Tree decompositions with bounded independence number and their algorithmic applications

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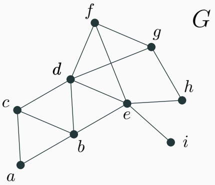
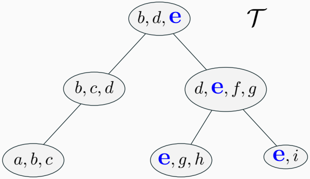
The **treewidth** of a graph G, denoted by $\text{tw}(G)$, measures how similar the graph is to a tree.

A **tree decomposition** of a graph G is a collection $\mathcal{T} = (X_t)_{t \in V(T)}$ of **bags** (subsets of $V(G)$) arranged into a tree T such that

- each vertex of G is in a bag,
- for each edge of G, both endpoints are in a bag, and
- for each vertex v of G, the nodes of T whose bags contain v form a subtree.





The **width** of $\mathcal{T}$ is the size of the largest bag minus one.
The **treewidth** of G is the minimum width over all possible tree decompositions.

Tree decompositions are a great tool for **dynamic programming**.

In particular, this is the case for MAXIMUM WEIGHT INDEPENDENT SET:

- We are given a graph $G = (V, E)$ and a weight function $w : V \to \mathbb{Q}_+$.
- The task is to find an independent set I in G of maximum possible weight $w(I)$, where

$$w(I) = \sum_{x \in I} w(x)\,.$$

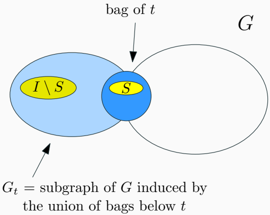
independent set (or: stable set): a set of pairwise non-adjacent vertices

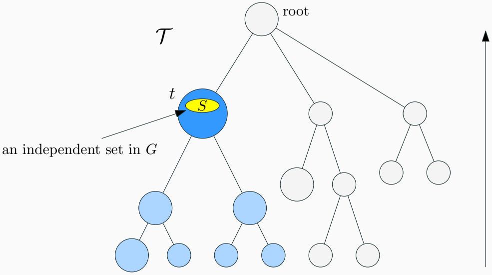
In general, the MAXIMUM WEIGHT INDEPENDENT SET problem is strongly NP-hard and also NP-hard to approximate on n-vertex graphs to within a factor of $n^{1-\epsilon}$ for every $\epsilon > 0$ (Zuckerman, 2007).

Dynamic programming for MAXIMUM WEIGHT INDEPENDENT SET:

Root the tree decomposition and traverse it bottom-up.

For each node t and each independent set S contained in the bag of t, we compute the maximum weight of an independent set I in G_t that agrees with S in the bag.

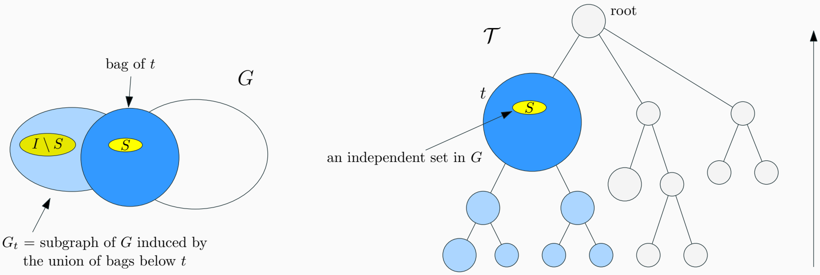




If each bag has at most a constant number k of vertices,
we only need to examine $\leq 2^k$ choices for S.
This computation can be done recursively, using separation properties of tree decompositions.

This idea can be generalized:

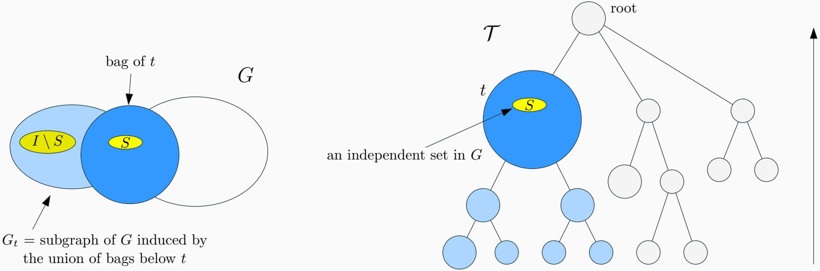
Suppose that in each bag we only need to examine **a polynomial number** of choices for S, say $\mathcal{O}(n^k)$ where $n = |V(G)|$ and k is a constant.



If G is given with such a tree decomposition, we still obtain a polynomial-time algorithm for Maximum Weight Independent Set.

In particular, this is the case if bags may be large, but each bag can intersect an optimal solution in a bounded number of vertices

= **each independent set contained in a bag is small**.



This approach was suggested independently by Maria Chudnovsky in a number of talks (Stony Brook, October 2020; Berlin, April 2021; Charles University, May 2021; invited talk at IWOCA 2021).

Tree-independence number: definition

The **independence number** of a graph G, denoted by $\alpha(G)$, is the maximum size of a independent set in G.

The **independence number** of a tree decomposition of a graph G is the maximum, over all bags of the decomposition, of the independence number of the subgraph of G induced by the bag.

The **tree-independence number** of G is the minimum independence number over all tree decompositions.

Notation: tree-$\alpha(G)$

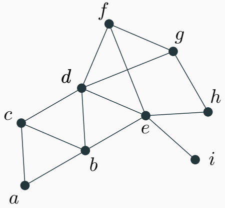
Similar invariants were studied in the literature with respect to various properties of the bags:

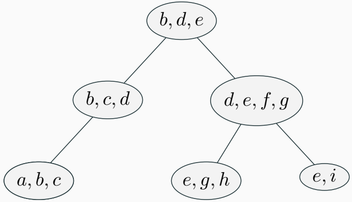
- connectivity properties (connected tree-width, Müller, 2012; Diestel and Müller, 2018, bag-connected treewidth, Jégou and Terrioux, 2014),
- metric properties (tree-length, Dourisboure and Gavoille, 2007, tree-breadth, Dragan and Köhler, 2014),
- chromatic properties (tree-chromatic number, Seymour, 2016).

Basic results about the tree-independence number

tree-$\alpha(G) \leq 1$: G has a tree decomposition in which each bag is a clique

This happens if and only if G is chordal.





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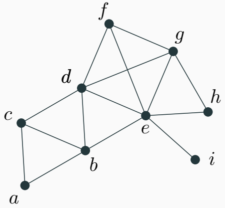
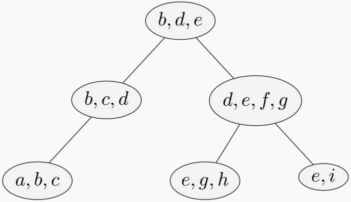




Many known graph classes have bounded tree-independence number:

- Graph classes of bounded treewidth:

$$\text{tree-}\alpha(G) \leq \mathrm{tw}(G) + 1\,.$$

  (This bound is sharp: for every positive integer $k \neq 2$, there exists a graph G such that $\text{tree-}\alpha(G) = \mathrm{tw}(G) + 1 = k$.)

- Graph classes of bounded independence number:

$$\text{tree-}\alpha(G) \leq \alpha(G)\,.$$

- Classes of graphs in which all minimal separators are of bounded size.

(Skodinis, 1999)

- Intersection graphs of connected subgraphs of graphs with treewidth t (Bodlaender, Gustedt, Telle, 1998)

$$\text{tree-}\alpha(G) \leq t + 1\,.$$

  Such graphs admit tree decompositions in which each bag can be covered by $t + 1$ cliques.

- ***H*-graphs**, that is, the intersection graphs of connected subgraphs of a subdivision of a fixed multigraph H.

  Introduced in 1992 by Biró, Hujter, and Tuza,

  studied more recently by
  Chaplick and Zeman (2017),
  Chaplick, Töpfer, Voborník, and Zeman (2017),
  Fomin, Golovach, and Raymond (2020),
  Chaplick, Fomin, Golovach, Knop, and Zeman (2021).

Computing the tree-independence number of a given graph is **NP-hard**.

- We reduce from the INDEPENDENT SET problem:
  given a graph G, the graph G' obtained from two copies of G by adding all edges between them satisfies

$$\text{tree-}\alpha(G') = \alpha(G)\,.$$

  - $\text{tree-}\alpha(G') \leq \alpha(G') = \alpha(G)$
  - $\text{tree-}\alpha(G') \geq \alpha(G)$, since every tree decomposition has a bag containing a vertex and all its neighbors.

Further properties of tree-α:

- Does not increase under **vertex deletions and/or edge contractions**.
- Behaves well with respect to **clique cutsets**:

  Let G_1 and G_2 be disjoint graphs except that they agree in a clique C, and let G be their union. Then

$$\text{tree-}\alpha(G) = \max\{\text{tree-}\alpha(G_1), \text{tree-}\alpha(G_2)\}\,.$$

An algorithmic application:

Theorem

For every integer $k \geq 1$, Maximum Weight Independent Set *is solvable in time $\mathcal{O}(|V(G)|^{k+1} \cdot |V(T)|)$ if the input vertex-weighted graph G is given with a tree decomposition $\mathcal{T} = (X_t)_{t \in V(T)}$ with independence number at most k.*

A generalization: independent packing problems

$\mathcal{H}$: a set of connected graphs

Given a graph G, denote by $\mathcal{H}(G)$ the following graph:

- the vertices are the subgraphs of G isomorphic to a member of $\mathcal{H}$
- an edge between two vertices means that the corresponding subgraphs of G either have a vertex in common or there is an edge between them.

Special cases:

- if $\mathcal{H} = \{K_1\}$, then $\mathcal{H}(G) = G$
- if $\mathcal{H} = \{K_2\}$, then $\mathcal{H}(G)$ is the square of the line graph of G

In 2006, **Cameron and Hell** identified several graph classes that are closed under this transformation, including the class of chordal graphs.

In terms of the tree-independence number, this means that the transformation preserves **tree-independence number one**.

We generalize this result by showing that this transformation does not increase the tree-independence number:

$$\text{tree-}\alpha(\mathcal{H}(G)) \leq \text{tree-}\alpha(G)\,.$$

If $\mathcal{H}$ is a fixed **finite** family of connected graphs and we are given a graph G along with a tree decomposition $\mathcal{T}$ with independence number at most k, we show how to compute in polynomial time the graph $\mathcal{H}(G)$ and a tree decomposition $\mathcal{T}'$ of $\mathcal{H}(G)$ with independence number at most k.

The Maximum Weight Independent $\mathcal{H}$-Packing problem on a graph G corresponds to the MWIS problem in the graph $\mathcal{H}(G)$.

This is a common generalization of:

- the Independent $\mathcal{H}$-Packing problem,
- the Maximum Weight Independent Set problem,
- the Maximum Weight Induced Matching problem,
- the Dissociation Set problem, and
- the k-Separator problem.

Theorem

Let $\mathcal{H}$ be a finite set of connected non-null graphs and let r be the maximum number of vertices of a graph in $\mathcal{H}$. Then, for every $k \geq 1$, the Maximum Weight Independent $\mathcal{H}$-Packing *problem is solvable in time $\mathcal{O}\left(|V(G)|^{r(k+1)} \cdot |V(T)|\right)$ if the input graph G is given with a tree decomposition $\mathcal{T} = (X_t)_{t \in V(T)}$ with independence number at most k.*

Connections with (tw, ω)-boundedness

A graph class $\mathcal{G}$ is **(tw, ω)-bounded** if there exists a function f such that for every graph $G \in \mathcal{G}$ and every induced subgraph G' of G, we have $\mathrm{tw}(G') \leq f(\omega(G'))$.

The **clique number** of a graph G, denoted by $\omega(G)$, is the maximum size of a clique in G.

Chaplick and Zeman showed in 2017 that, for every positive integer k, there exist linear-time FPT algorithms for the k-Clique and List k-Coloring problems in any (tw, ω)-bounded graph class having a computable binding function.

Open question: Does every (tw, ω)-bounded graph class have a **polynomial** binding function?

Ramsey's theorem implies that this is the case for graph classes with bounded tree-independence number.

Our main questions:

Which graphs admit a tree decomposition such that each bag induces a subgraph with bounded independence number?

Can such a tree decomposition be computed in polynomial time?

The motivation is algorithmic:

When available, such a tree decomposition can be used to solve various independent set and packing problems in polynomial time.

Our results:

We consider **six graph containment relations**

(subgraph, topological minor, minor + their induced variants).

For each of them we completely characterize
graph classes of bounded tree-independence number
defined by a single forbidden graph with respect to the relation.

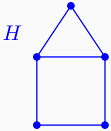
For each of the obtained bounded cases, we show that a tree decomposition with small independence number can be computed efficiently.

As our main result, we obtain an infinite family of graph classes that admit polynomial-time algorithms for the MAXIMUM WEIGHT INDEPENDENT SET problem.

- These classes generalize the class of chordal graphs (for which polynomial-time solvability of MAXIMUM WEIGHT INDEPENDENT SET was given by Frank in 1976).
- We answer a question of Beisegel, Chudnovsky, Gurvich, M., and Servatius (WADS 2019) by showing that MAXIMUM WEIGHT INDEPENDENT SET is solvable in polynomial time in the class of 1-perfectly orientable graphs.

Characterizations

Examples of containments:



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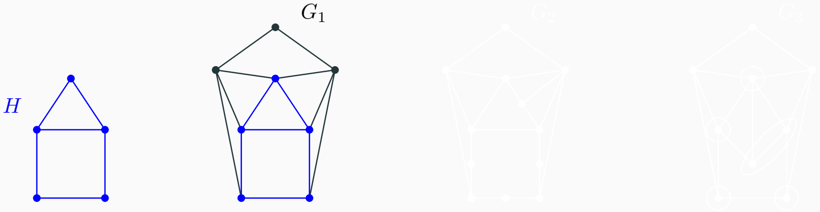
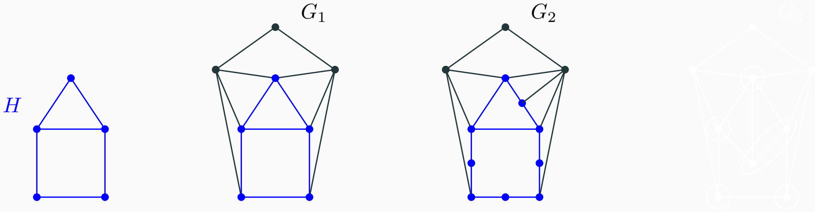


$H \subseteq_{is} G_1$ — induced subgraph

$H \subseteq_{tm} G_2$ — induced topological minor

$H \subseteq_{im} G_3$ — induced minor

Examples of containments:



$H \subseteq_{is} G_1$ $H \subseteq_{itm} G_2$ $H \subseteq_{im} G_3$

induced subgraph | induced topological minor | induced minor

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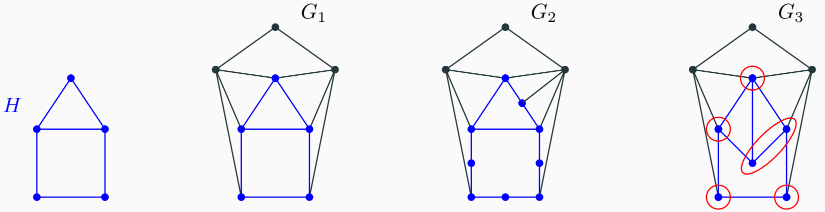

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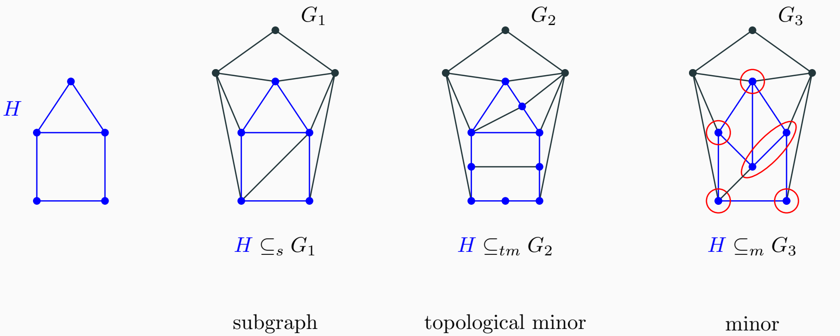


Graphs H for which the class of graphs excluding H has bounded tree-independence number:

	Non-induced	Induced
Subgraph	$H \in \mathcal{S}$	P_3 or edgeless
Topological minor	H is subcubic and planar	C_4, K_4^-, or edgeless
Minor	H is planar	W_4, K_5^-, $K_{2,q}$ for some $q \in \mathbb{N}$

$\mathcal{S}$ is the class of graphs whose connected components are either paths or subdivisions of the claw ($K_{1,3}$).

That these are conditions are **necessary** follows from our previous work (SIDMA 2021), where for each relation we completely characterize graphs H for which the class of graphs excluding H (wrt the relation) is (tw, ω)-bounded.

Excluding H as a **subgraph, topological minor, or minor**:

bounded tree-independence number is equivalent to **bounded treewidth**.

Excluding H as an **induced subgraph** or **induced topological minor**:

bounded tree-independence number is equivalent to **bounded independence number**, except for a small number of well structured graph classes:

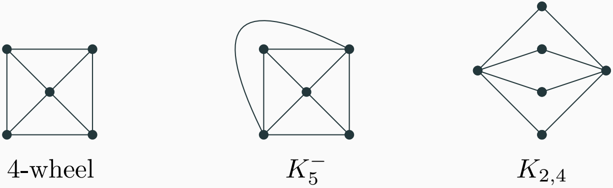
- disjoint unions of complete graphs (tree-$\alpha \leq 1$),
- chordal graphs (tree-$\alpha \leq 1$), or
- block-cactus graphs
  (every block is either complete or a cycle; tree-$\alpha \leq 2$).

The most interesting case: induced minors

Now, let H be excluded wrt the **induced minor** relation.

We have two main cases:

H is the W_4 or K_5^-:	tree-α(H-induced-minor-free graphs) ≤ 4
$H = K_{2,q}$ for some $q \geq 2$:	tree-α(H-induced-minor-free graphs) $\leq 2(q-1)$

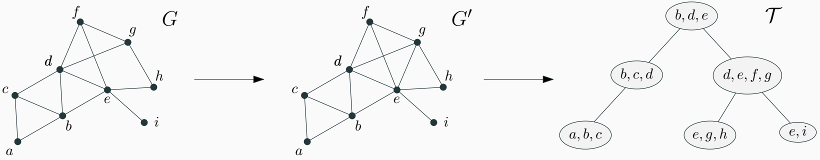


Proof idea for $H = K_{2,q}$ for $q \geq 2$

Let $\mathcal{G}$ be the class of $K_{2,q}$-induced-minor-free graphs, $G \in \mathcal{G}$.

1. We compute a **minimal triangulation** G' of G (= we add edges to G in a minimal way to make it chordal).
   - Many polynomial algorithms are known for this task, for example, an algorithm running in time $\mathcal{O}(|V(G)|^{\mu} \log |V(G)|)$, where $\mu < 2.37286$ is the matrix multiplication exponent (Heggernes, Telle, and Villanger, 2005).

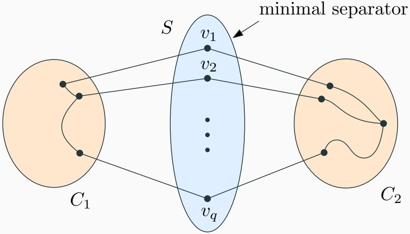
2. We compute a **clique tree** of G' = a tree decomposition of G' having exactly the maximal cliques of G' as bags
   - Can be done in time $\mathcal{O}(|V(G')| + |E(G')|)$ (e.g., Berry and Simonet, 2017)



3. A clique tree of G' is also a tree decomposition of G.

   Following the approach of Bouchitté and Todinca, 2002, we show that each bag (which is a **potential maximal clique** of G) is either a clique in G or can be covered by two minimal separators.

4. **Observation:** Since G is $K_{2,q}$-induced-minor-free, each minimal separator induces a subgraph with independence number less than q.



We thus obtain a tree decomposition with independence number at most $2(q-1)$.

Proof idea for $H \in \{W_4, K_5^-\}$

Fix $H \in \{W_4, K_5^-\}$ and let $\mathcal{G}$ be the class of H-induced-minor-free graphs.

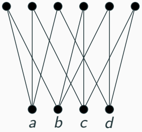
We prove structural properties of graphs in $\mathcal{G}$, leading to conceptually simpler proofs of (tw, ω)-boundedness of these two graph classes compared to our earlier proofs, which rely on graph minors theory.

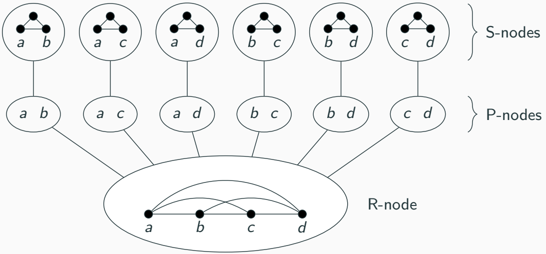
1. We characterize the **3-connected graphs** in $\mathcal{G}$.
2. In linear time we **reduce the problem to the 3-connected case**, using **block-cutpoint trees** and **SPQR trees**.

We show that every graph in $\mathcal{G}$ has a tree decomposition in which each bag is a clique plus at most three vertices.

The reduction to triconnected components using SPQR trees is involved.

- Unlike the treewidth, the tree-independence number can increase after subdividing an edge.
- Boundedness of the tree-independence number of the triconnected components of a graph is only a necessary but not a sufficient condition for boundedness of the tree-independence number of the graph itself.





Consequences for the MWIS problem

The Maximum Weight Independent Set problem has been extensively studied in **hereditary** graph classes, that is, graph classes closed under induced subgraphs.

What if a single graph H is excluded as a minor, induced minor, or induced topological minor?

- If a **non-planar graph** H is excluded (wrt to any of these three relations), then the problem is NP-hard, since the class contains the class of all planar graphs.
- If a **planar graph** H is excluded as a **minor**, then the treewidth is bounded and the problem is solvable in linear time.

What if a planar graph H is excluded as an induced minor?

- If the k-vertex path P_k is excluded as an induced minor (= as an induced topological minor = as an induced subgraph), the problem is polynomial-time solvable for $k \leq 6$ (Grzesik, Klimošová, Pilipczuk, and Pilipczuk, SODA 2019) and open for all $k \geq 7$.

- If the k-vertex cycle C_k is excluded as an induced minor (= as an induced topological minor), the problem is polynomial-time solvable for $k \leq 5$ (Abrishami, Chudnovsky, Pilipczuk, Rzążewski, and Seymour, SODA 2021) and open for all $k \geq 6$.

- The problem is solvable in subexponential time (Korhonen, arXiv 2022).

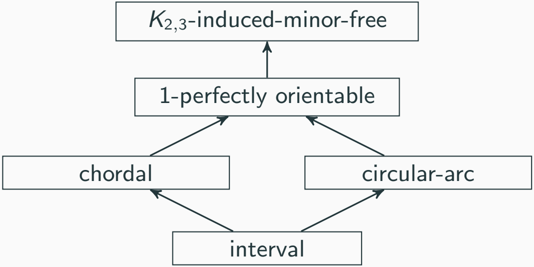
We obtain that Maximum Weight Independent Set is solvable in:

- time $\mathcal{O}(|V(G)|^3)$ for W_4-induced-minor-free graphs,
- time $\mathcal{O}(|V(G)|^3)$ for K_5^--induced-minor-free graphs,
- time $\mathcal{O}(|V(G)|^{2q})$ for $K_{2,q}$-induced-minor-free graphs, for all $q \geq 2$.

The algorithms are **robust**: we do not need to know if the input graph belongs to the class.

- If it does not, the algorithm either correctly solves the problem or correctly detects that the graph is not in the class.

Corollary: MAXIMUM WEIGHT INDEPENDENT SET is solvable in polynomial time in the class of 1-perfectly orientable graphs.



This answers an open question of Beisegel, Chudnovsky, Gurvich, M., and Servatius (2019).

Further recent developments

M., Rzążewski (2022+):

1. For every odd k,

$$\text{tree-}\alpha(G^k) \leq \text{tree-}\alpha(G)\,.$$

   - For even k, the k-th powers of chordal graphs contain all possible graphs as induced subgraphs.

2. Extension of the result for the Maximum Weight Independent $\mathcal{H}$-Packing problem to the Maximum Weight Distance-d Independent $\mathcal{H}$-Packing problem, for even d.

   - Eto, Guo, Miyano, 2014: for all odd $d \geq 3$, Distance-d Independent Set is NP-complete on chordal graphs.

M., Rzążewski (2022+):

The following problems are solvable efficiently given a tree decomposition with bounded independence number:

- finding a smallest feedback vertex set (set of vertices intersecting all cycles),
- finding a largest q-colorable induced subgraph (for constant q),
- finding a largest induced subgraph of treewidth at most q (const. q).

More generally, the subgraph can satisfy some fixed $CMSO_2$ formula.

Dallard, Fomin, Golovach, Korhonen, M. (2022+):

Theorem

For every positive integer k, there is an $n^{\mathcal{O}(k)}$ algorithm that, given an n-vertex graph G, either computes a tree decomposition with independence number $\mathcal{O}(k)$ or correctly determines that tree-$\alpha(G) > k$.

Conclusion

We introduced the **tree-independence number** of a graph, an invariant that is:

- an algorithmically useful generalization of the treewidth and independence number,
- monotone under induced minors.

We provided some initial results regarding boundedness, resp. unboundedness of this invariant.

Open questions

Question 1:
Does every (tw, ω)-bounded graph class have bounded tree-independence number?

Question 2:
Is there a "graph searching" characterization of the tree-independence number?

Question 3 (asked first by **Oum**):
Is there a bramble-like characterization of the tree-independence number?

- **Wiederrecht** (during GRASTA 2022): an approximate bramble-like characterization

Question 4:
Is the Independent Set problem solvable in polynomial time in the class of H-induced-minor-free graphs for any planar graph H?

Thank you!

Questions?