The Game of Cops and Robber on String Graphs

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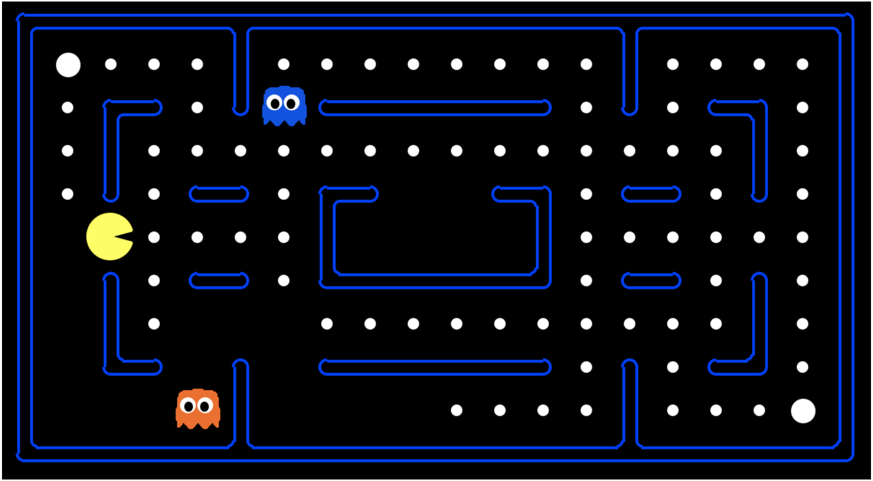
PacMan



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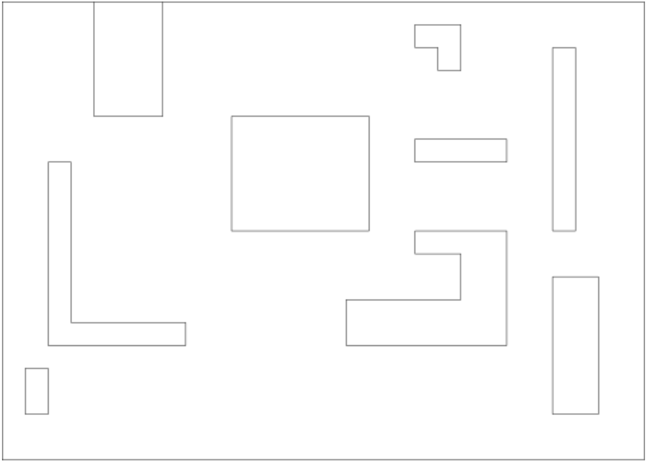

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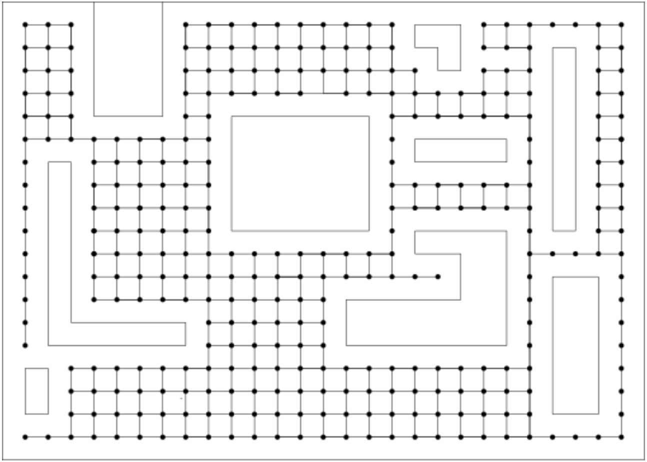

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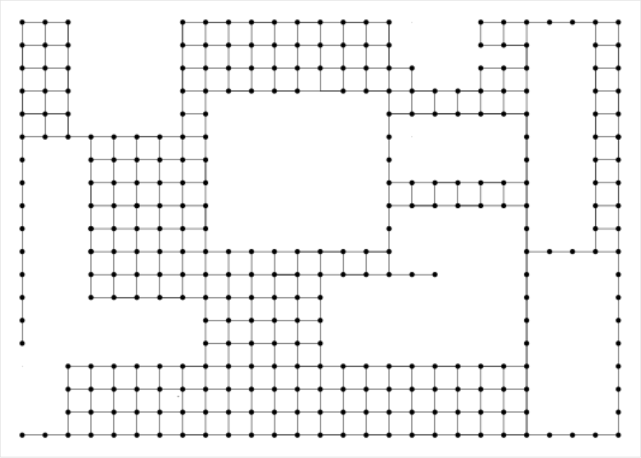

Cops and Robber

Playground

A simple, connected and finite graph.

Two teams

k cops and a single robber.

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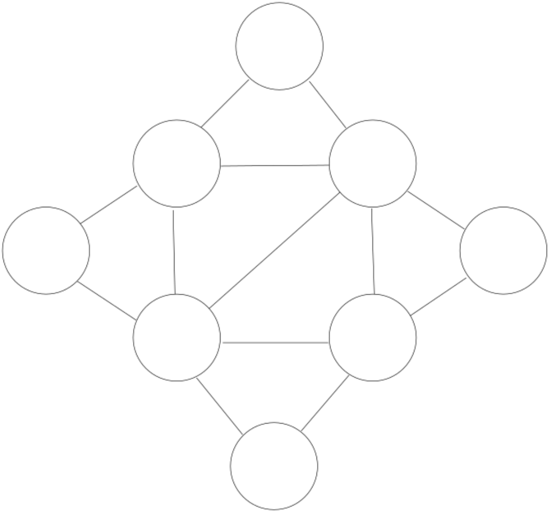
Winning

- Cops win if some cop and robber occupy the same vertex. (*Capture*)
- Robber wins otherwise.

Demo(Playground)



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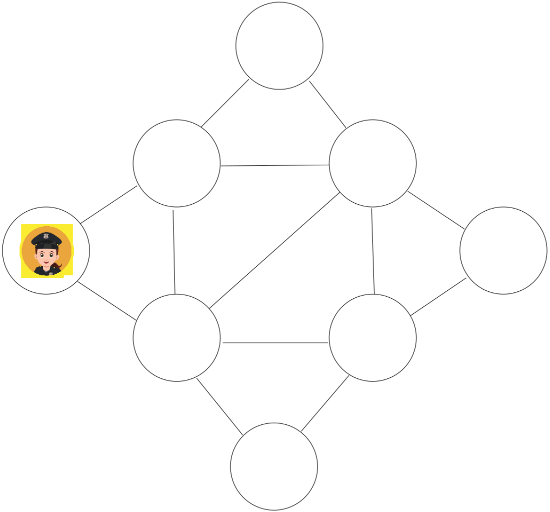

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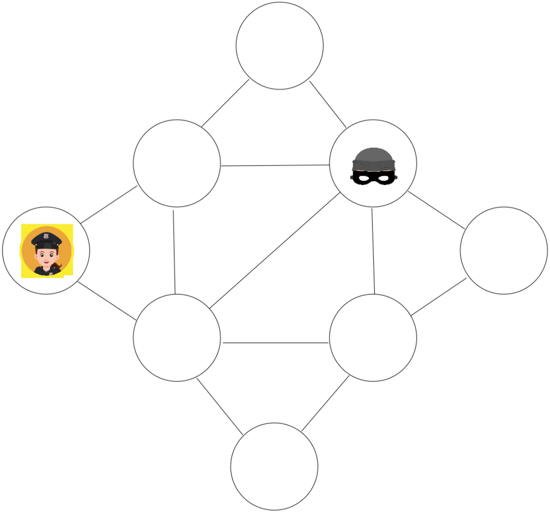
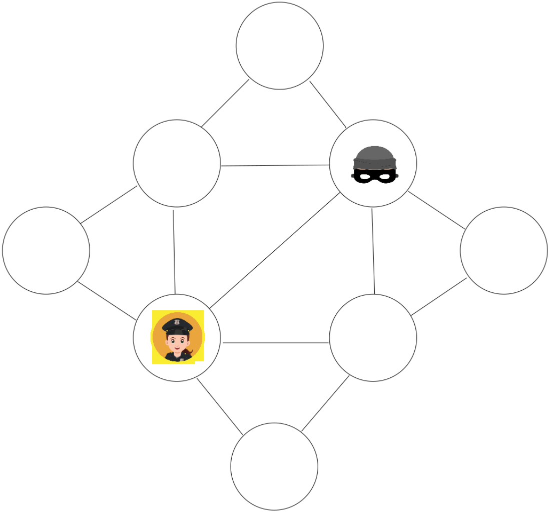


Demo



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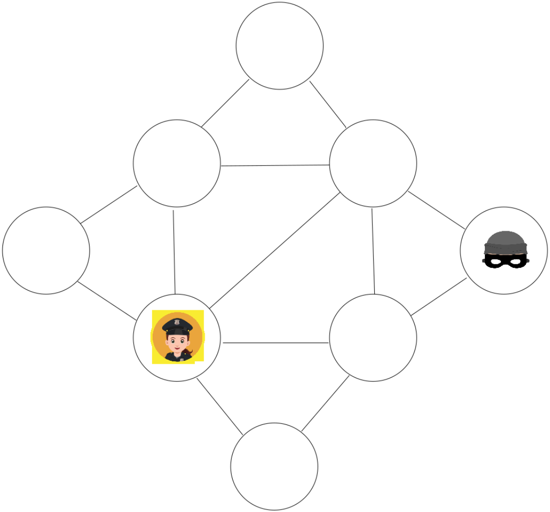

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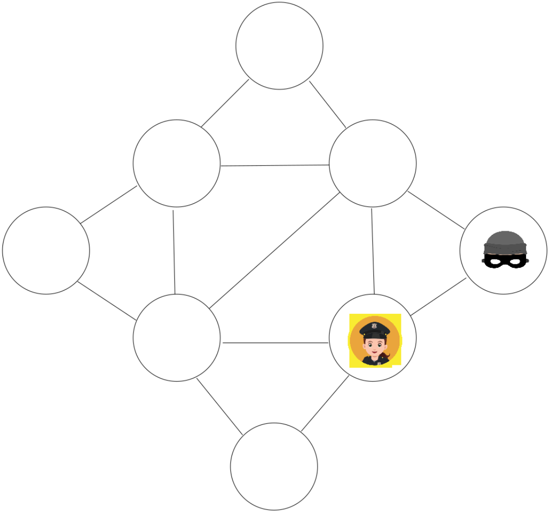

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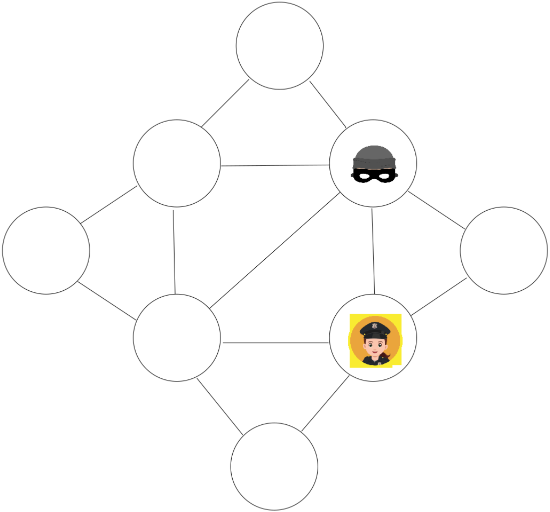
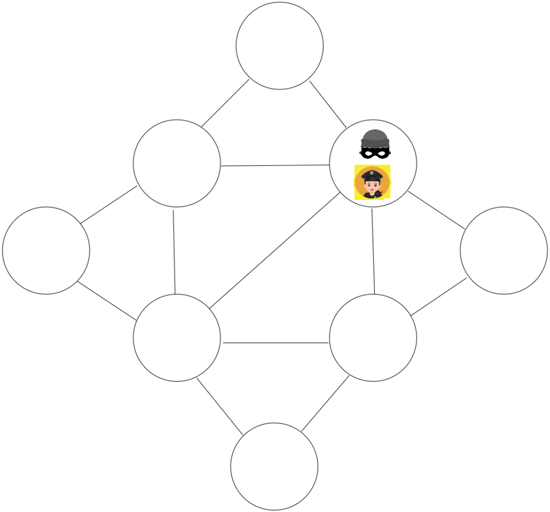


Demo(Capture)



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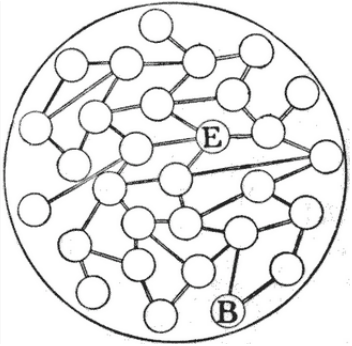
Prehistory

In the book *Amusements in Mathematics*, published in 1917, Henry Ernest Dudeney asked the following question.

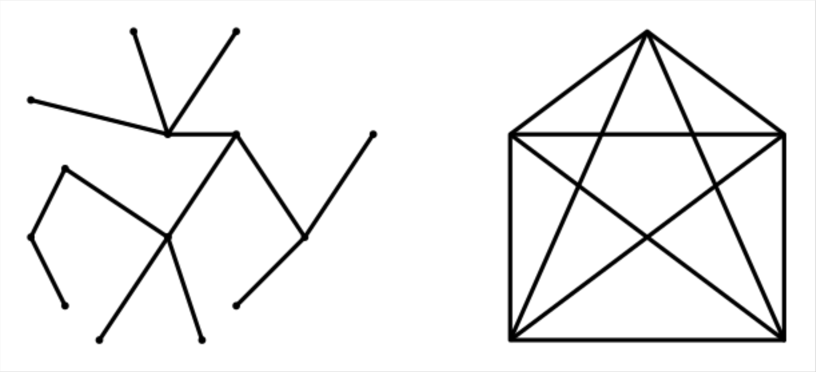


Origin

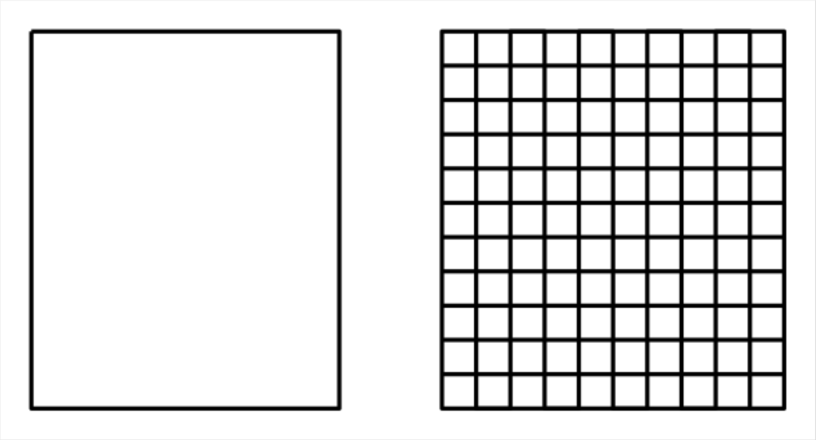
One Cop vs Robber

- First considered by A. Quilliot in his doctoral thesis in 1978.
- Considered independently by Nowakowski and Winkler in 1983.
- Both characterized the *cop-win* graphs, where one cop can win.

Some Copwin Graphs



Some Not Copwin Graphs



Cop number

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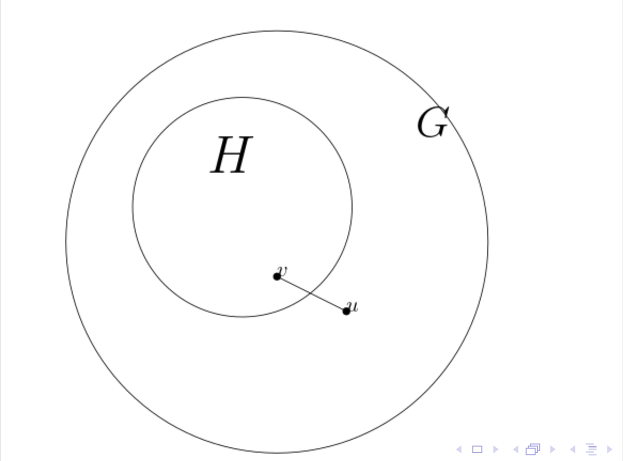
Cop number

- is the minimum number of cops required to capture a robber in the graph.
- is denoted by $c(G)$.
- is upper bounded by *domination number*.

Definitions

Guarding a subgraph

Let H be a subgraph of G. Cops *guard* H if $\mathcal{R}$ cannot enter H without being captured.



Cops and Robber on Planar graphs

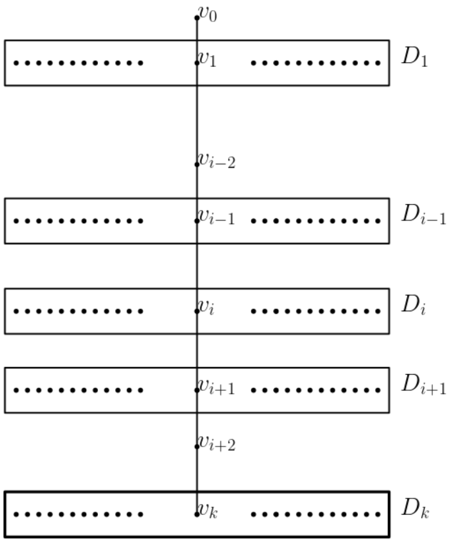
Lemma [Aigner and Fromme, 1984]

Let P be an isometric path in G. Then one cop can guard P.

Theorem [Aigner and Fromme, 1984]

Let $\mathcal{P}$ be the class of planar graphs. Then $c(\mathcal{P}) = 3$.

Guarding isometric paths



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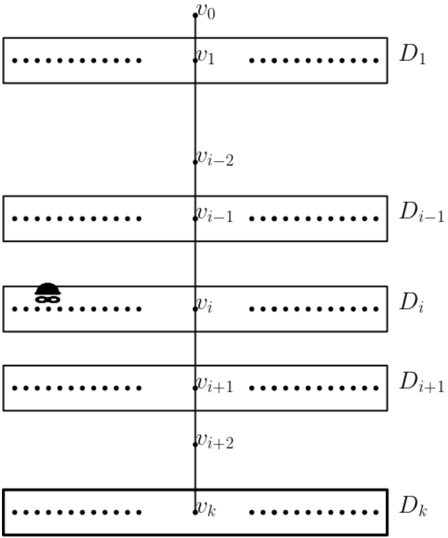

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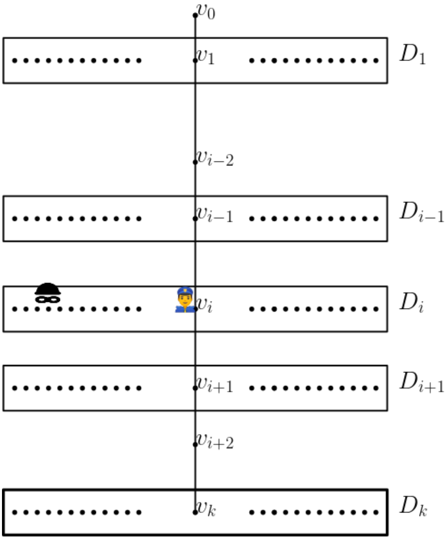

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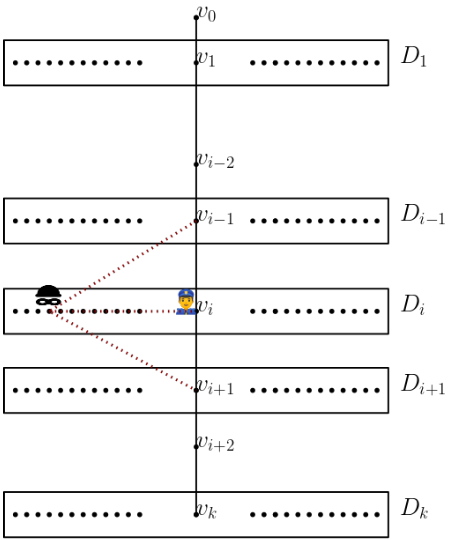

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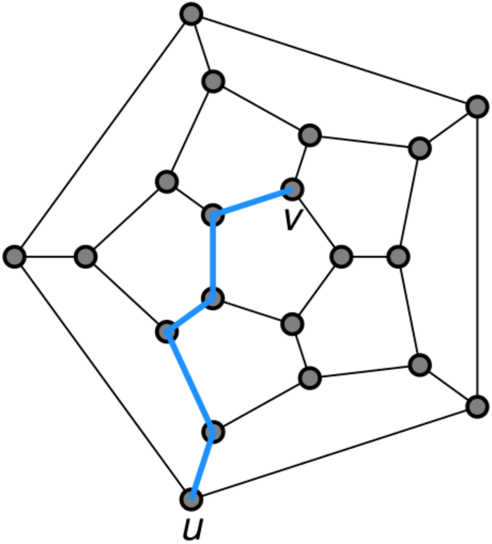

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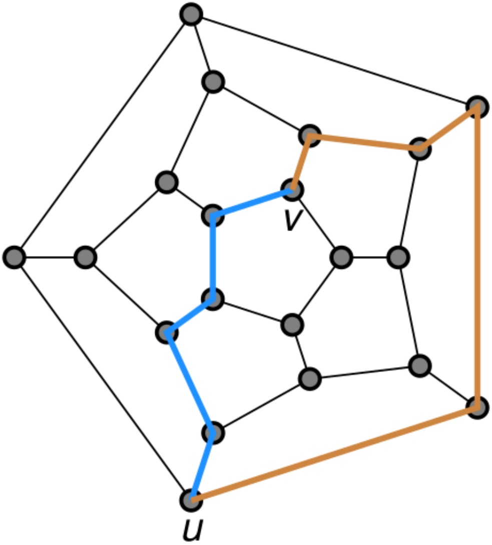

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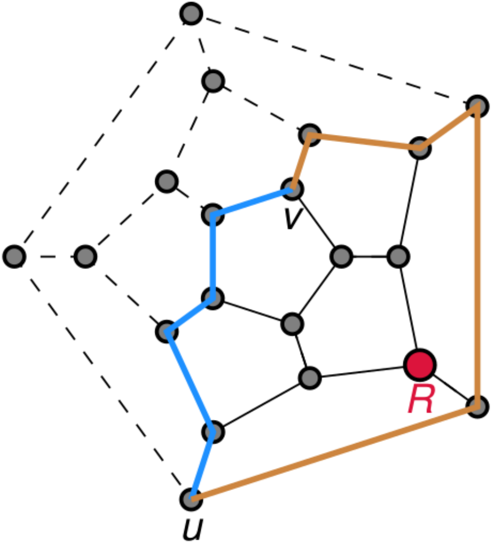

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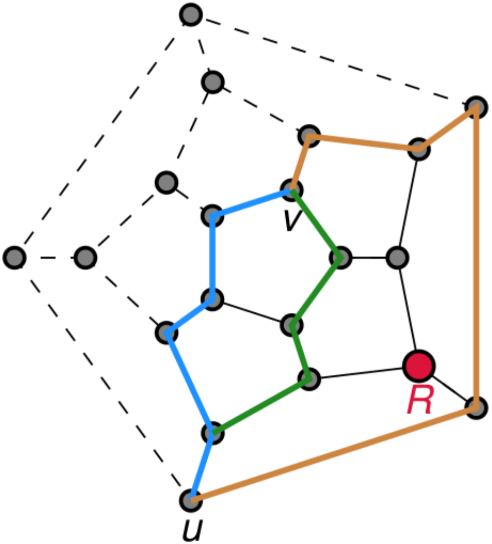

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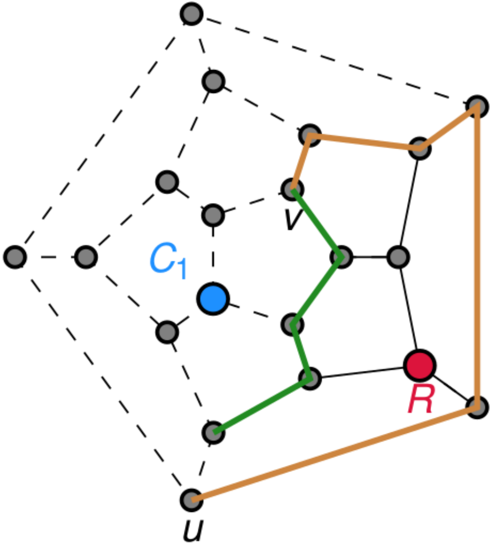

Preventing Robber to Cross an Isometric Path

Lemma 1 [GGJKK, 2018]

Let P be an isometric path in G. Then 5 cops can guard $N[P]$.

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Lemma [BDFM, 2012]

Let P be an isometric path in a unit-disk graph G. Then three cops can prevent the robber to cross P.

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Theorem [BDFM, 2012]

Let G be a unit-disk graph. Then $c(G) \leq 9$.

Guarding the Neighbourhood of Unique Isometric Paths

Unique isometric path

A u, v-path P is the *unique isometric path* if all other u, v-paths are longer than P.

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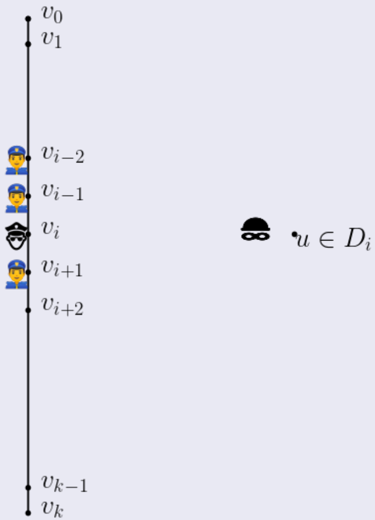
Lemma 2 [Das and G.]

Let P be a unique isometric path in G. Then 4 cops can guard $N[P]$.

Guarding the Neighbourhood of Unique Isometric Paths

Proof



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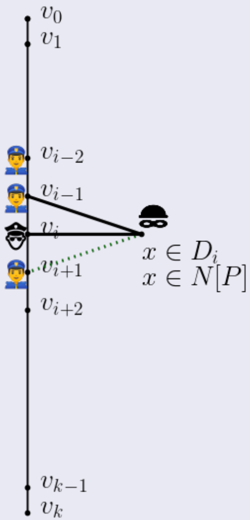

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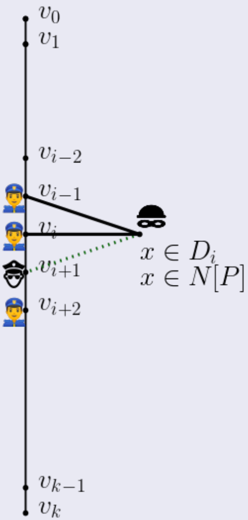

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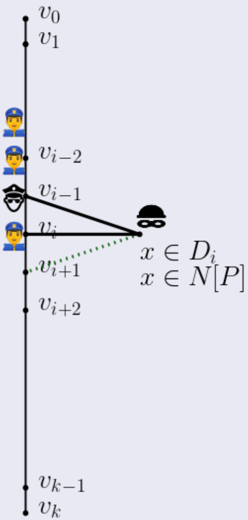

Cops and Robber on String graphs

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Capturing the robber on String Graphs

- Guard a isometric u, v-path such that u is a top-most and v is a bottom-most vertex.

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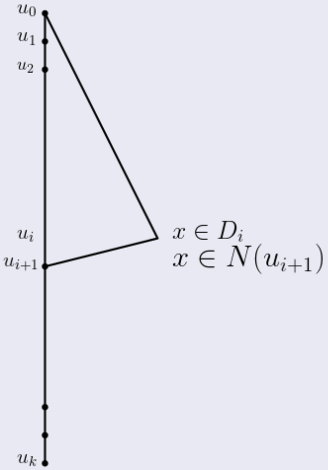
- Guard a isometric u, v-path such that u is a top-most and v is a bottom-most vertex.
- Extend the isometric path in robber region.

Extending an isometric path



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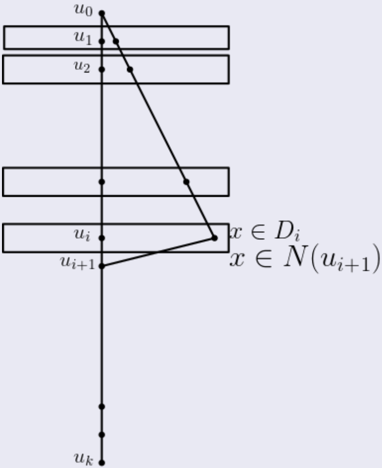

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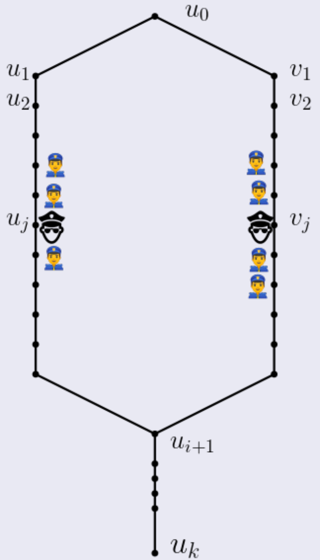

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Theorem 1 [Das and G.]

Let G be a string graph. Then $c(G) \leq 13$.

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A string graph G with girth 5 and cop number k is k-degenerate and hence $k+1$ colorable.

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A string graph G with girth 5 and cop number k is k-degenerate and hence $k+1$ colorable.

Corollary 1 [Das and G.]

If G is a girth 5 string graph, then G is 13-degenerate and 14-colorable.

Fully active Cops and Robber

Introduction

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- Each cop has to move in every round to an adjacent vertex.
- Cop number of a graph G is denoted by $c_A(G)$.
- For a graph G, $c_A(G) \leq 2c(G)$.
- Goromikov et al. (2018) asked if there is some constant k such that for any planar graph G, $c_A(G) \leq k$.

Fully active Cops and Robber on Planar graphs

Observation 1

Let P be an isometric path of a planar graph G. Two cops can guard P.

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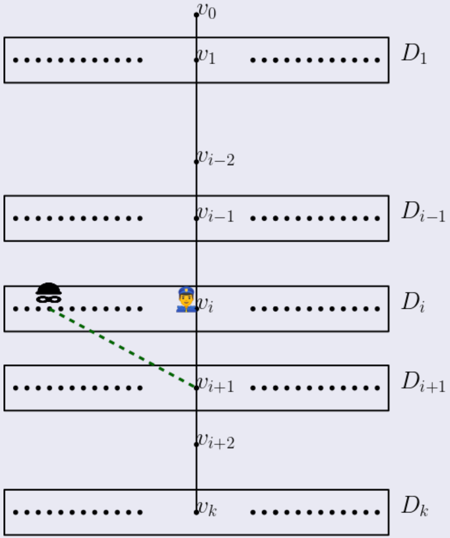
Let P be an isometric path of a planar graph G. Two cops can guard P.

Lemma 2 [Das and G.]

Let P be a unique isometric path in a graph G. Then 1 active cop can guard P.

Guarding Isometric Paths using Active Cops

Proof

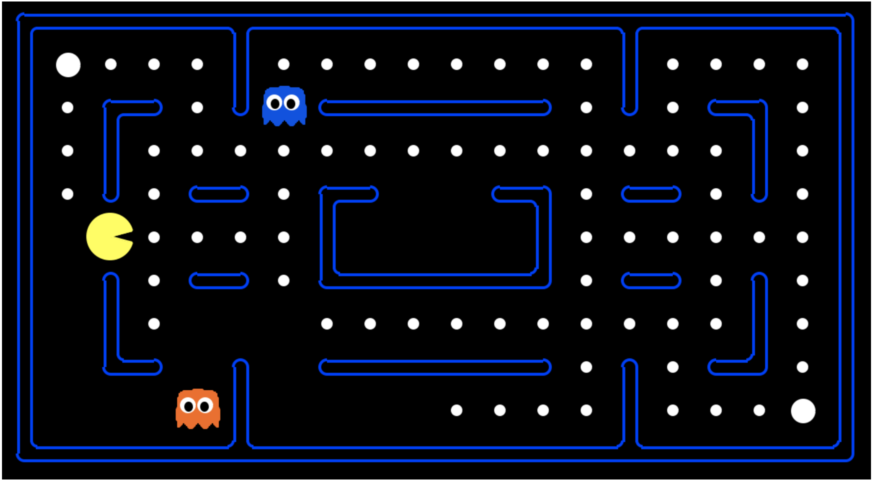


Fully active Cops and Robber on Planar graphs

Theorem 3 [Das and G.]

Let G be a planar graph. Then $c_A(G) \leq 4$.

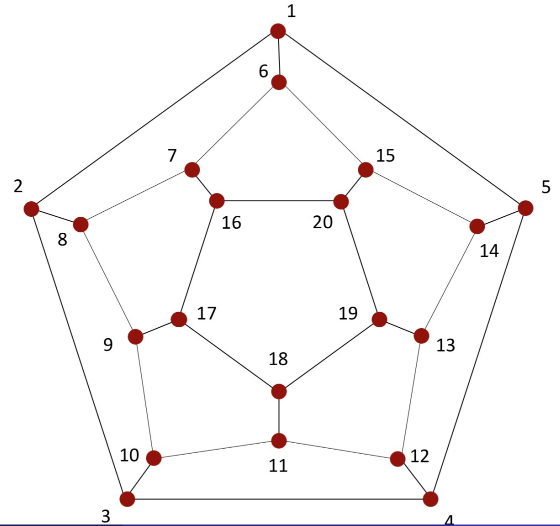
PacMan



PacMan and Cops and Robber on Subgraphs of Grids

Joret et al.(2010) showed that subdividing each edge of G an equal number of times does not decrease $c(G)$.

Partial Grids



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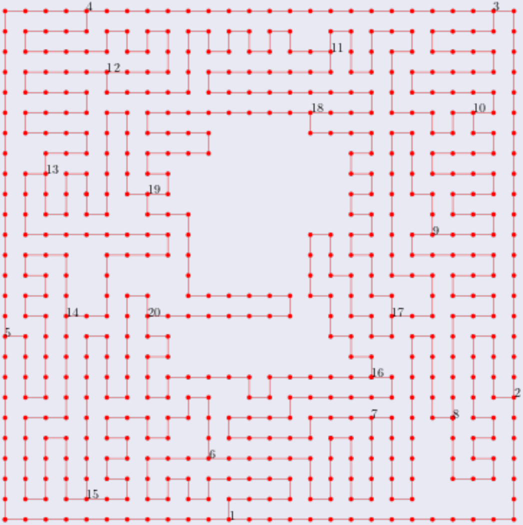

Open Problems

Cops and Robber on String graphs of genus g

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- Using our results, it gives $c(G) \leq 10g + 13$.
- Let C be an isometric cycle of G. What is the least k such that k cops can guard $N[C]$? If $k < 10$, then $kg + 13$ cops have an obvious winning strategy.

Open Problems

Active Cops and Robber on Planar graphs

Is there a planar graph G such that $c_A(G) = 4$?

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String Graphs with High Cop Number

Constructing string graphs of high cop number.

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Meyniel's Conjecture

For any graph G, $c(G) = O(\sqrt{n})$.

Thank You