Incidence, a scoring positional game on graphs

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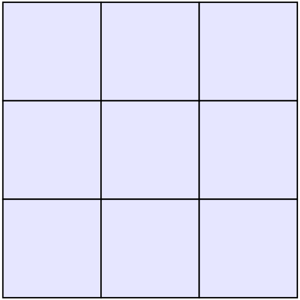
Summary

Definition of Positional game

Maker-Maker



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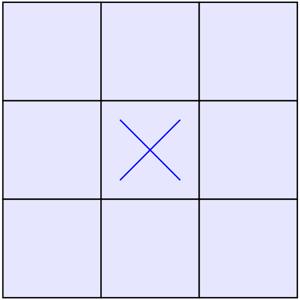

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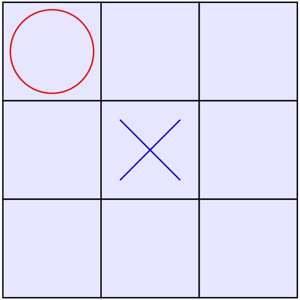

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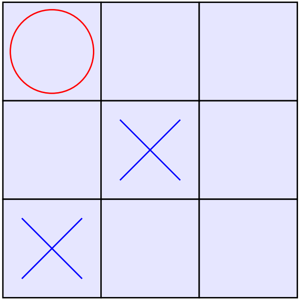

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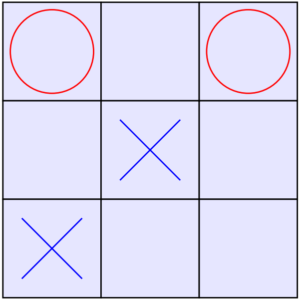

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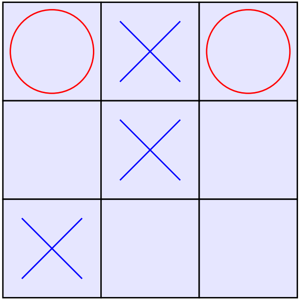

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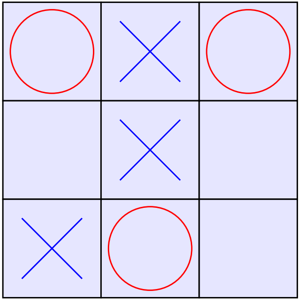

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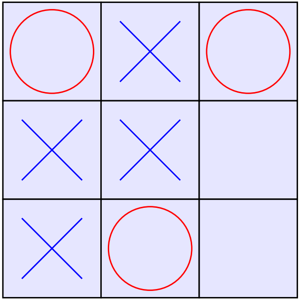

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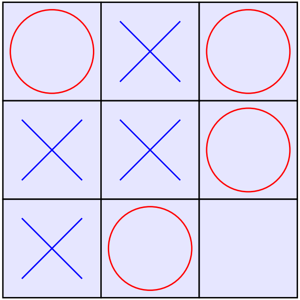

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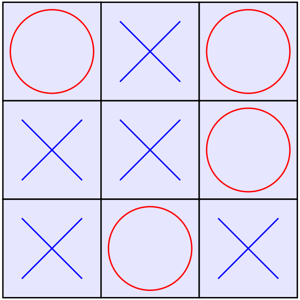

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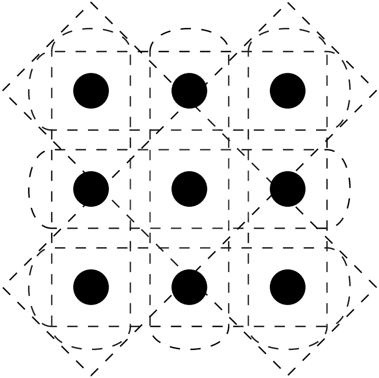

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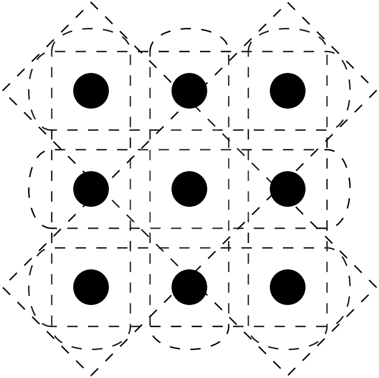

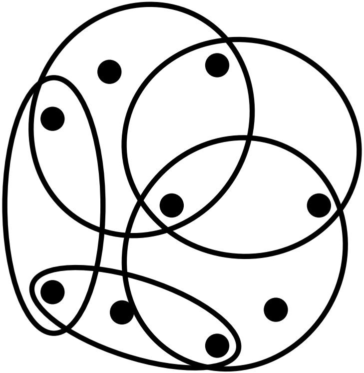

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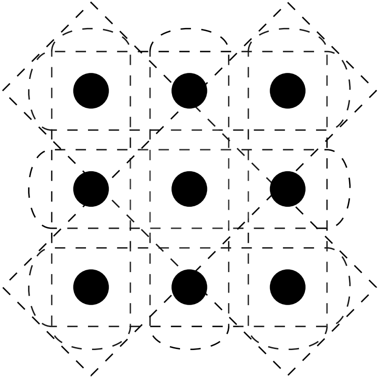

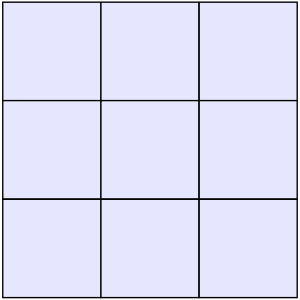

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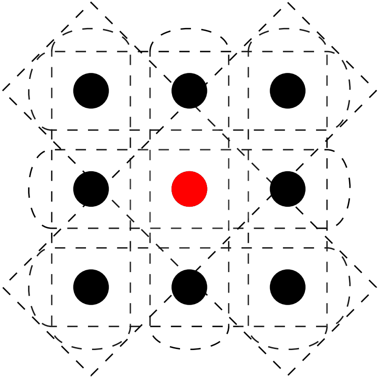

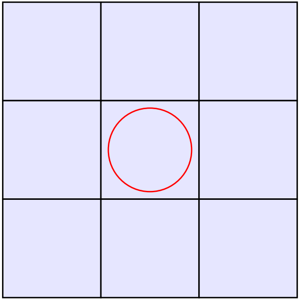

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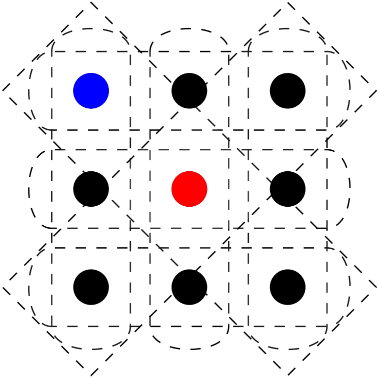

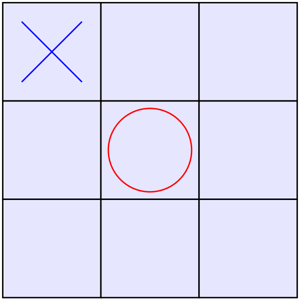

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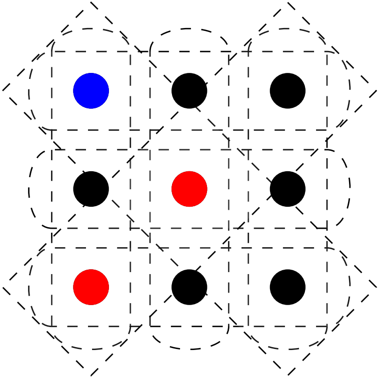

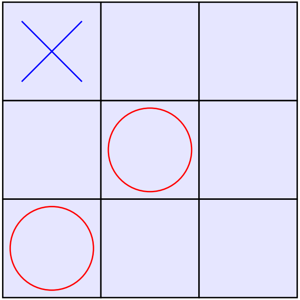

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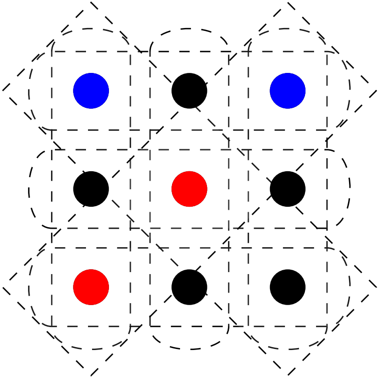

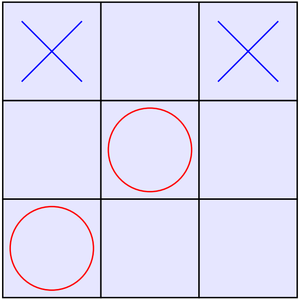

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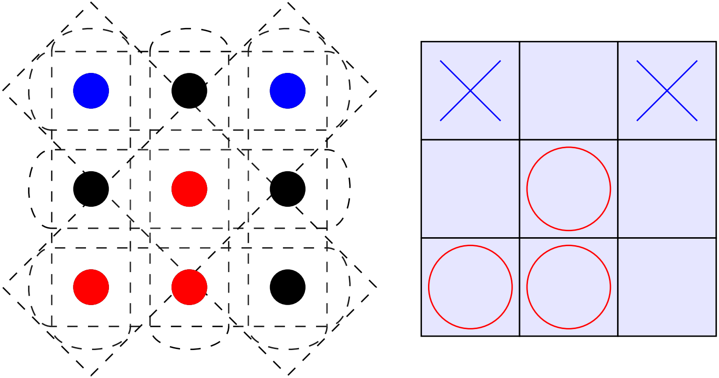

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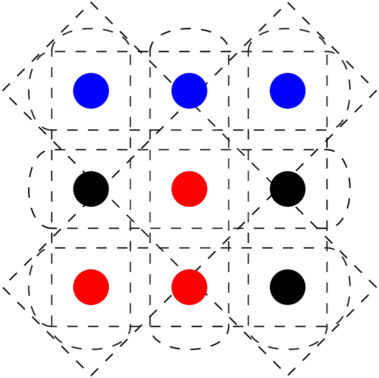

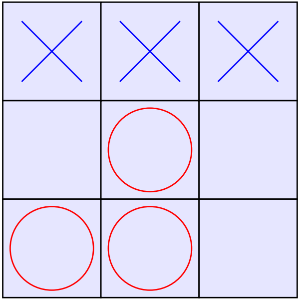

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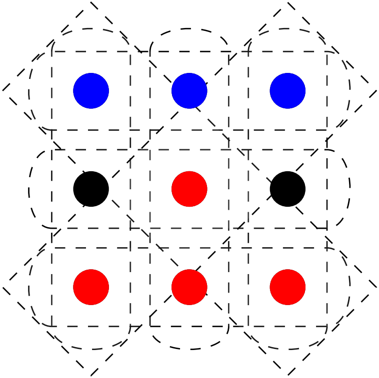

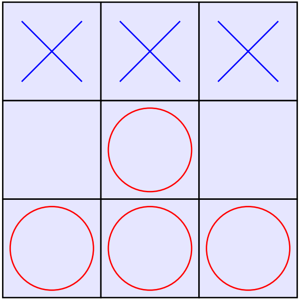

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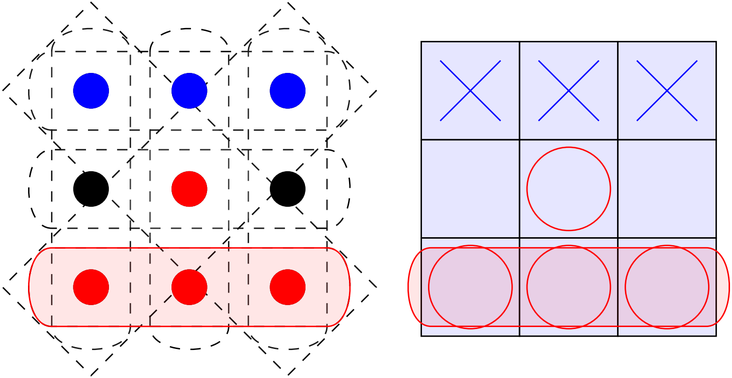

General Framework of positional game

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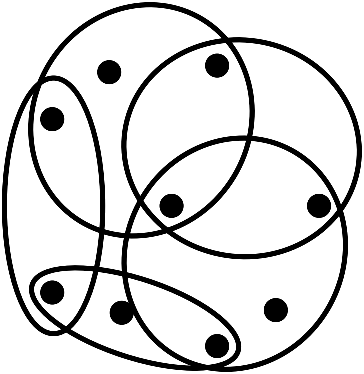
- Can we introduce scores in positional games ?

From positional games to scoring positional games



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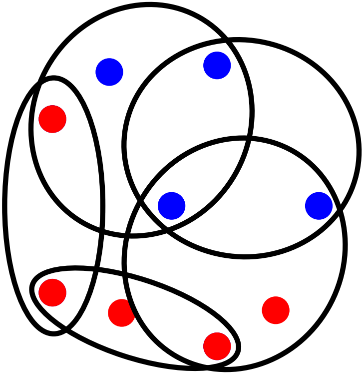

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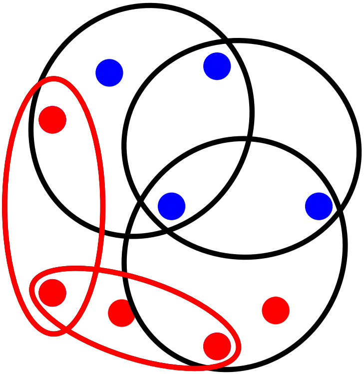

Final score (Maker-Breaker) : $s(G) = 2$

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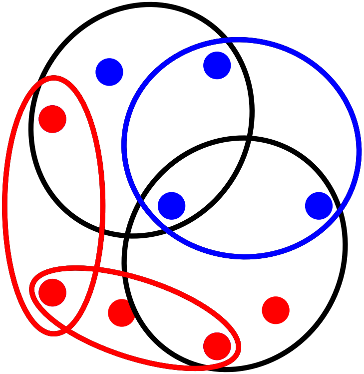
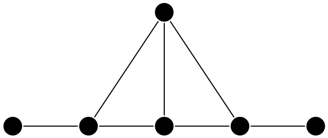


Final score (Maker-Maker) : $s_{MM}(G) = 2 - 1 = 1$

Presentation of Incidence



Definition (Incidence)

Entry : A graph G.
Score ($s(G)$ or $s_{MM}(G)$: Number of edge with both extremity taken by a player.

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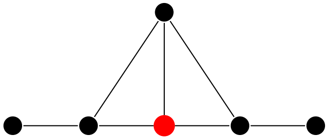

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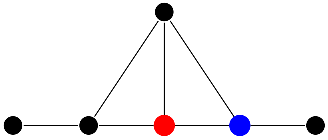

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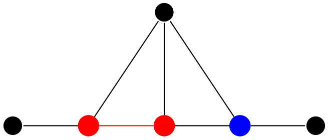

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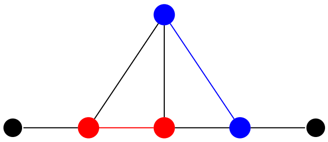

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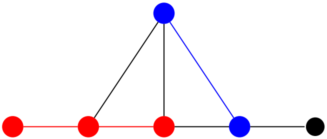

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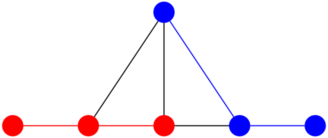

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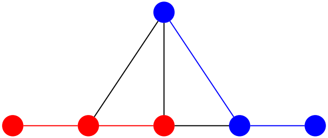


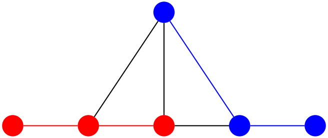
Score Maker-Maker :

$s_{MM}(G) = \#($  $) - \#($  $)$

$= 2 - 2$

$= 0.$

Presentation of Incidence



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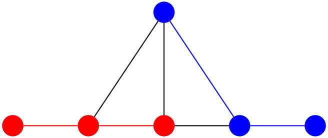

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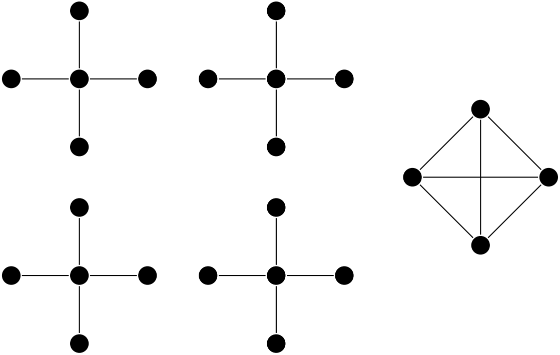
Problem

Given a graph G and an integer k, do we have $s_{MM}(G) \geq k$ (resp. $s(G) \geq k$).

Strategy in Maker-Maker



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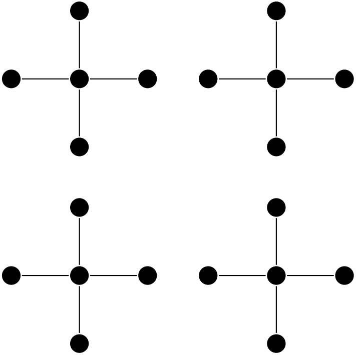

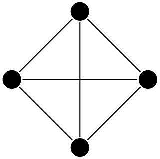

Strategy : play the available vertex of maximal degree.

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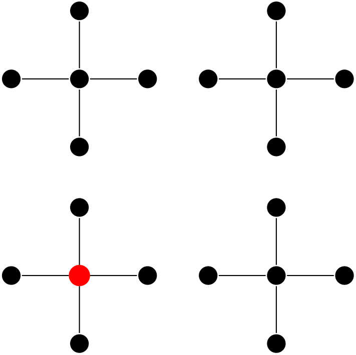

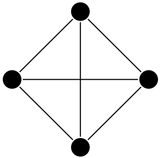

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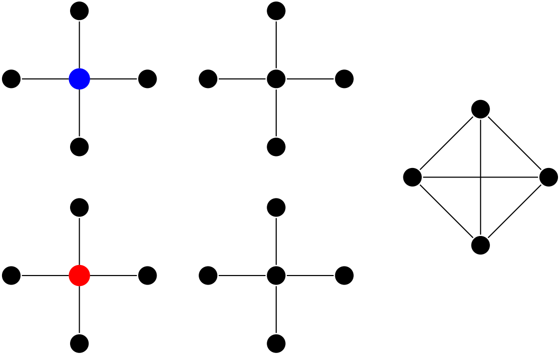

Strategy : play the available vertex of maximal degree.

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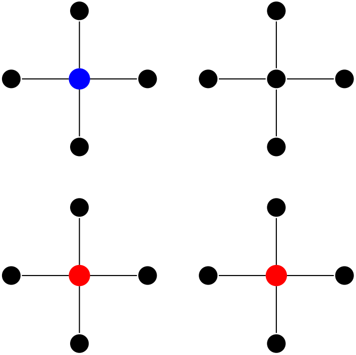

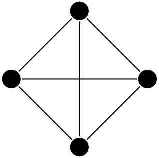

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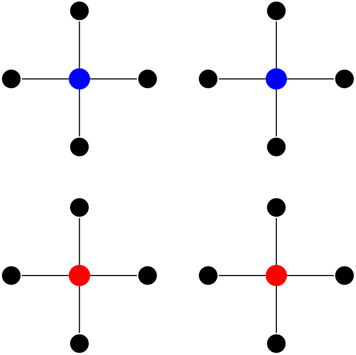

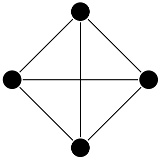

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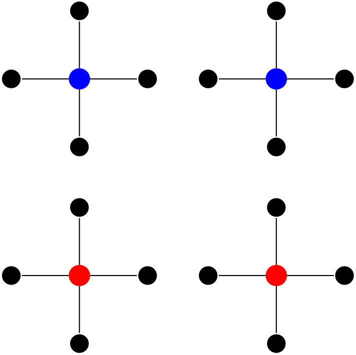

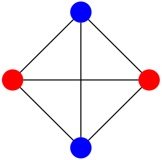

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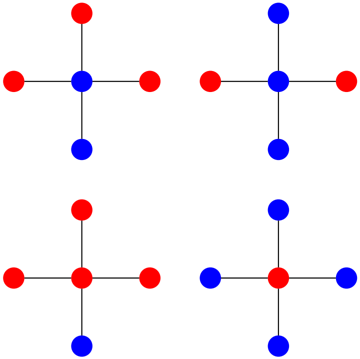

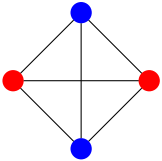

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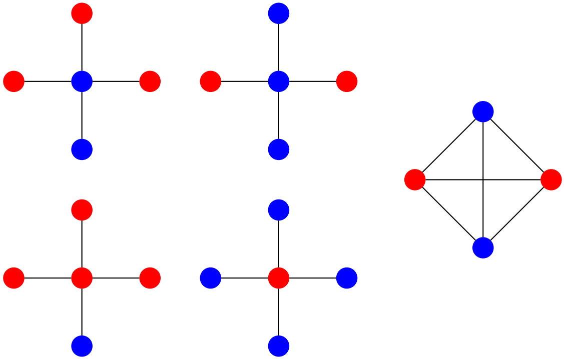
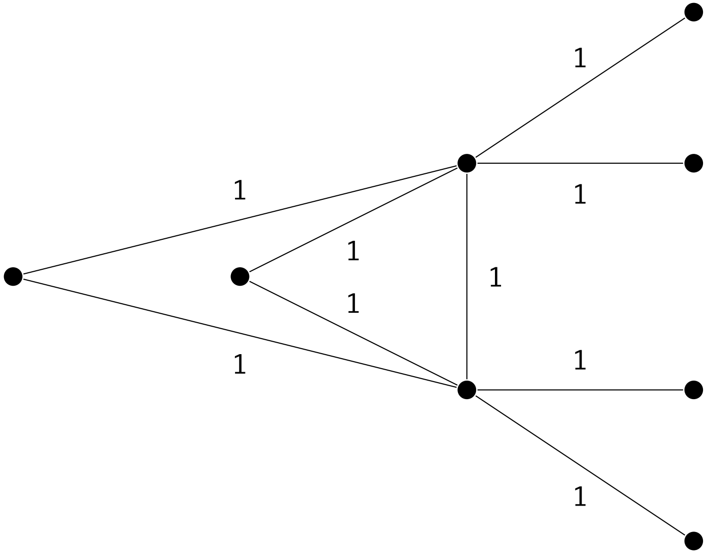


Strategy : play the available vertex of maximal degree.

$s_{MM}(G) = 0$

Incidence is Polynomial in Maker-Maker



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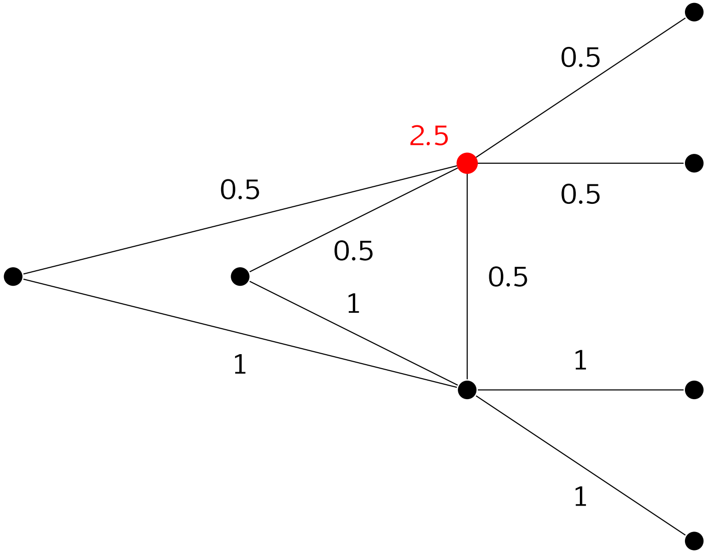

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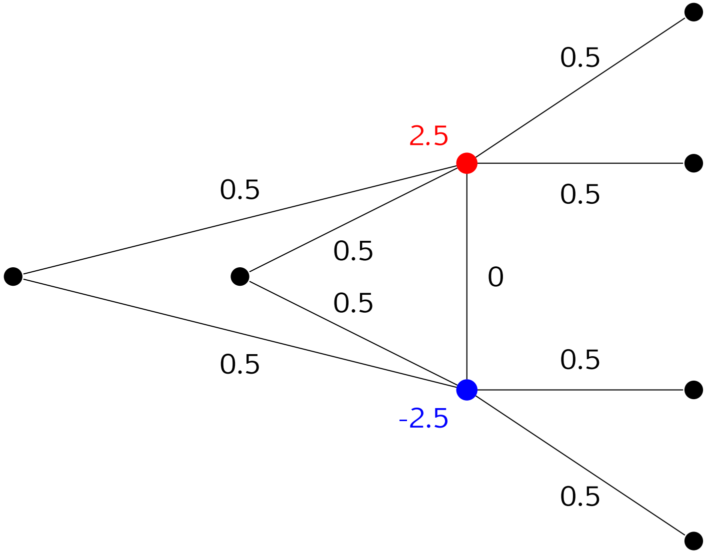

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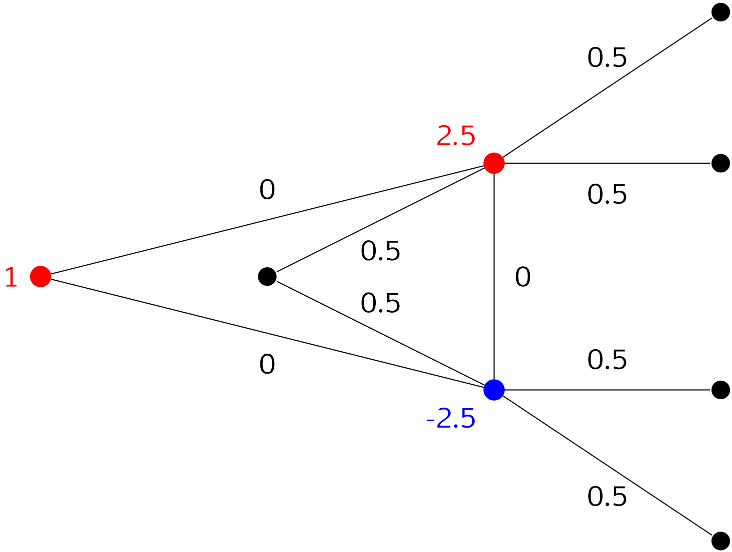

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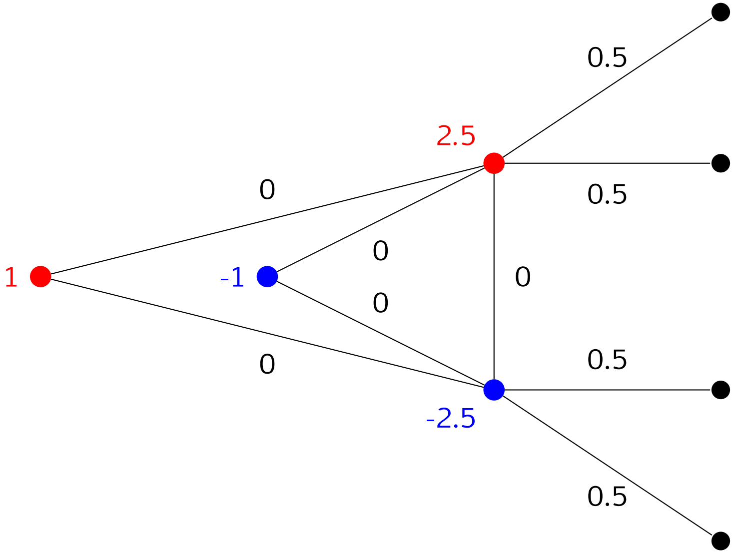

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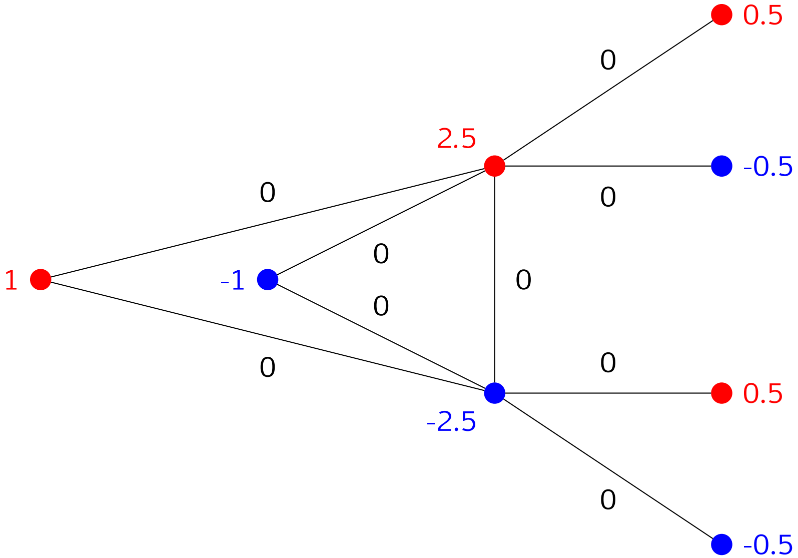
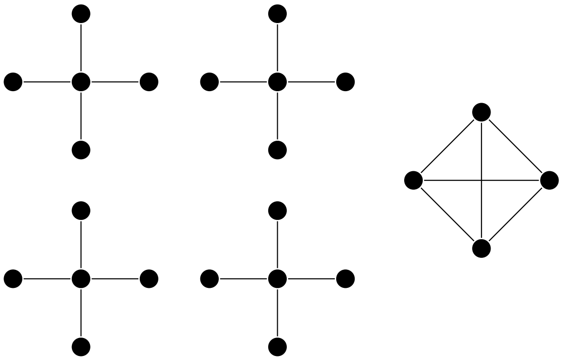
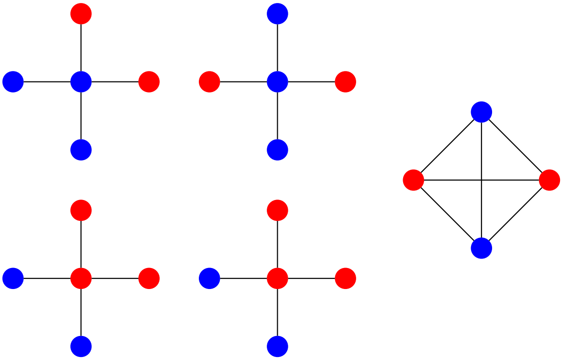


Greedy strategy in Maker-Breaker



Greedy strategy in Maker-Breaker



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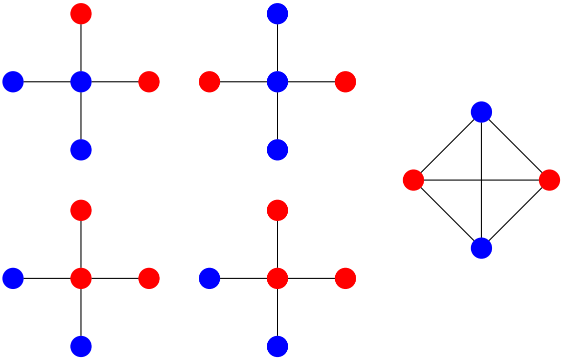
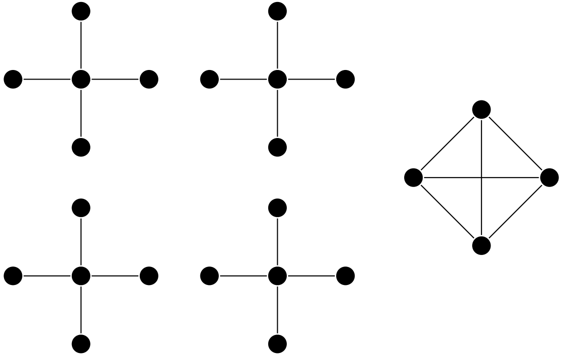


Final score : 5

Another strategy ?



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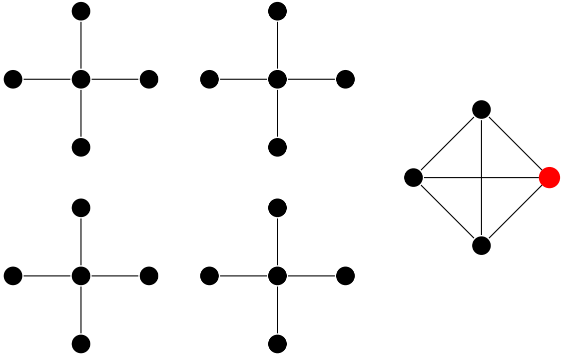

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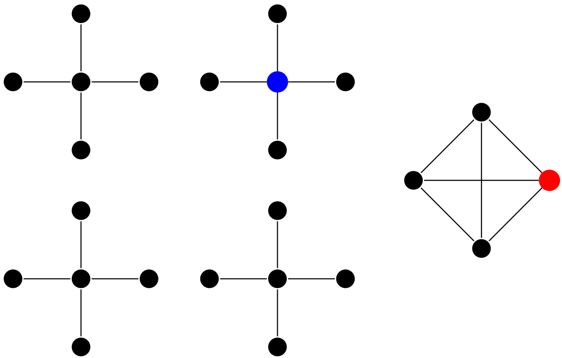

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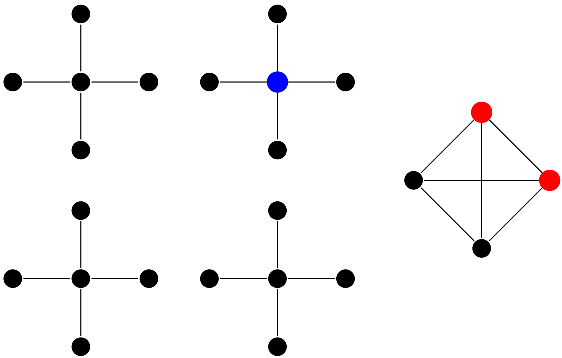

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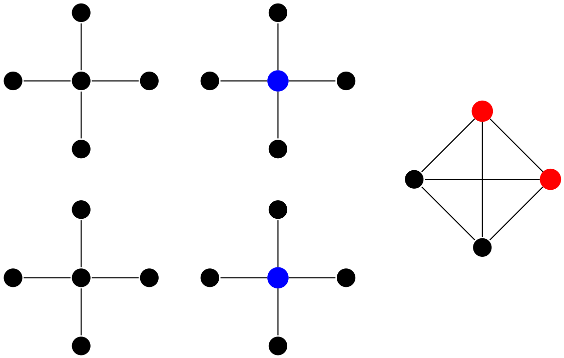

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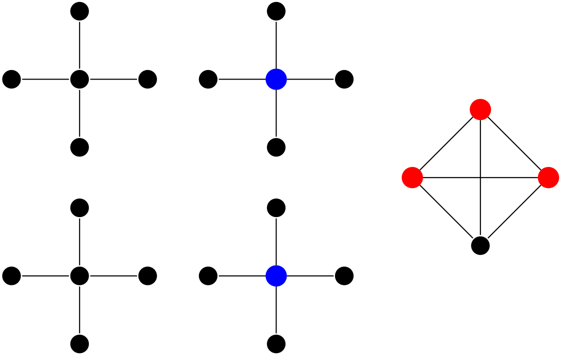

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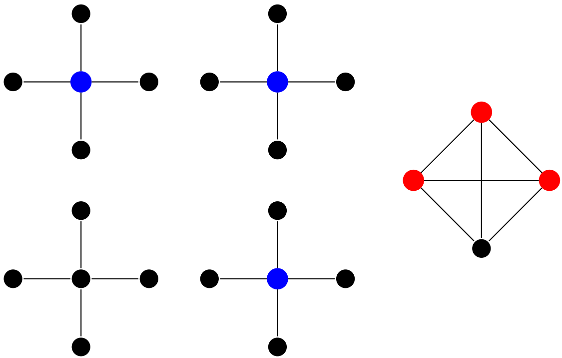

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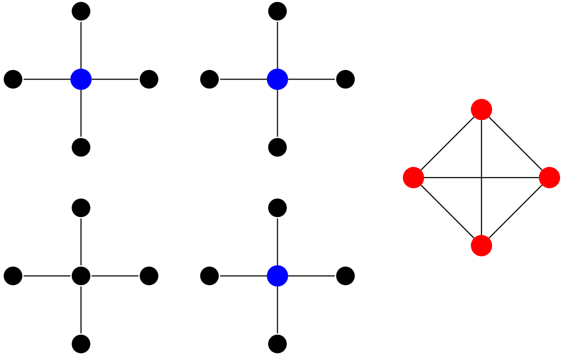

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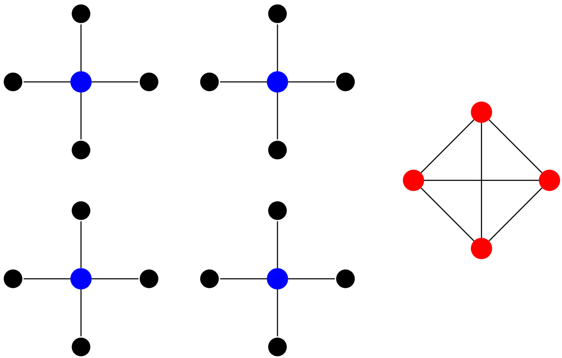

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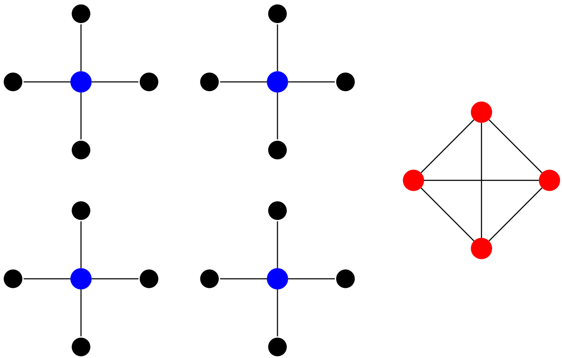

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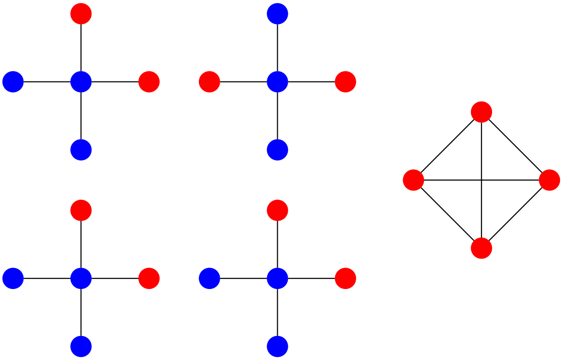
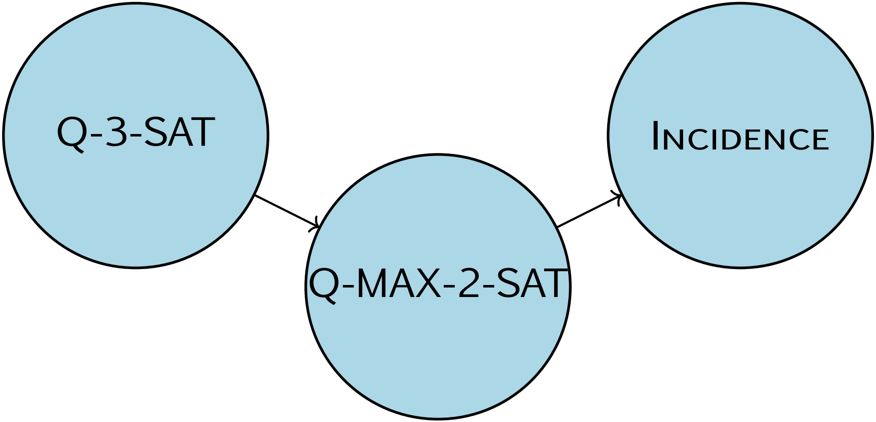


Final score : 6

Reduction



Q-MAX-2-SAT is PSPACE-complete

Sketch of the proof.

Inspired of the NP-hardness reduction of MAX-2-SAT from Williams (2008).

- Let ϕ be a 3-SAT formula of n clauses. We construct ψ a 2-SAT formula of $10n$ clauses.
- $C = x \vee y \vee z \in \phi$.

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- $(x), (y), (z); (c), (\neg x \vee \neg y), (\neg x \vee \neg z), (\neg y \vee \neg z), (\neg c \vee x), (\neg c \vee y), (\neg c \vee z)$

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- Let $k = 7n$. ϕ is satisfied if and only if k clauses of ψ are.

$\triangleleft$

Reduction to INCIDENCE

Theorem

Maker-Breaker INCIDENCE *is* PSPACE*-complete.*

Sketch of the proof. Instance : $\varphi = \exists x_{2n+1} \forall x_{2n} \exists x_{2n-1} \ldots \exists x_1 \psi$

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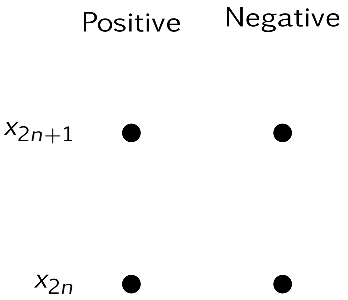

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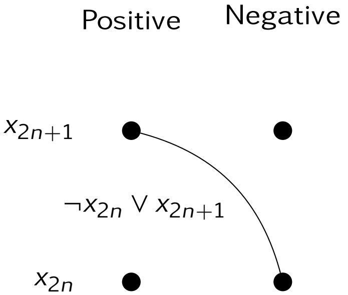

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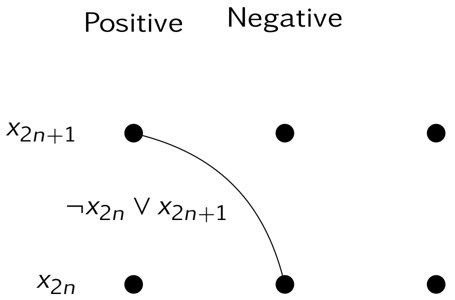

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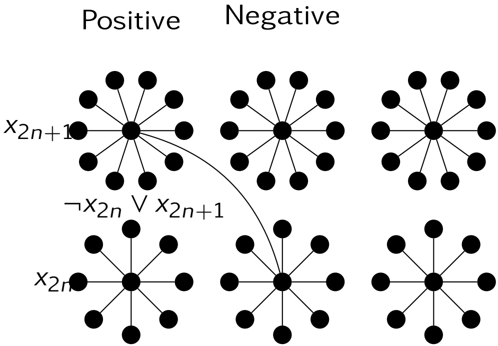

C vertices to force moves. it gives N points to maker.

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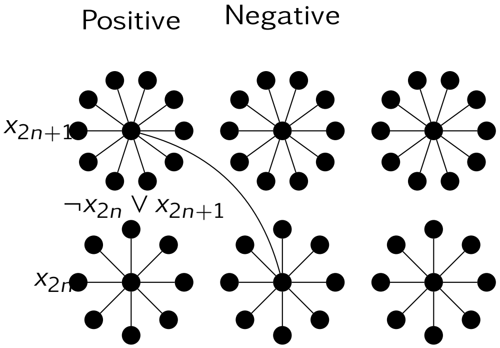

C vertices to force moves. it gives N points to maker.

$s(G) < N + |\psi| - k$ if and only if k clauses of φ are satisfied.

◁

The main result

Theorem

Let P_n be a path of order n. For $n \geq 0$ We have:

$$s(P_n) = \left\lfloor \frac{n+2}{5} \right\rfloor$$

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Sketch of the proof

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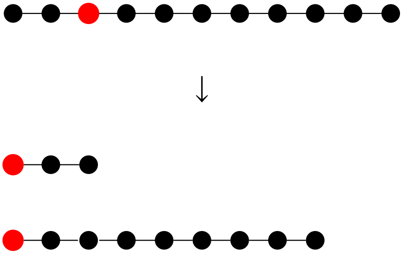

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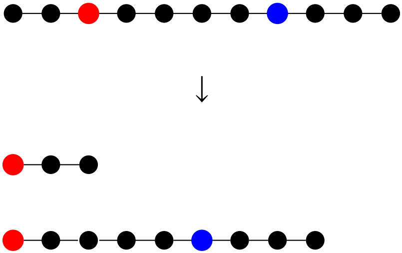

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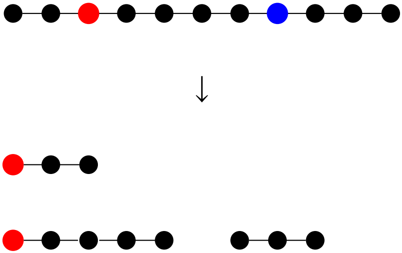

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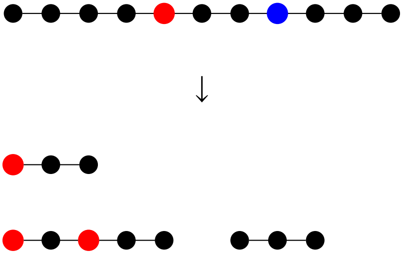

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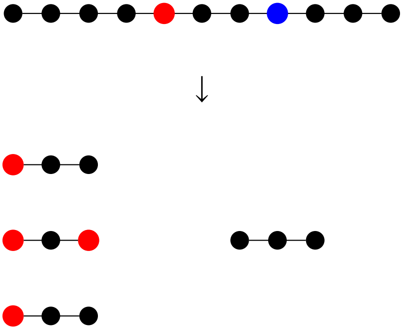

Edrös-Selfridge's theorem

Theorem (Erdös-Selfridge (1973))

In a Maker-Breaker positional played on a hypergraph $H = (V, E)$, if $\sum_{e \in E} 2^{-|e|} < \frac{1}{2}$, then H is a Breaker win.

Incidence

Theorem

Let G be a graph with n vertices and m edges. $s(G) \geq \frac{m}{4} - \frac{n}{8}$

Sketch of the proof.

◁

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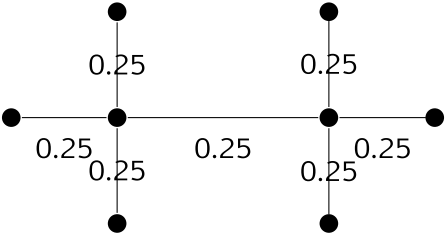

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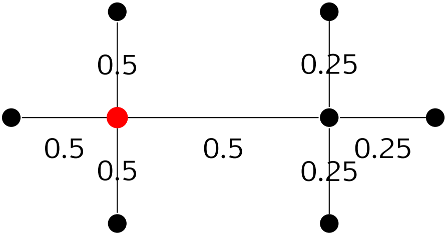

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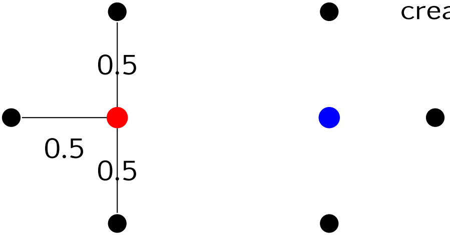

After 2 moves, $P(G)$ decreases by at most $\frac{1}{4}$

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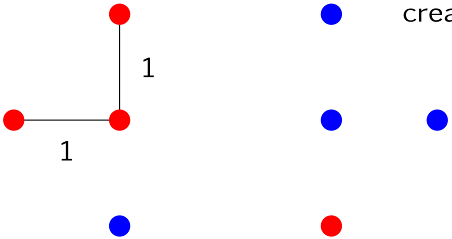

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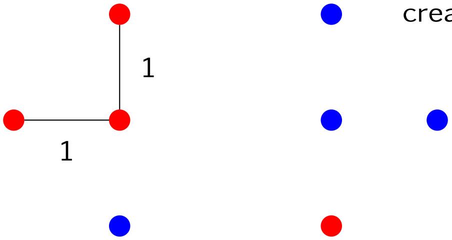

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$$s(G) = P(\emptyset)$$

◁

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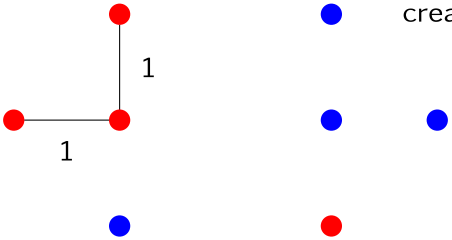

After 2 moves, $P(G)$ decreases by at most $\frac{1}{4}$

$$\begin{aligned} s(G) &= P(\emptyset) \\ &\geq P(G) - \frac{1}{4} \cdot \frac{n}{2} \\ &= \frac{m}{4} - \frac{n}{8} \end{aligned}$$

◁

Open questions

- The score on many graph classes is open.

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- A link with the largest connected subgraph game.

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- The score on many graph classes is open.
- A link with the largest connected subgraph game.
- The complexity of the maker-maker convention on 3-regular hypergraphs ?

Thank you !