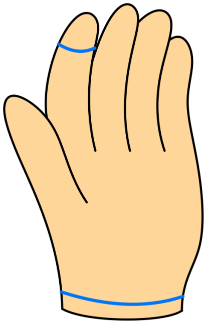
Homotopy height: searching a planar graph with closed curves

Arnaud de Mesmay

CNRS, LIGM, Université Gustave Eiffel

Based on joint works with T. Biedl, G. R. Chambers, E. W. Chambers, D. Eppstein, T. Ophelders, R. Rotman.

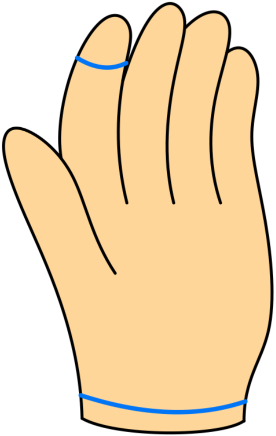


What is this talk about?

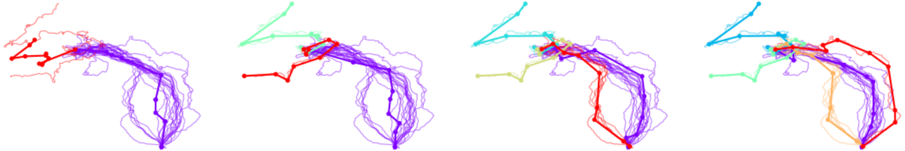
We study how much *stretch* is needed to go continuously from one curve to another.

- A *homotopy* is a continuous deformation of the top blue curve into the bottom blue curve that stays on the surface of the hand.
- An optimal homotopy is one that **minimizes** the length of the **longest** curve.

Informally, how stretchable must a rubber band be to fit around my hand?



Motivation I: Similarity measure for curves



- How to compute whether two trajectories are similar to each other?
  → Hausdorff distance, Fréchet distance . . .

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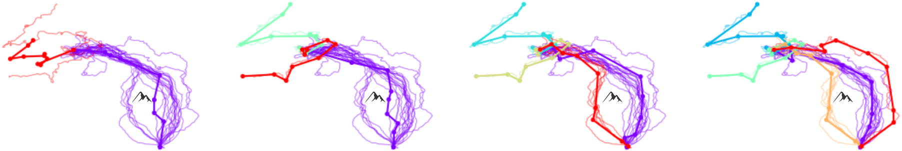
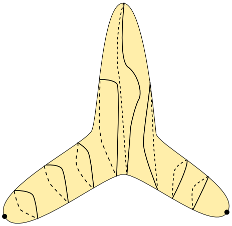


- How to compute whether two trajectories are similar to each other?
  $\rightarrow$ Hausdorff distance, Fréchet distance . . . but they do not see the mountain.
  $\rightarrow$ Distances that require to sweep the entire area between two curves.

Motivation II: Quantitative homotopy theory

- Optimal homotopies provide a way to find *geodesics* [Birkhoff'1917].



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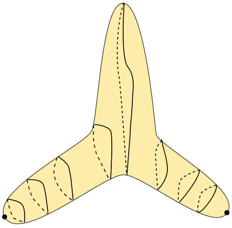

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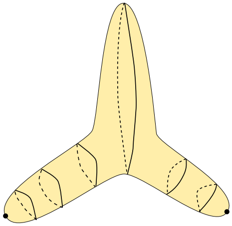

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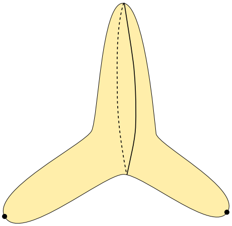

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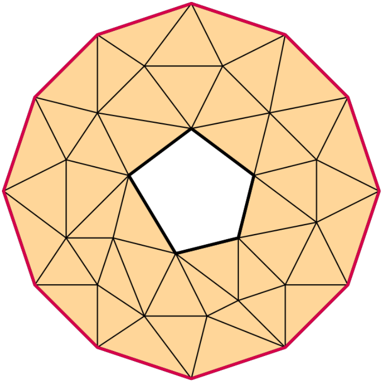
Motivation III: Graph searching on planar graphs

- Similar to *searching games* on graphs (cops and robbers, node searching) where we sweep a graph to find a fugitive.
  $\rightarrow$ Connections with width parameters (path-,tree-,branchwidth).
- Here the rule is that the cops hold hands to form a connected closed curve sweeping an annulus.
- Many other variants (sweeping a disk or the sphere).



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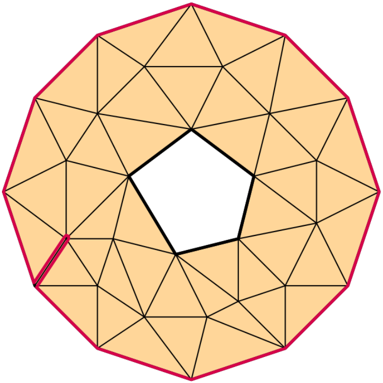

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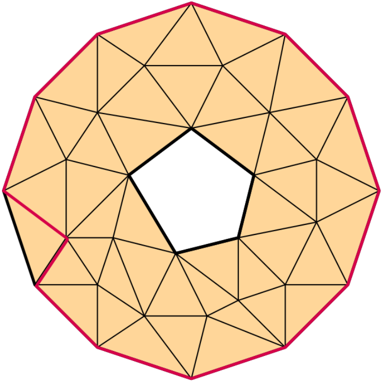

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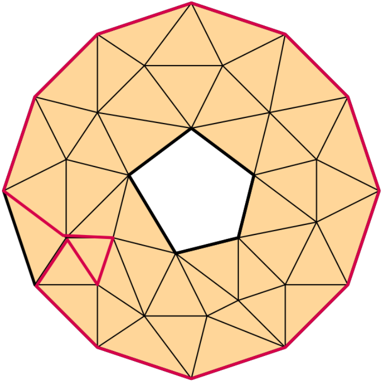
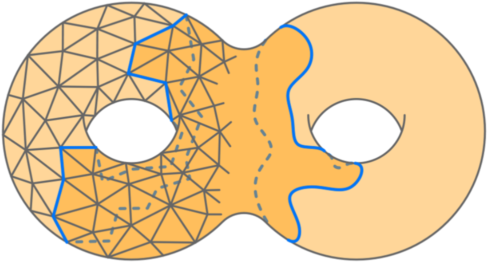


- We can work on the plane or on any surface with or without boundary.
- We can work with a discrete or a continuous (Riemannian) metric.



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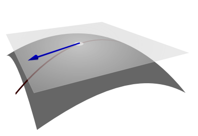
Riemannian metric

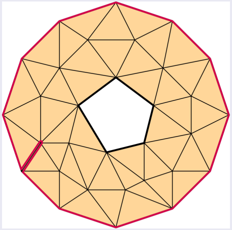
Scalar product m on the tangent space $\to$ Riemannian length $|\gamma|_m$.



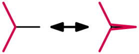
Homotopy: Continuous map h between two curves.

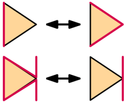
Discrete metric

Edge-weighted triangulation G $\to$ Length $|\gamma|_G$



Homotopy: Sequence h of *edge spikes* and *face flips*





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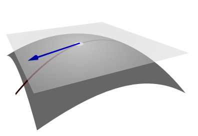

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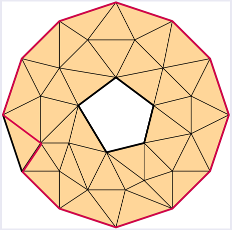

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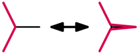

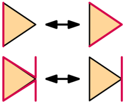

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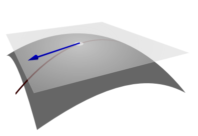

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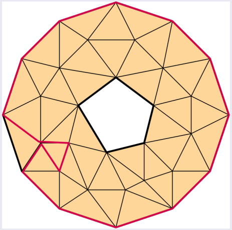

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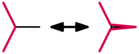
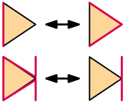




Homotopy Height

γ and γ' are two homotopic disjoint simple closed curves on a surface.

Height of a homotopy

The *height* of a homotopy h between γ and γ' is the maximal length of the intermediate curves:

$$Height(h) = \max_t |h(t, \cdot)|$$

Homotopy Height

The *Homotopy Height* between γ and γ' is the smallest possible height of a homotopy between γ and γ':

$$HH(\gamma \to \gamma') = \inf_{h:\gamma\to\gamma'} Height(h) = \inf_{h:\gamma\to\gamma'} \max_t |h(t, \cdot)|$$

A homotopy of minimal height is called *optimal*.

Main questions

Computational question

How to *compute* the homotopy height between two input curves?

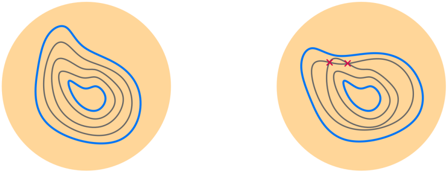
Even *brute-forcing* the problem is non trivial, as optimal homotopies may a priori be very complicated.

Mathematical question

Does there always exist an optimal homotopy that is not too complicated?

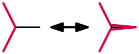
Two reasonable conjectures:

1. There always exists a homotopy that is an *isotopy*, i.e., curves stay *simple*.
2. There always exists an isotopy that is *monotone*. (no recontamination)



Isotopies

In the discrete setting, we allow for tangencies in simple curves.



Theorem ([G. Chambers, Liokumovich '14])

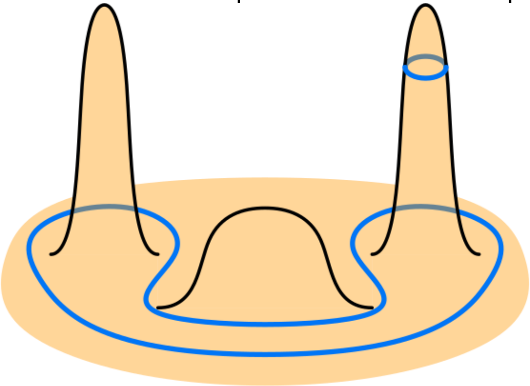
In the Riemannian setting, for γ and γ' two non-contractible simple closed curves, and a homotopy $\gamma \to \gamma'$ of height L, there exists an isotopy from γ to γ' of height $L + \varepsilon$, for any $\varepsilon > 0$.

- The proof applies verbatim to the discrete setting.
- The need for ε comes from arbitrarily small perturbations, which are not needed there.

The (beautiful) proof analyzes carefully all the *resolutions* of the intermediate curves and finds a path there using the handshaking lemma.

Monotonicity

However, there are sometimes *no* optimal monotone isotopies.



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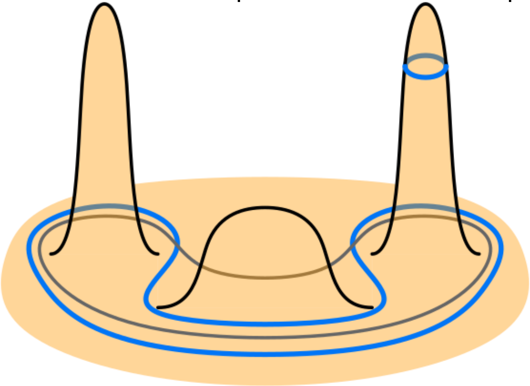

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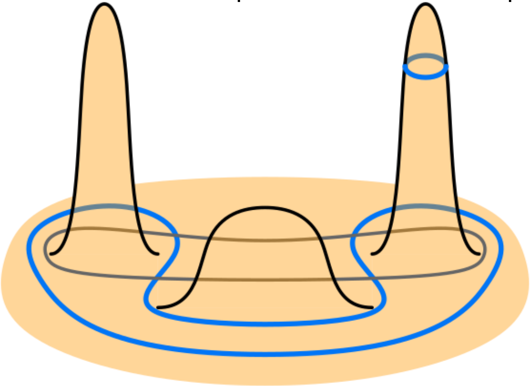

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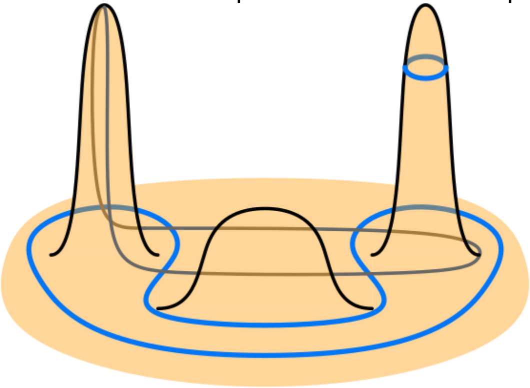

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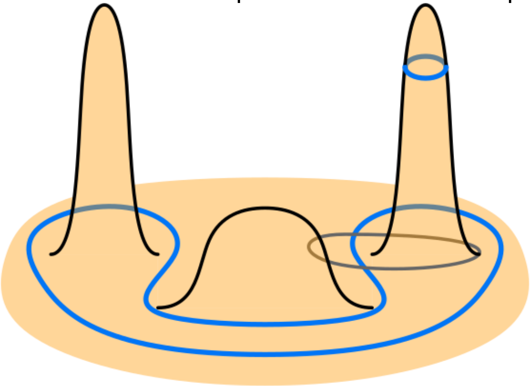

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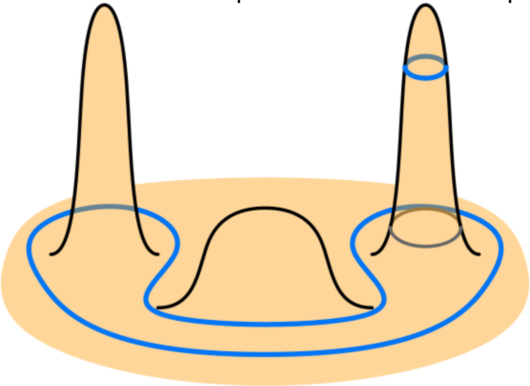

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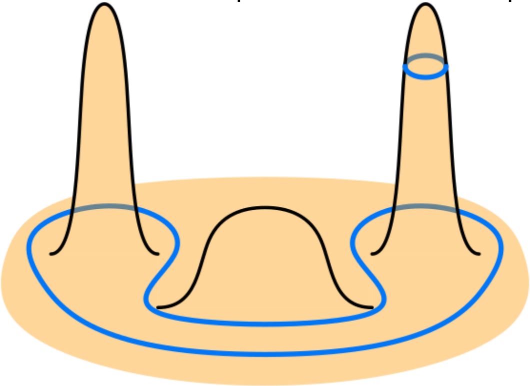

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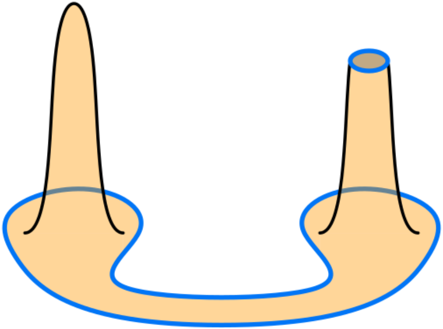

What if the curves form the boundary of the surface?

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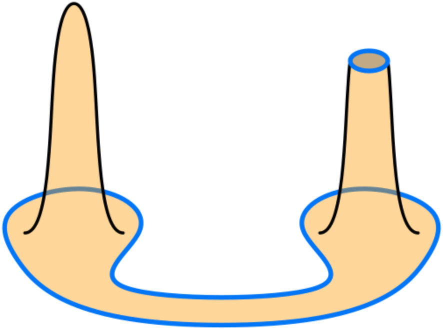

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Theorem ([E. Chambers, Letscher '09])

Let A be a discrete or Riemannian annulus with boundaries γ and γ'. Then there exists an optimal homotopy between γ and γ' that is a monotone isotopy.

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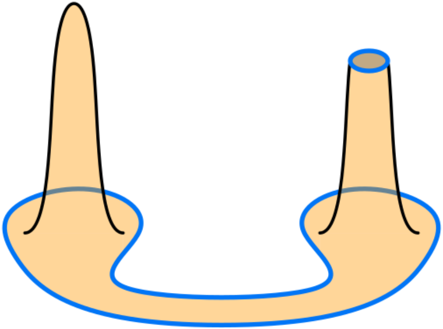

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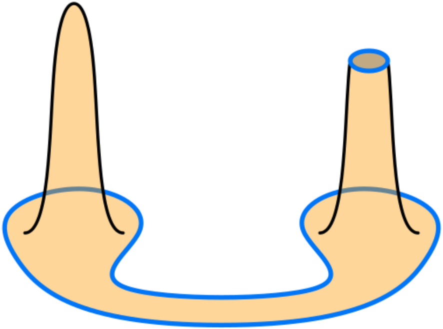

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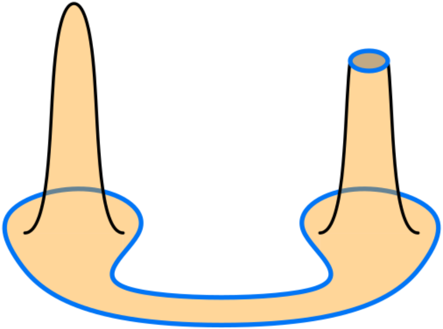

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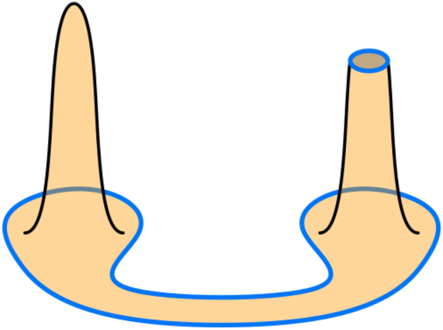

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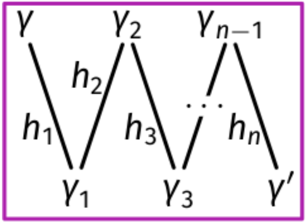
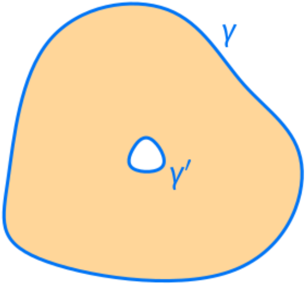
Idea of the proof

We first decompose an optimal isotopy into maximal monotone isotopies $h_1, \ldots, h_n$, forming a *zigzag* Z of order n.





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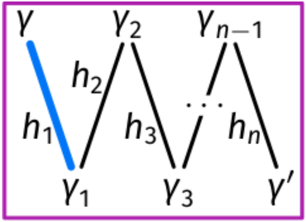

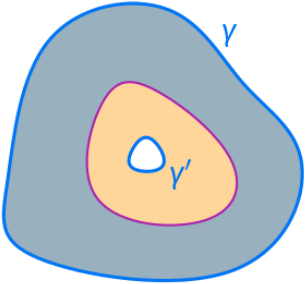

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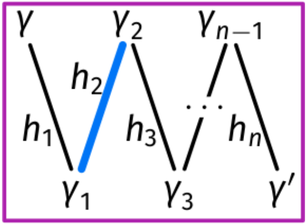

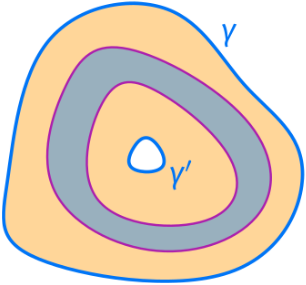

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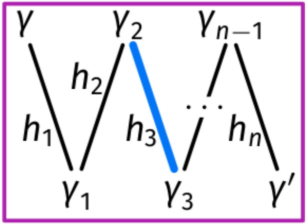

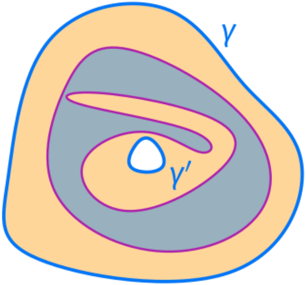

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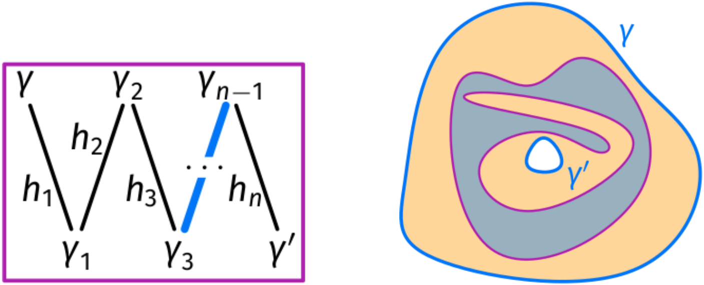

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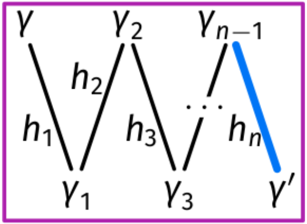

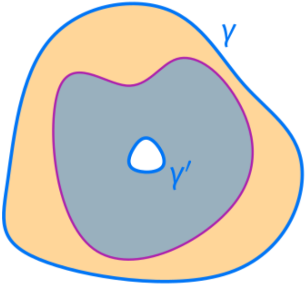

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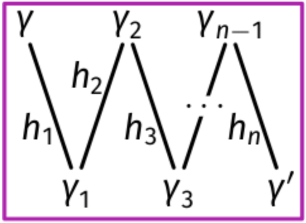
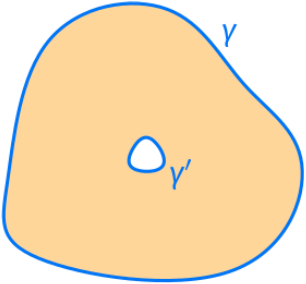


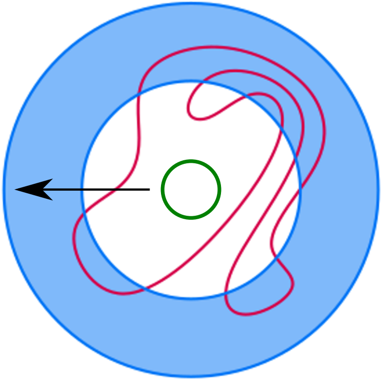


Shortcutting monotone isotopies

Lemma

Let A be an annulus with boundaries γ and γ', and δ be the shortest non-contractible curve in A. Then a monotone isotopy between γ and γ' of height L can be transformed into an isotopy between γ and δ of height L.

Proof: any intermediate curve can be shortcut at δ.



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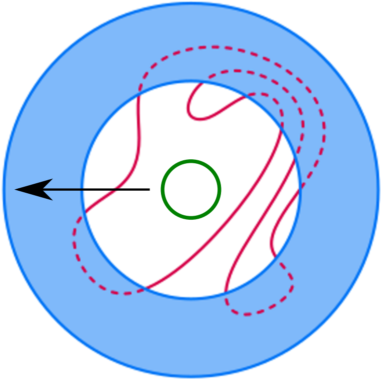

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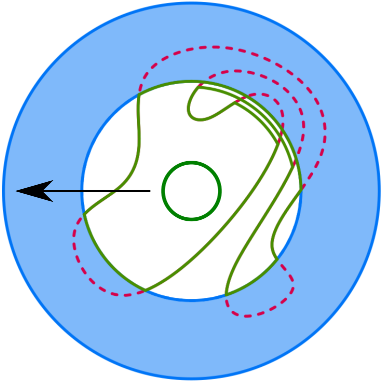

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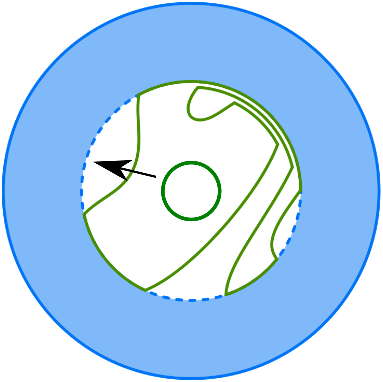

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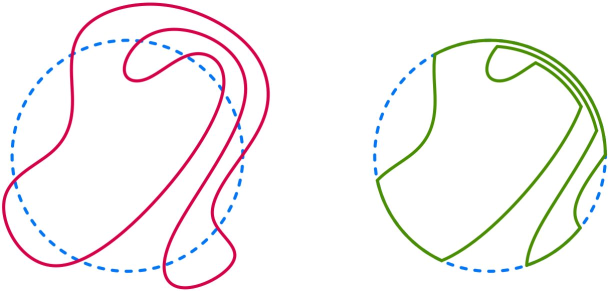

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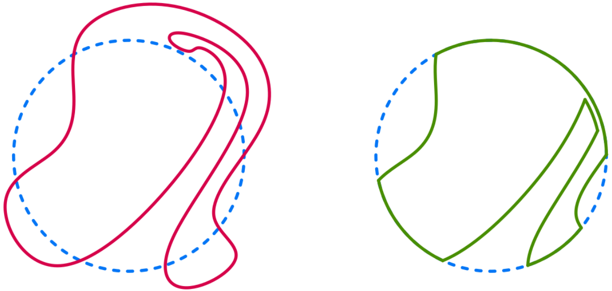

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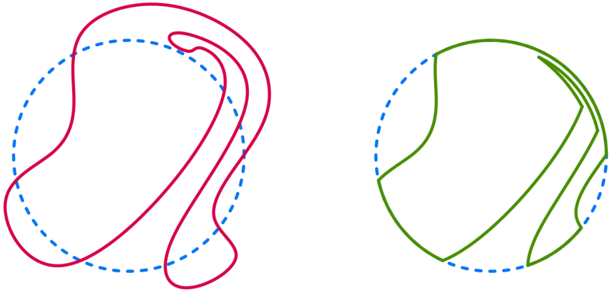

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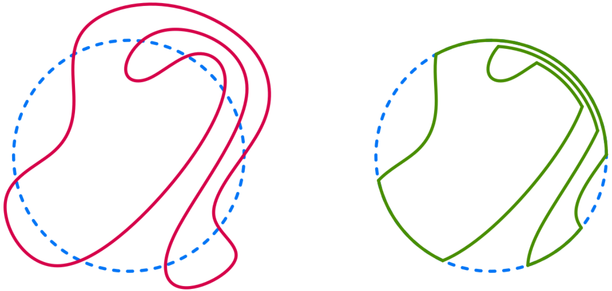

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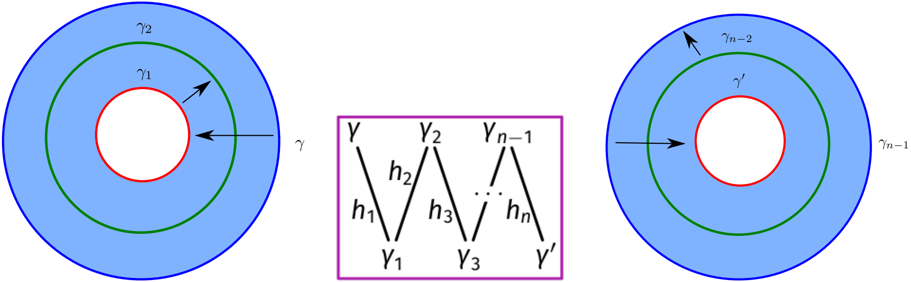
An inductive argument?

Lemma

One can modify Z without increasing its height so that γ_1 is the shortest non-contractible curve in the annulus between γ_1 and γ_2.



- We would like to iterate the argument, obtaining that for all i, γ_i is the shortest non-contractible curve in the annulus $A(\gamma_i, \gamma_{i+1})$.
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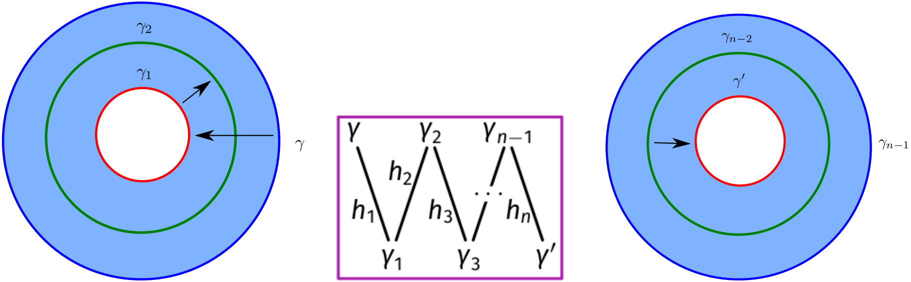

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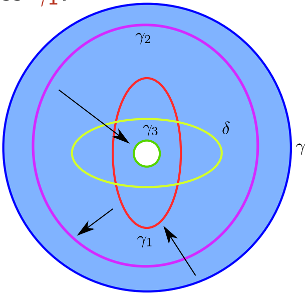
. . . that requires other operations to work.

- But the shortcutting might already fail for γ_2, because the δ between γ_2 and γ_3 might cross γ_1.



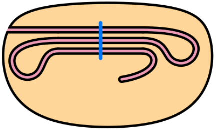
$\rightarrow$ A more technical surgery is used to circumvent this.

Computing the homotopy height

Homotopy Height in an annulus

Input: Discrete annulus with boundaries γ and γ', integer L.
Output: Is the homotopy height between γ and γ' at most L?

We know that there exists an optimal homotopy that is an *isotopy* and that is *monotone*. But it may still be exponentially long!



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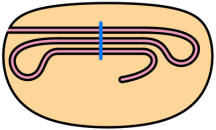


Theorem ([Chambers, dM, Ophelders '17])

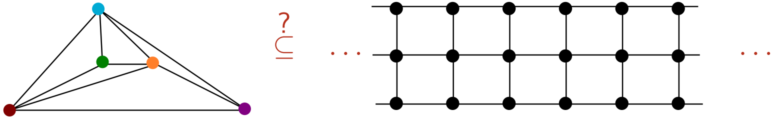
There exists an optimal monotone homotopy where every edge is spiked at most 3 times.

Corollary

The problem Homotopy Height in an annulus *is in* ***NP****.*

Grid minors and grid majors

- Graph width parameters have important connections with graph minor theory, what about homotopy height?
- The *grid-major height*/*gridwidth* of a graph G is the smallest H so that G is a minor of a $H \times W$ grid **for some** W.



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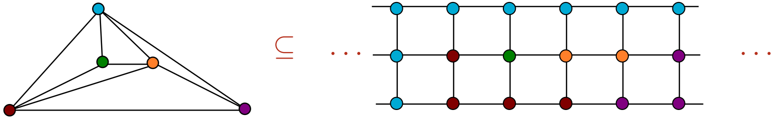
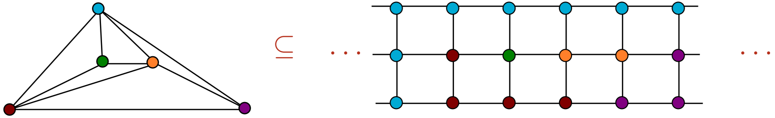


Homotopy height and grid-major height

Theorem (Biedl, E. Chambers, Eppstein, dM, Ophelders'19)

The homotopy height of a triangulated graph equals the grid-major height.*



*: variant with moving endpoints on the outerface with edge-slides.

- Grid-major height is obviously minor-closed.

Corollary

Computing the homotopy height is fixed-parameter tractable (parameterized by the output).*

I do not know a practical algorithm for this.

Summary and open problems

There exist optimal homotopies which are:

- *isotopies*
- *monotone*
- *of polynomial length*

This places the algorithmic problem in **NP**. There is also an (unknown) FPT algorithm via a characterization by grid majors.

Many remaining questions:

1. Is it **NP**-hard?
2. Explicit FPT algorithms?
3. Approximation algorithms? Best known is $O(\log n)$.
4. What can we say/compute when the homotopy is allowed to move outside of the initial curves?

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Thank you! Questions?