Monotonicity in directed cops and robber games

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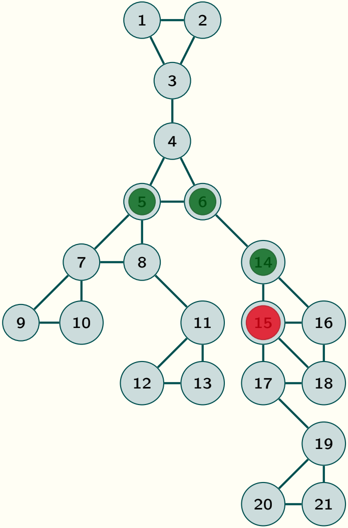
Treewidth game on undirected graphs



Game rules:

- *k* cops, one robber
- robber runs along cop free paths
- cops fly
- cops want to capture robber

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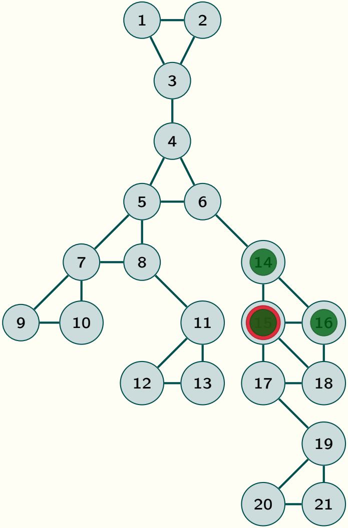

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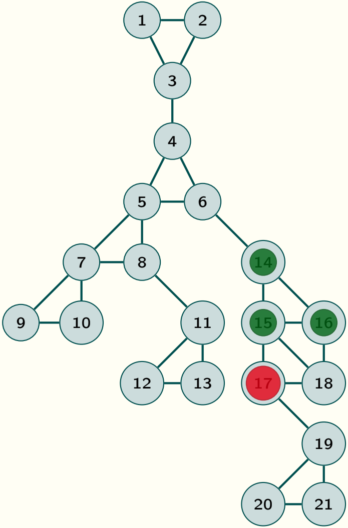

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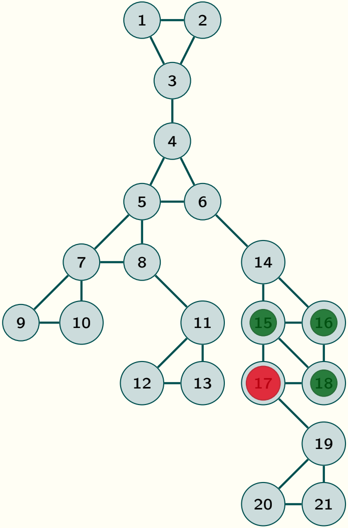

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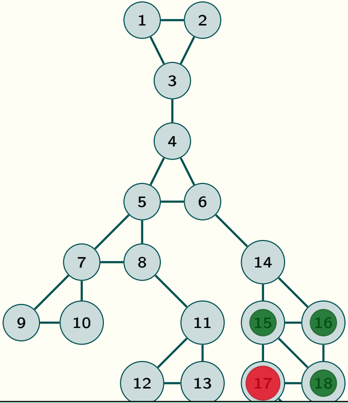

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Treewidth $=$ the minimal number of cops capturing the robber -1

Generalisations to directed graphs

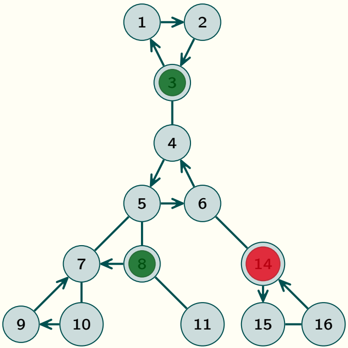
(1) DAG-width, a similar game
(2) Kelly-width, inert and invisible robber
(3) directed treewidth game

DAG-width game

the same game as for treewidth, just now

- the robber runs along directed paths
- cops win only robber-monotone plays:

vertices that have been occupied
by a cop
remain unreachable for the robber for ever



Robber-monotonocity

- important for algorithmical applications (DAG-decompositions),
- but not for treewidth: no additional cops needed to win robber-monotonically [Robertson, Thomas]
- Non-(robber-)monotone DAG-width game
  - a technical notion
  - often: capturing the is preserved, the monotonicity not
  - e.g.: a winning cop strategy for a Kelly-width game translates to a winning cop strategy for the non-monotone DAG-width game [Hunter, Kreutzer]
- at least a $4k/3$-gap for DAG-width [Kreutzer, Ordyniak]
- an interesting open question [Berwanger et al.]: is there a function f such that if k cops capture the robber, then $f(k)$ cops capture the robber in a robber-monotone way independently of the graph?

Fighting for monotonicity in DAG-width

non-mon. $\lneq_{?}$ rob.-mon.

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Theorem. If k cops win in a weakly monotone way, then $O(k^2)$ cops win (strongly) robber-monotonically.

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Theorem. If k cops win in a weakly monotone way, then $O(k^2)$ cops win (strongly) robber-monotonically.

- This covers the (only known) examples by Kreutzer and Ordyniak.
- The translation of strategies from Kelly-width game to DAG-width leads to *weakly monotone* strategies, so $\mathrm{DAG}^2 \leq O(\mathrm{Kelly})$, conjectured in a stronger form by [Hunter, Kreutzer].

The weakly monotone game

As the robber-monotone game, but now
vertices that have been occupied
by a cop that was placed in the robber component
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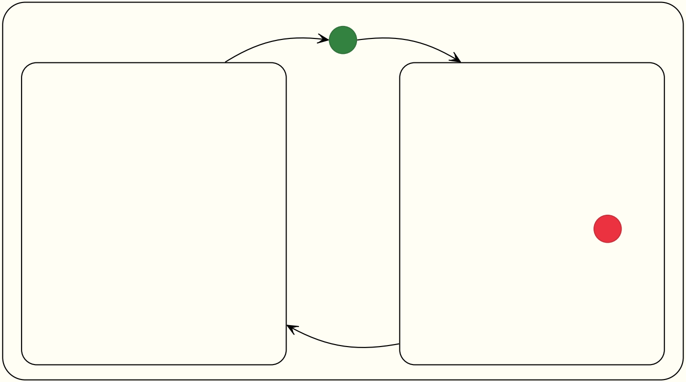

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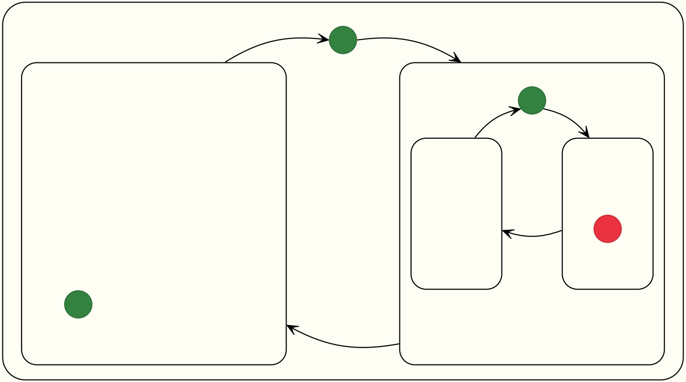

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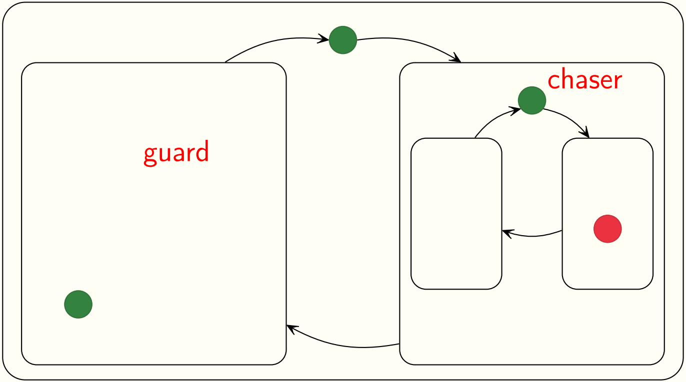


Theorem

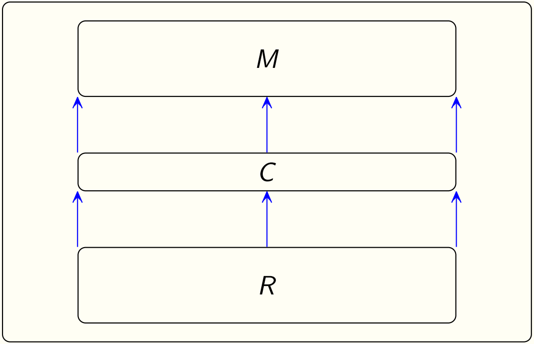
If k cops win in a weakly monotone way, then $O(k^2)$ cops win (strongly) robber-monotonically.

Proof sketch

1. When the robber does not change his component:
   a shy robber game.
2. If the robber changes his component.

Proof sketch

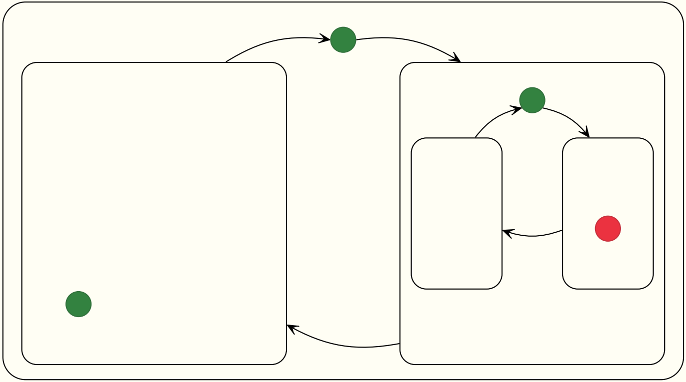
1. Shy robber game: Minimally blocking strategies.
   1.1 C blocks $R \to M$.
   1.2 Minimize $|C|$.
   1.3 Among all such C choose one as close to M as possible (so C blocks as few vertices as possible).



   1.4 Place chasers as before and guards minimally blocking.
   1.5 Then we have strong monotonicity.

Proof sketch

2. The robber changes his component.
   2..1 Change strategy for weak game:



   2..2 Perform the main trick.

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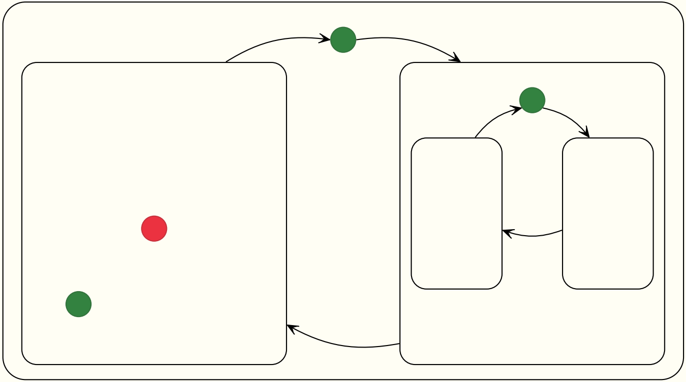

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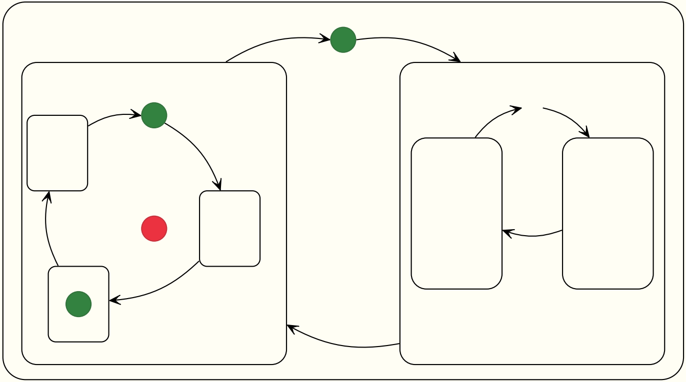

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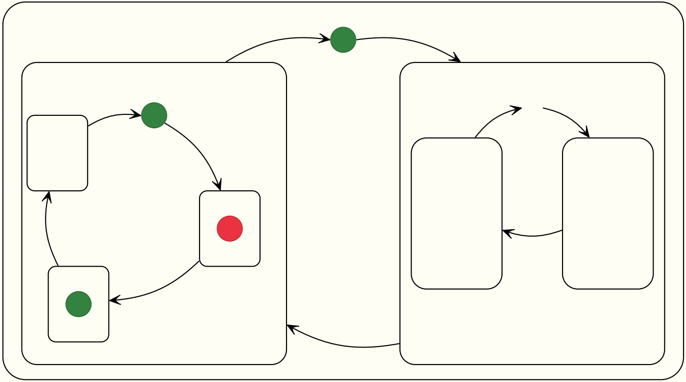

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Directed treewidth game

As the DAG-width game, but better for the cops:

- the robber not allowed to leave his SCC
- no robber-monotonicity needed (but can be enforced)

Cop-monotonicity: reoccupying vertices is forbidden for cops.

Theorem. There exist graphs G_n where 4 cops capture the robber, but n cops are needed to so in a cop-monotone way.