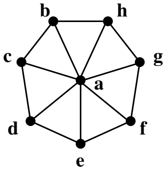
Similarity of treewidth and MM-width by a cops and robber game

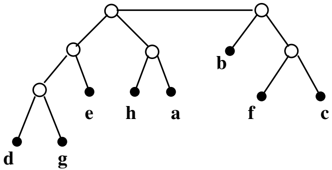
Jan Arne Telle
presenting results of Martin Vatshelle and Sigve H. Sæther

Department of Informatics, University of Bergen, Norway

Branch decompositions

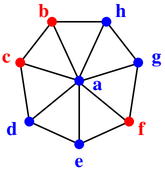
Branch decomposition (rooted) of graph G is a pair (T, δ)
-T is a ternary tree (binary) and
-δ is a bijection between leaves of T and vertices of G (or $E(G)$)

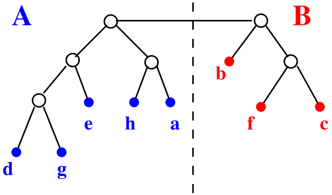




-one cut of G for each edge of T

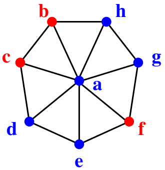
Defining a width parameter using a cut function

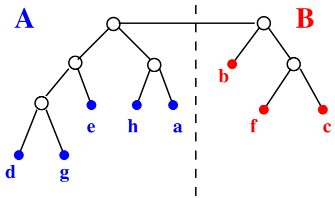




- Cut function (symmetric) $f : 2^{V(G)} \rightarrow N$

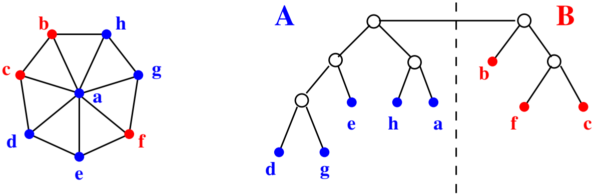
Defining a width parameter using a cut function





- Cut function (symmetric) $f : 2^{V(G)} \to N$
- $fwidth(T,\delta) = \max_{uv \in E(T)}\{f(A_u)\} = \max_{u \in V(T)}\{f(A_u)\}$

Defining a width parameter using a cut function



- Cut function (symmetric) $f : 2^{V(G)} \rightarrow N$
- $fwidth(T, \delta) = \max_{uv \in E(T)}\{f(A_u)\} = \max_{u \in V(T)}\{f(A_u)\}$
- $fwidth(G) = \min_{(T,\delta)}\{fwidth(T, \delta)\}$

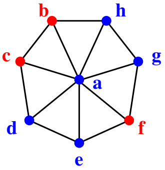
Examples

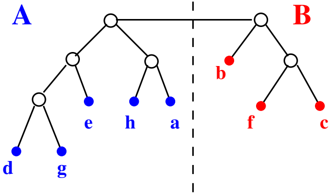
Carving-width

Rank-width

Boolean-width

MM-width $mmw(G)$:
cut function $mm(A) =$ size of Maximum Matching of $G(A, \overline{A})$





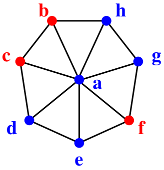
Examples

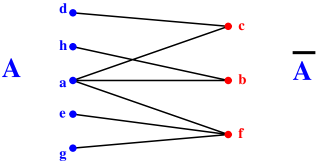
Carving-width

Rank-width

Boolean-width

MM-width $mmw(G)$:
cut function $mm(A) =$ size of Maximum Matching of $G(A, \overline{A})$





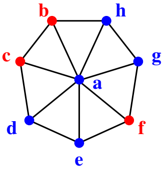
Examples

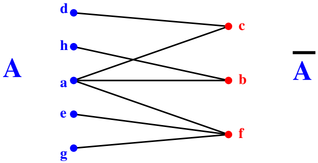
Carving-width

Rank-width

Boolean-width

MM-width $mmw(G)$:
cut function $mm(A) =$ size of Maximum Matching of $G(A, \overline{A})$





Example:
$mmw(K_n) = n/3$ (ternary tree and max matching of $K_{a,b}$ is $\min(a, b)$).

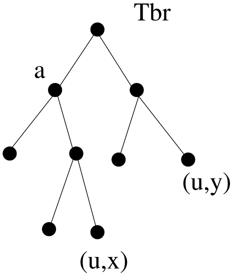
Rest of talk

1. $mmw(G) \leq tw(G) + 1$ [Vatshelle'12]
2. $tw(G) \leq 3mmw(G)$ by non-monotone cop strategy [Vatshelle'12]
3. This strategy can be made monotone [Sæther'13]
4. mm cut function is submodular [Sæther'13]

$mmw(G) \leq brw(G)$

Branchwidth

$brw(G)$ defined by cut function $br : 2^{E(G)} \to N$, with $br(E_a) = |midset(E_a, \overline{E_a})|$ = number of vertices both in edge mapped to a-subtree and in edge not mapped to a-subtree.



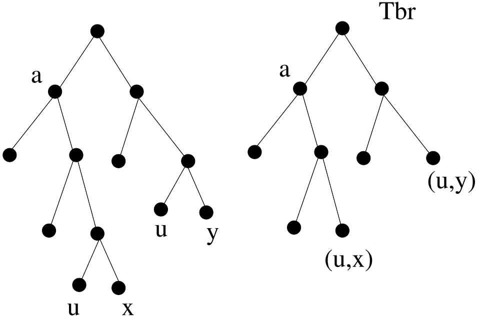
.

$mmw(G) \leq brw(G)$

Branchwidth

$brw(G)$ defined by cut function $br : 2^{E(G)} \rightarrow N$, with
$br(E_a) = |midset(E_a, \overline{E_a})|$ = number of vertices both in edge mapped to a-subtree and in edge not mapped to a-subtree.

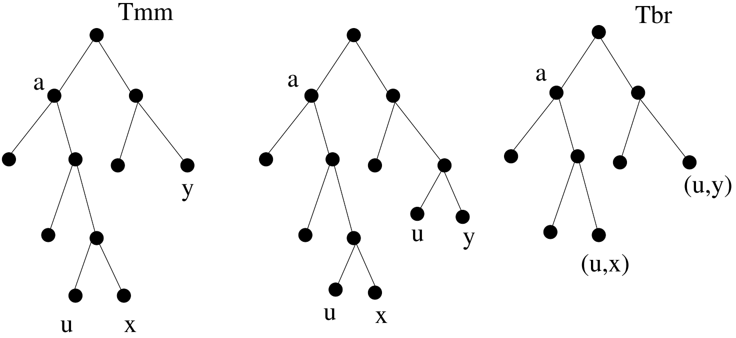
.



$mmw(G) \leq brw(G)$

Branchwidth

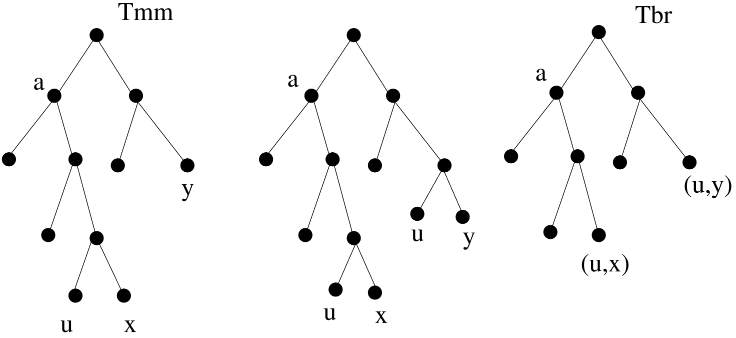
$brw(G)$ defined by cut function $br : 2^{E(G)} \rightarrow N$, with $br(E_a) = |midset(E_a, \overline{E_a})|$ = number of vertices both in edge mapped to a-subtree and in edge not mapped to a-subtree.



$mmw(G) \leq brw(G)$

Branchwidth

$brw(G)$ defined by cut function $br : 2^{E(G)} \rightarrow N$, with $br(E_a) = |midset(E_a, \overline{E_a})|$ = number of vertices both in edge mapped to a-subtree and in edge not mapped to a-subtree.



- Assume (u, y) in matching M of $G(A_a, \overline{A_a})$ of Tmm. Then either u or y in mid-set of $(E_a, \overline{E_a})$ of Tbr. Thus $mmw(G) \leq brw(G) \leq tw(G) + 1$

$tw(G) \leq 3mmw(G) - 1$

Treewidth $tw(G)$ is number of cops (-1) needed to capture robber when:

- Robber is visible and moves fast along cop-free paths
- Cops move by helicopter

$tw(G) \leq 3mmw(G) - 1$

Treewidth $tw(G)$ is number of cops (-1) needed to capture robber when:

- Robber is visible and moves fast along cop-free paths
- Cops move by helicopter

For G, given (Tmm, δ) of MM-width k, here is $3k$-cop strategy on G:

- Start at root of Tmm and move down:

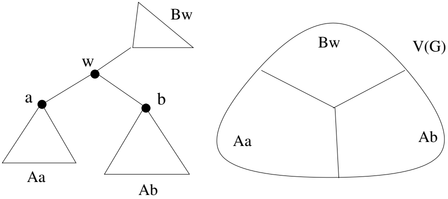
$tw(G) \leq 3mmw(G) - 1$

Treewidth $tw(G)$ is number of cops (-1) needed to capture robber when:

- Robber is visible and moves fast along cop-free paths
- Cops move by helicopter

For G, given (Tmm, δ) of MM-width k, here is $3k$-cop strategy on G:

- Start at root of Tmm and move down:
- When reaching w with children a, b: know robber in $A_w = A_a \cup A_b$ and we have cops on a Minimum Vertex Cover C_w of $G(A_w, B_w)$. By König's Theorem $|C_w| \leq k$. C_w is a separator.



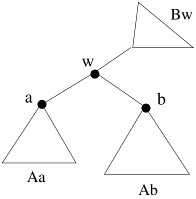
$tw(G) \leq 3mmw(G) - 1$

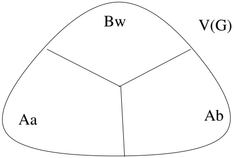
Treewidth $tw(G)$ is number of cops (-1) needed to capture robber when:

- Robber is visible and moves fast along cop-free paths
- Cops move by helicopter

For G, given (Tmm, δ) of MM-width k, here is $3k$-cop strategy on G:

- Start at root of Tmm and move down:
- When reaching w with children a, b: know robber in $A_w = A_a \cup A_b$ and we have cops on a Minimum Vertex Cover C_w of $G(A_w, B_w)$. By König's Theorem $|C_w| \leq k$. C_w is a separator.





- Add $\leq k$ cops on C_a and $\leq k$ cops on C_b

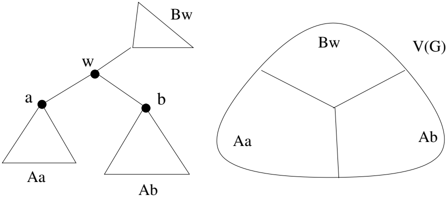
$tw(G) \leq 3mmw(G) - 1$

Treewidth $tw(G)$ is number of cops (-1) needed to capture robber when:

- Robber is visible and moves fast along cop-free paths
- Cops move by helicopter

For G, given (Tmm, δ) of MM-width k, here is $3k$-cop strategy on G:

- Start at root of Tmm and move down:
- When reaching w with children a, b: know robber in $A_w = A_a \cup A_b$ and we have cops on a Minimum Vertex Cover C_w of $G(A_w, B_w)$. By König's Theorem $|C_w| \leq k$. C_w is a separator.



- Add $\leq k$ cops on C_a and $\leq k$ cops on C_b
- wlog robber in A_a: Move to a and keep cops only on C_a

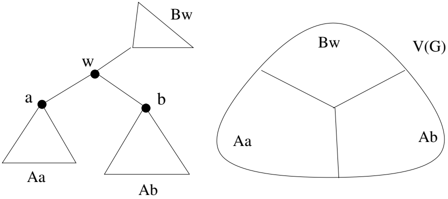
$tw(G) \leq 3mmw(G) - 1$

Treewidth $tw(G)$ is number of cops (-1) needed to capture robber when:

- Robber is visible and moves fast along cop-free paths
- Cops move by helicopter

For G, given (Tmm, δ) of MM-width k, here is $3k$-cop strategy on G:

- Start at root of Tmm and move down:
- When reaching w with children a, b: know robber in $A_w = A_a \cup A_b$ and we have cops on a Minimum Vertex Cover C_w of $G(A_w, B_w)$. By König's Theorem $|C_w| \leq k$. C_w is a separator.

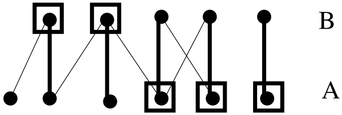


- Add $\leq k$ cops on C_a and $\leq k$ cops on C_b
- wlog robber in A_a: Move to a and keep cops only on C_a

Non-monotone since vertex x could go in/out/in of the Vertex Covers.

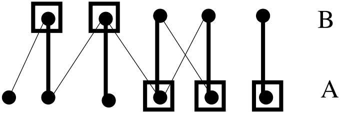
Monotone strategy [Sæther]

Given max matching M of $G(A, B)$ define König Vertex Cover $C(M)$:
-For every edge in M put A-vertex in $C(M)$; unless robber can then escape from A, i.e. unless there is an alternating path containing the edge and starting in an unsaturated A-vertex; if so put B-vertex in $C(M)$.



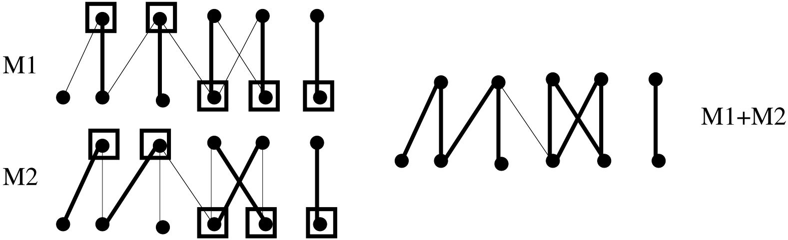
Monotone strategy [Sæther]

Given max matching M of $G(A, B)$ define König Vertex Cover $C(M)$:
-For every edge in M put A-vertex in $C(M)$; unless robber can then escape from A, i.e. unless there is an alternating path containing the edge and starting in an unsaturated A-vertex; if so put B-vertex in $C(M)$.



Fact

If $M1, M2$ are max matchings then $C(M1) = C(M2)$



Using König Vertex Covers $3k$-cops strategy is monotone

Robber is always on A-side, and A-side shrinks as we move down Tmm.

Cop movement legal if we keep all cops on old A-side and add no new cops on old B-side.

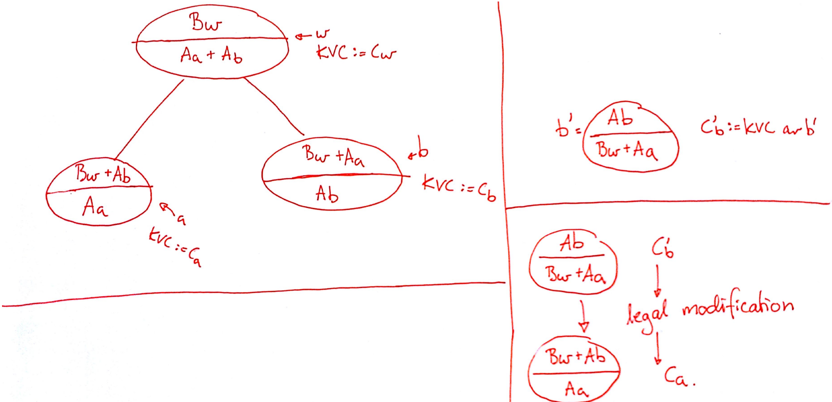
Combining legal movements gives monotone strategy.

Using König Vertex Covers $3k$-cops strategy is monotone

Robber is always on A-side, and A-side shrinks as we move down Tmm.
Cop movement legal if we keep all cops on old A-side and add no new cops on old B-side.
Combining legal movements gives monotone strategy.

Lemma

Moving cops from Cw to $Cw \cup Ca \cup Cb$ for $G(Aw, Bw)$ is legal.
Moving from $Cw \cup Ca \cup Cb$ for $G(Aw, Bw)$ to Ca for $G(Aa, Ba)$ legal.



Max matching is a submodular cut function

Recall: for $A \subseteq V(G)$ define $mm(A) =$size of max matching of $G(A, \overline{A})$

Lemma

For $A, B \subseteq V(G)$ *have* $mm(A) + mm(B) \geq mm(A \cup B) + mm(A \cap B)$

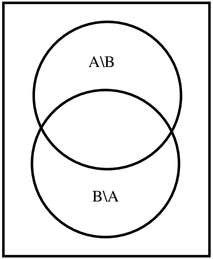
Max matching is a submodular cut function

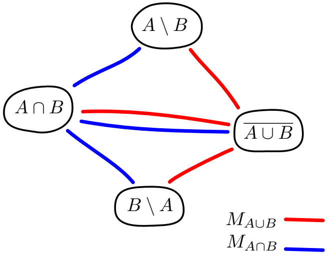
Recall: for $A \subseteq V(G)$ define $mm(A)$ =size of max matching of $G(A, \overline{A})$

Lemma

For $A, B \subseteq V(G)$ have $mm(A) + mm(B) \geq mm(A \cup B) + mm(A \cap B)$

For any matchings $M_{A\cup B}$ and $M_{A\cap B}$ there exists matchings M_A and M_B such that $M_A \dot{\bigcup} M_B = M_{A\cup B} \dot{\bigcup} M_{A\cap B}$ (as multisets). Note $M_{A\cup B} \dot{\bigcup} M_{A\cap B}$ forms vertex-disjoint paths and cycles. Let P be such. Show matchings N_A and N_B on the same edges as P, then take disjoint union of these to get M_A and M_B. Edges of P alternate Blue and Red, so at most one vertex v of P in $A \setminus B \cup B \setminus A$, say wlog $v \in B \setminus A$. Then $P \cap M_{A\cap B}$ is a matching of A and $P \cap M_{A\cup B}$ is a matching of B.





Use of MM-width

MM-width has been used to define a parameter between treewidth and clique-width [ST'14]

Graphs of MM-width at most k are closed under minors.
For $k = 1$ the set of Minimal Forbidden Minors is $\{C_4\}$.
What about larger k?

Other uses of MM-width?

Use of MM-width

MM-width has been used to define a parameter between treewidth and clique-width [ST'14]

Graphs of MM-width at most k are closed under minors.
For $k = 1$ the set of Minimal Forbidden Minors is $\{C_4\}$.
What about larger k?

Other uses of MM-width?

Thank you!

Thank you!