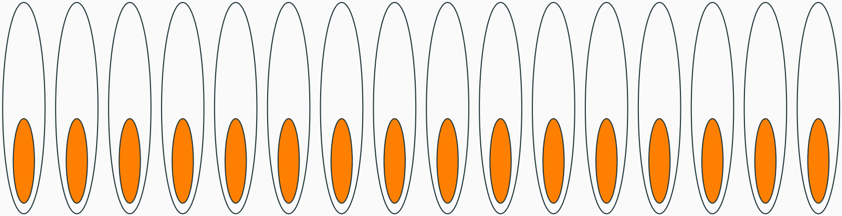
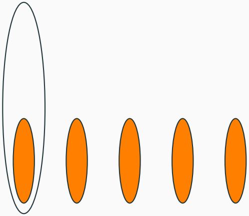


Edge periodic temporal graphs



The Game of Cops and Robbers

Line of Research

- *A Game of Cops and Robbers on Graphs with Periodic Edge-Connectivity*
  Thomas Erlebach, Jakob T. Spooner - SOFSEM 2020
- *A Timecop's Work Is Harder Than You Think*
  Nils Morawietz, Carolin Rehs, Mathias Weller - MFCS 2020
- *A Timecop's Chase Around the Table*
  Nils Morawietz, Petra Wolf - MFCS 2021

The Game of Cops and Robbers

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- Cops win if they can catch the robber, i.e., move to vertex occupied by robber.
- If k cops can catch robber on G, call G k-cop-winning.
  For $k = 1$ call G cop-winning.

Timecops

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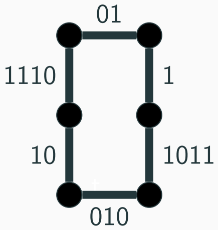

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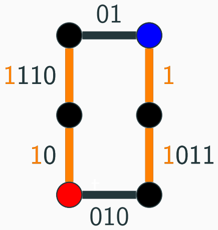

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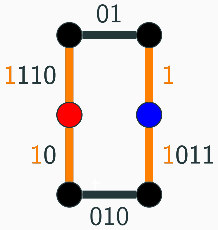

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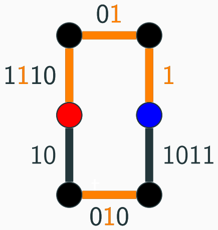

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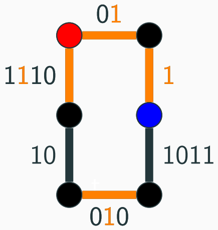

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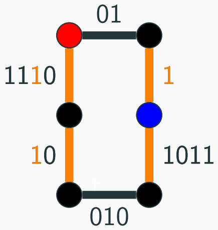

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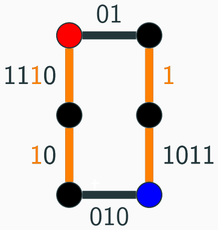

Periodic Cops & Robbers

[Erlebach, Spooner 2020]:

- For EPG: Algo in time $\mathcal{O}(\mathrm{lcm}(L_{\mathcal{G}}) \cdot n^3)$ (implicit EXPTIME upper bound).
- Open Question: Is the problem NP-hard?

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- Family of EPCs giving lower bounds with a gap of $\frac{1}{4}$ times upper bound.

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[Morawietz, Rehs, Weller 2020]:

- Lower Bounds for EPG: NP-hard and W[1]-hard in $|G|$ for EPGs even if the underlying graph G has a constant size VC or a constant distance to clique.

Our Results

New [Morawietz, Wolf 2021]:

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- **Periodic Cops & Robbers** is NP-hard and W[1]-hard in $|G|$ for directed and undirected cycles.
- PSPACE upper bound for **Periodic Cops & Robbers** for general EPGs.

Length Required to Ensure Robber-Winning EPCs

E&S: An EPC $\mathcal{G}$ is robber-winning if $|V| \geq 2 \cdot \ell \cdot \mathrm{lcm}(L_{\mathcal{G}})$ where $\ell = 1$ if $\mathrm{lcm}(L_{\mathcal{G}}) \geq 2 \cdot \max(L_{\mathcal{G}})$ and $\ell = 2$ otherwise.

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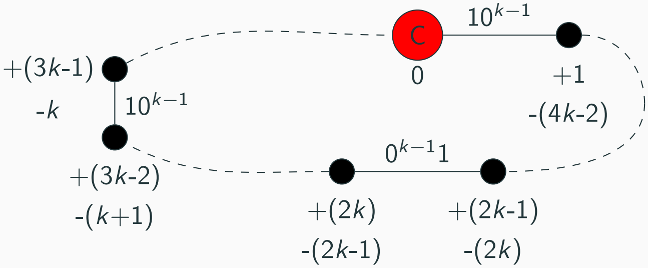
Theorem

For every $k \geq 3$, $\ell \in \{1, 2\}$, there exists a cop-winning EPC $\mathcal{G} = (V, E, \tau)$ with $\max(L_\mathcal{G}) = k$ and $2 \cdot \ell \cdot \text{lcm}(L_\mathcal{G}) - 1$ vertices.

Case $\ell = 2$, $\text{lcm}(L_\mathcal{G}) = k$:



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Case $\ell = 2$,
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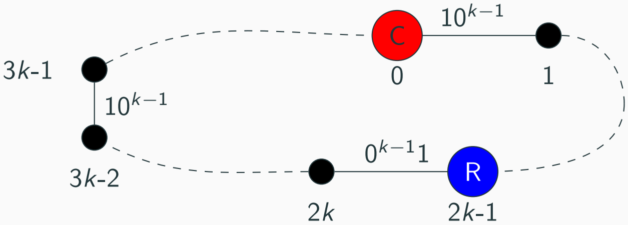
Time step t = start



Situation after moving in time step t.

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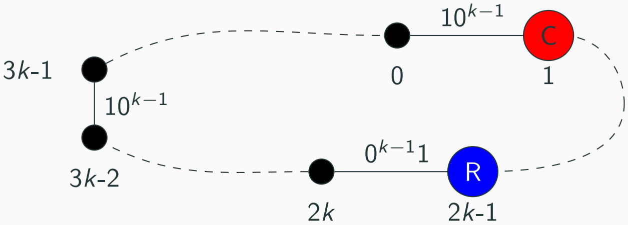
Time step $t = 0$



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Case $\ell = 2$,
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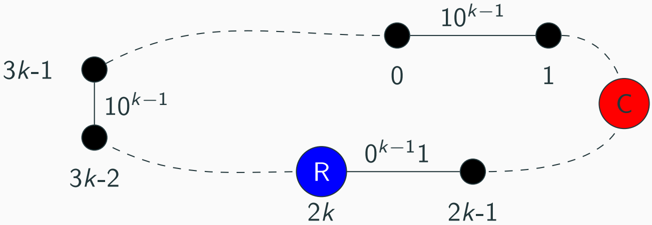
Time step $t = k - 1$



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Case $\ell = 2$,
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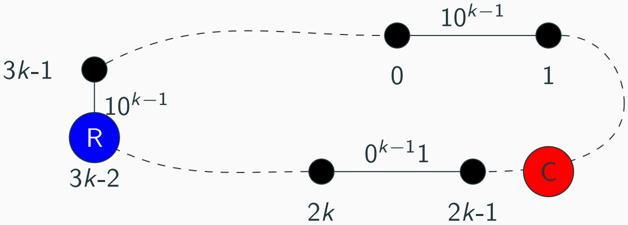
Time step $t = 2k - 3$



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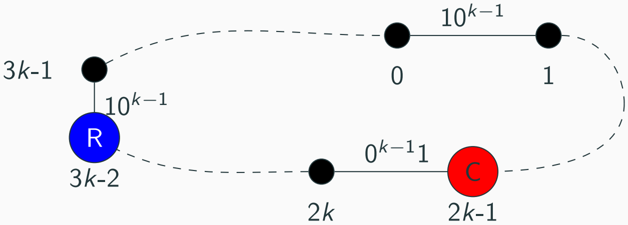

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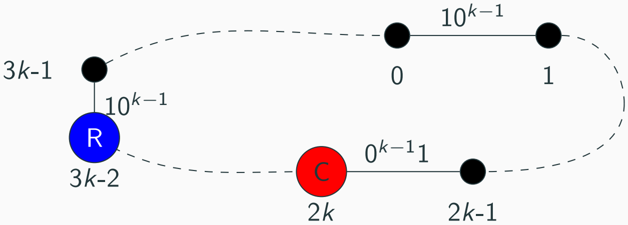

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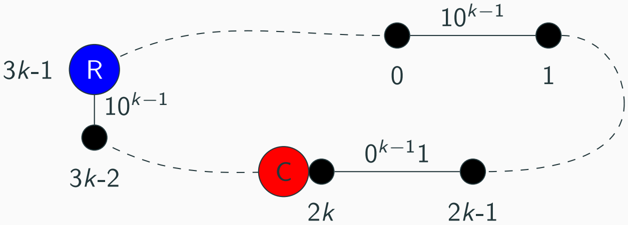

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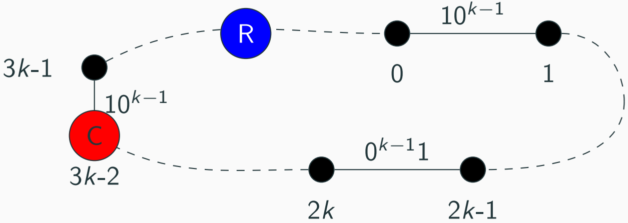
Time step $t = 3k - 3$



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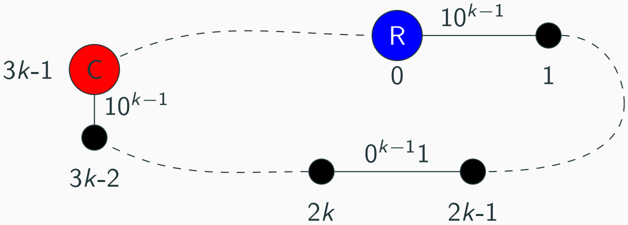

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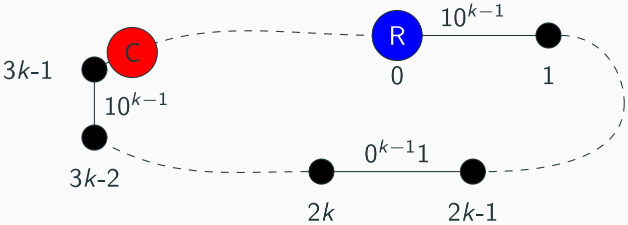
Time step $t = 3k + 1$



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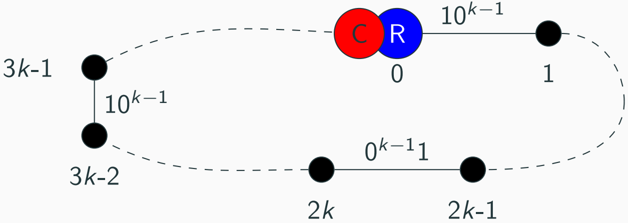

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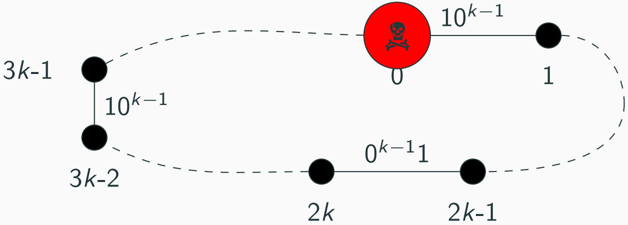

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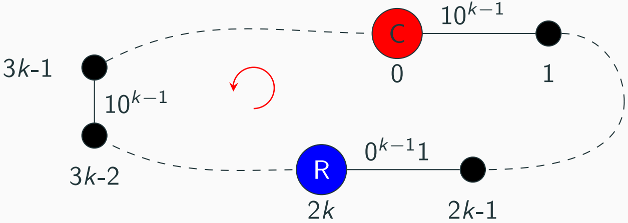
Time step t = start



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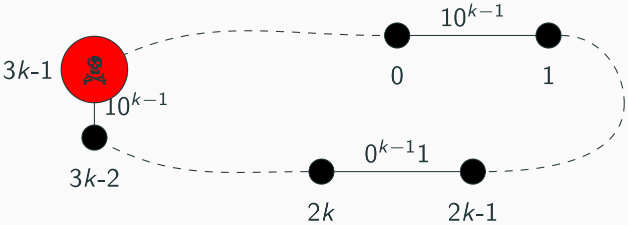
Time step $t = 5k$



Situation after moving in time step t.

Computational Complexity

Periodic Character Alignment

Periodic Character Alignment (PCA)
Input: A set $X \subseteq \{0,1\}^*$ of finite binary sequences
Question: Is there an index $i \geq 0$ such that $x[i]^\circ = 1$ for all $x \in X$, where $x[i]^\circ := x[i \bmod |x|]$?

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10101010101010101010...

Periodic Character Alignment

Periodic Character Alignment (PCA)
Input: A set $X \subseteq \{0,1\}^*$ of finite binary sequences
Question: Is there an index $i \geq 0$ such that $x[i]^\circ = 1$ for all $x \in X$, where $x[i]^\circ := x[i \bmod |x|]$?

$$
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00100010001000100010\ldots \\
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\end{array}
$$

$\uparrow$
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$$\begin{array}{l} 01001\,0\mathbf{1}00101001\mathbf{0}1001\ldots \\ 0010\,00\mathbf{1}0001000\mathbf{1}00010\ldots \\ 10\,1010\mathbf{1}0101010\mathbf{1}01010\ldots \\ \qquad\quad \uparrow \\ \qquad\quad i{=}6 \end{array}$$

Periodic Character Alignment is NP-complete and W[1]-hard in parameter $|X|$, [Morawietz, Rehs, Weller 2020].

Reduction from Multi-Colored Clique [Morawietz et al. 2020]

Multi-Colored Clique (MCC)
Input: An integer k and an undirected k-partite graph $G = (V_1 \uplus V_2 \uplus \ldots \uplus V_k, E)$.
Question: Is there a clique of size k in G?

Reduction from Multi-Colored Clique [Morawietz et al. 2020]

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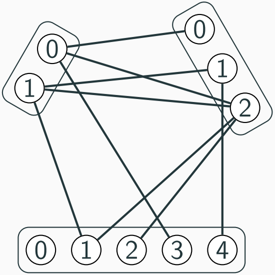

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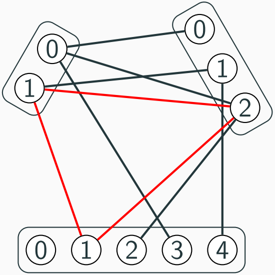

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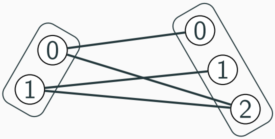

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Let $a \in \{0, 1\}$ and $b \in \{0, 1, 2\}$: there is a position $i \in \{0, 1, \ldots, 5\}$ such that $i \mod 2 = a$ and $i \mod 3 = b$.

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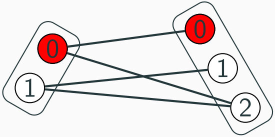

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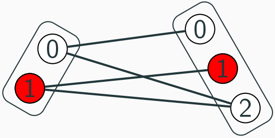

11????

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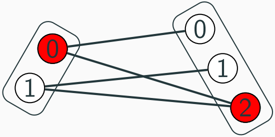

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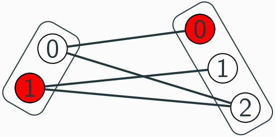

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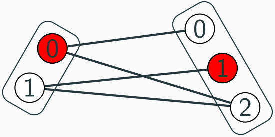

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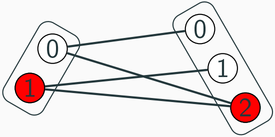

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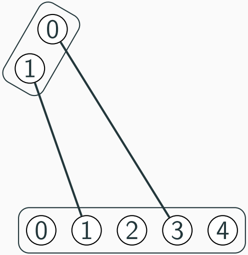

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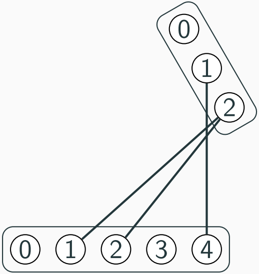

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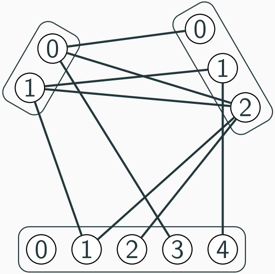

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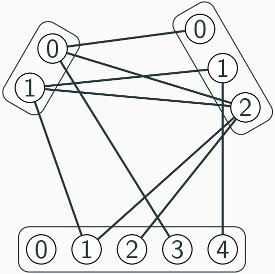

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↑
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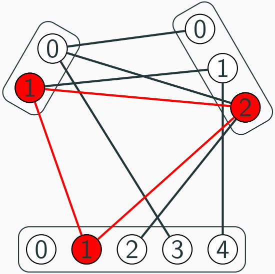

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Relation to Tally Intersection

0 1 0 0 1

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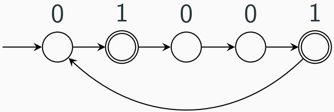

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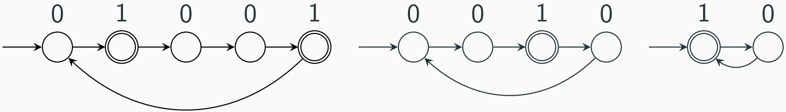

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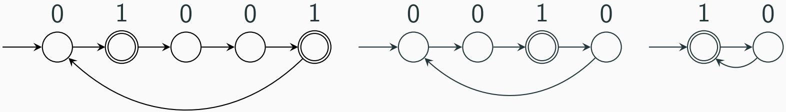

Tally Intersection $\hat{=}$ PCA

Question: is there a word $w \in \Sigma^*$ which is accepted by all the automatons of X?

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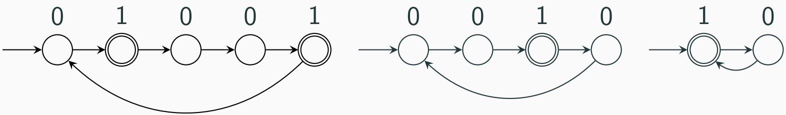

Tally Intersection $\hat{=}$ **PCA**

Question: is there a word $w \in \Sigma^*$ which is accepted by all the automatons of X?

Theorem

Tally Intersection *and* **PCA** *are W[1]-hard when parameterized by* $|X|$.

No $f(|X|) \cdot n^{\mathcal{O}(1)}$ time algorithm for any computational function f, unless FPT $=$ W[1].

It's Hard to Run Around a Table

Theorem

Periodic Cops & Robbers *on directed and undirected EPCs is NP-hard and W[1]-hard in the size of the underlying graph.*

(Undirected case:) Reduction from **PCA**

Input: A set $X \subseteq \{0,1\}^*$ of finite binary sequences

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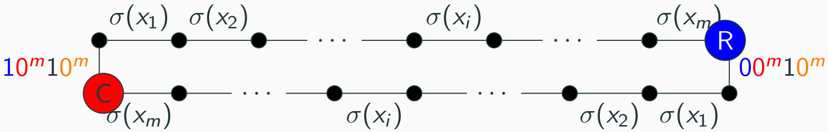
$t \cdot (2m+3) + m + 1$



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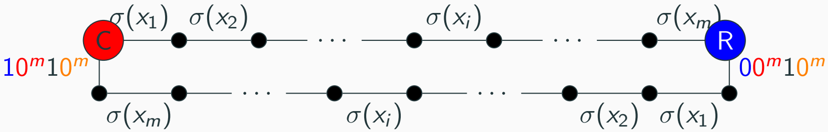

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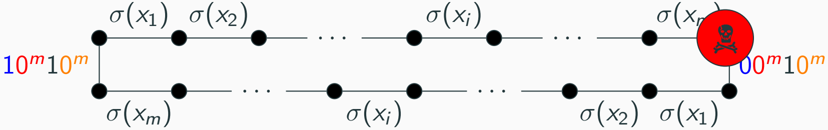

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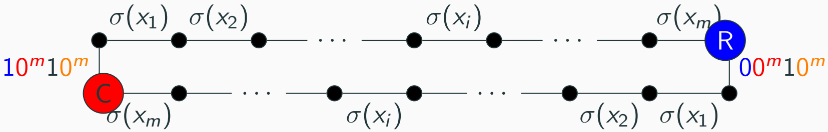

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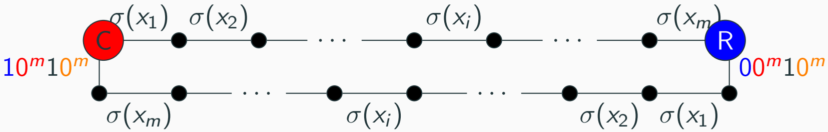

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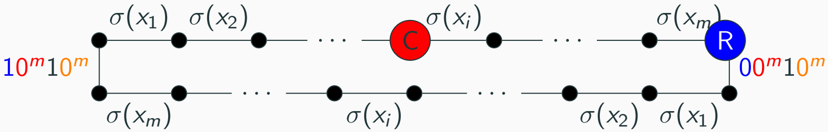

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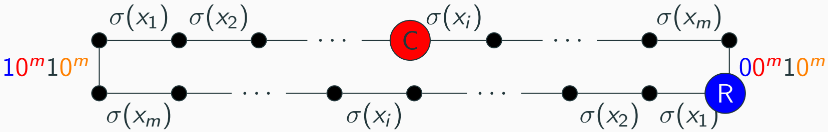

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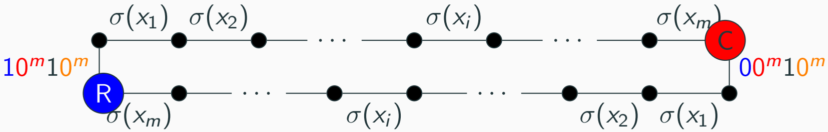

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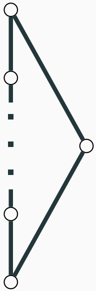

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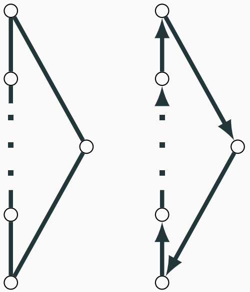

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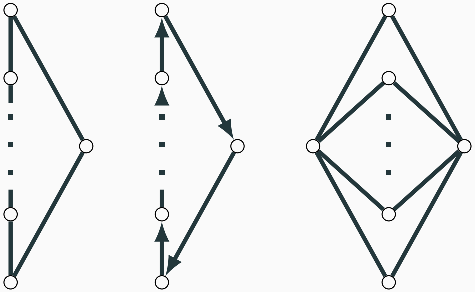

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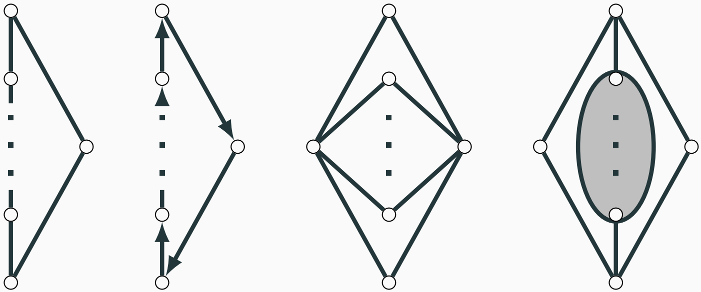

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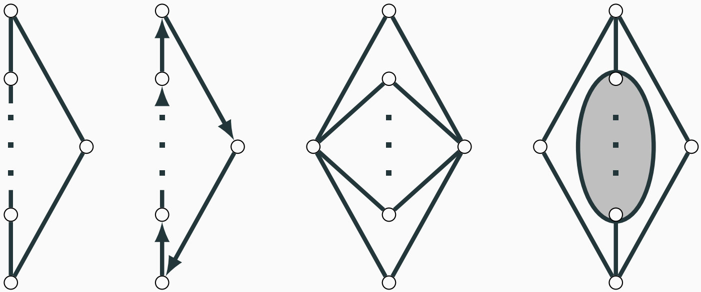

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General Complexity Upper Bound

Theorem

The **Periodic Cops & Robbers** *problem on EPGs is in PSPACE.*

General Complexity Upper Bound

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The **Periodic Cops & Robbers** *problem on EPGs is in PSPACE.*

- If EPG $\mathcal{G}$ is cop-winning, then cop wins within $n^2 \cdot \text{lcm}(\mathcal{G})$ moves.
- Enroll the graph $2 \cdot n^2 \cdot \text{lcm}(\mathcal{G})$ times to get leveled DAG.

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- Solve AND-OR GRAPH REACHABILITY problem on leveled DAG.
- Starting in level $\ell = 2 \cdot n^2 \cdot \mathrm{lcm}(L_{\mathcal{G}}) - 1$ and compute set of attractors for each level.
- $\mathcal{G}$ is cop-winning iff some vertex with $t = 0$ is an attractor.

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- Only keep levels and attracotrs with $\ell = i$ and $\ell = i - 1$ in memory.
- In each level ℓ only n^2 many vertices in this level $\rightarrow$ PSPACE.

Open Question

A Timecop's Chase Around the Table

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- Gap between NP-hardness and membership in PSPACE.

Open Question

A Timecop's Chase Around the Table

- Gap between NP-hardness and membership in PSPACE.
- Can we close this gap at least for EPCs?
- Problem: Certificate may not be small.
- Family of EPCs with exponential chase?

Finding Minors and Subgraphs

Short Traversal

EPG Short Traversal

Input: EPG $\mathcal{G} = (V, E, \tau)$, vertices $a, b \in V$, and $k \in \mathbb{N}$.

Question: Is there a time step t such that starting from a at time step t, we can reach b at the beginning of time step $t + k$ while traversing at most one edge per time step?

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EPG Short Traversal *is FPT with parameter* $|G| + \#1_{\text{max}}$.

$\#1_{\text{max}}$ = maximal number of ones in an edge label.

EPG Short Traversal: FPT in $|G| + \#1_{\max}$

Theorem

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- Iterate through all possible (a, b) paths in G.
- Path assumed to be vertex simple $\rightarrow$ bounded by $|G|$.
- For each path, check time constraint via ILP formulation.

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- For each path, check time constraint via ILP formulation.
- ILP Formulation:
  - Binary variable for each 1 in a edge label in current path.
  - Constraint: cross edges in order of the path.
  - Constraint: cross an edge only if it is present.
  - Constraint: cross each edge exactly once.

Minor Problems

EPG Minor
Input: EPG $\mathcal{G} = (V, E, \tau)$, graph $H = (V_H, E_H)$.
Question Is there a time step t, s.t. H is a minor of $\mathcal{G}(t)$?

EPG Minor-Free
Input: EPG $\mathcal{G} = (V, E, \tau)$, graph $H = (V_H, E_H)$.
Question: Is there a time step t, s.t. H is not a minor of $\mathcal{G}(t)$?

Minor EPGs vs. Classic Graphs

On classic graphs:

Both problems are FPT in the size of the minor.

Minor EPGs vs. Classic Graphs

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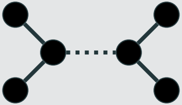
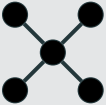
Theorem

EPG Minor *is NP-complete and W[1]-hard with parameter $|G|$ for*

- *fixed H containing a cycle*
- *fixed forest H with a connected component containing*



or



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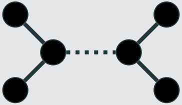

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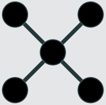
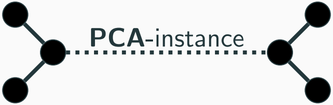


Proof:



Minor-Free EPGs vs. Classic Graphs

On classic graphs:

Both problems are FPT in the size of the minor.

Theorem

For every fixed minor containing at least one edge, **EPG Minor-Free** *is NP-complete and W[1]-hard with parameter* $|G|$.

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$\overline{x_1}$ $\overline{x_2}$ $\overline{x_3}$ $\overline{x_4}$ $\overline{x_5}$ $x_1, \ldots, x_5 \in$ **PCA**

H H H H H copies of minor

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Theorem

EPG Minor-Free *is* Σ_P^2*-complete.*

Reduction from ∃∀**3UNSAT** using reduction from **3SAT** to **Clique**.

Subgraph Problems

EPG Subgraph
Input: EPG $\mathcal{G} = (V, E, \tau)$, graph $H = (V_H, E_H)$.
Question: Is there a time step t, s.t. H is a subgraph of $\mathcal{G}(t)$?

EPG Subgraph-Free
Input: EPG $\mathcal{G} = (V, E, \tau)$, graph $H = (V_H, E_H)$.
Question: Is there a time step t, s.t. H is not a subgraph of $\mathcal{G}(t)$?

Subgraphs are meant to be **induced** subgraphs.

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On classic graphs:

The problems are NP-, resp. coNP-complete and XP in $|H|$.

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Theorem

EPG Sugraph *is NP-complete and W[1]-hard parameterized by* $|G|$*.*
This holds even if $\underline{H \text{ is a path and } G = H}$*.*

Proof: Put **PCA** on path.

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EPG Subgraph *can be solved in* $\mathcal{O}(n^h \cdot \max(L_{\mathcal{G}})^{(h^2)} \cdot 2^{\mathcal{O}(\sqrt{h \log h})})$ *where* h *is the number of vertices in* H*.*

Subgraph EPGs vs. Classic Graphs

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Proof:

- Iterate over subsets $W \subseteq V$ of size h in G
- For subgraph, sequence of snapshot graphs repeat after $\max(L_\mathcal{G})^{h^2}$ time steps
- For each $t \leq \max(L_\mathcal{G})^{h^2}$ check if $\mathcal{G}(t)[W] \cong H$

Subgraph-Free EPGs vs. Classic Graphs

On classic graphs:

The problems are NP-, resp. coNP-complete and XP in $|H|$.

Theorem

For (nontrivial) fixed subgraph H, **EPG Subraph-Free** *is NP-complete and W[1]-hard parameterized by $|G|$.*

Theorem

EPG Subgraph-Free *is Σ_P^2-complete.*

Both proofs similar to minor case.

FPT-Algorithms

$\#1_{\max}$ [$\#0_{\max}$] = maximal number of ones [zeros] in an edge label.

Theorem

EPG Subgraph(-Free) *and* **EPG Minor(-Free)** *are FPT with parameter* $|G| + \min(\#1_{\max}, \#0_{\max})$.

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EPG Subgraph(-Free) *and* **EPG Minor(-Free)** *are FPT with parameter* $|G| + \min(\#1_{\max}, \#0_{\max})$.

Proof:

- Iterate over all possible graphs $F \subseteq G$
- Check that F does (not) contain H
- If yes, check that $F = \mathcal{G}(t)$ for some t
  - Use FPT-algorithm for PCA as subroutine
  - **PCA** is FPT in total number of consecutive groups of 1's.
  - Number of consecutive groups of 1's is bounded by $|G| + \#1_{\max}$ and $|G| + \#0_{\max}$.

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EPG Subgraph(-Free) *and* **EPG Minor(-Free)** *are NP-hard even if G is a disjoint union of paths and* $\#1_{\text{max}} \in \mathcal{O}(1)$ *or* $\#0_{\text{max}} \in \mathcal{O}(1)$.

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Theorem

EPG Subgraph(-Free) *and* **EPG Minor(-Free)** *are W[1]-hard with parameter vertex cocver number of G even if* $\#1_{\text{max}} = 1$.

Summary

	EPGs	**classic graphs**
Minor	NP-c also for fixed H	FPT in $\|H\|$
Minor-free	Σ_P^2-c and NP-c for fixed H	FPT in $\|H\|$
Subgraph	NP-c and XP in $\|H\|$	NP-c and XP in $\|H\|$
Subgraph-free	Σ_P^2-c and NP-c for fixed H	coNP-c and XP in $\|H\|$

Thank You!

A Timecop's Chase Around the Table

Multi-Parameter Analysis of Finding Minors and Subgraphs in Edge Periodic Temporal Graphs