Patrolling Games – The Line

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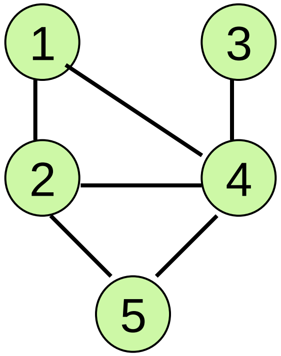
Outline

- Introduce Patrolling Games on a graph.
- Strategies and earlier results.
- The Discrete Line.
- The Continuous Line.

Patrolling Game on a Graph

Graph: Q=(N,E)
Nodes: N ={1,2,…,n}
Edges: E



T = time horizon of the game
t = 1,…,T

Players

Attacker: picks a node i and time τ to perform the attack and needs m uninterrupted periods at the node for the attack to be successful

Patroller: picks a walk w on the graph that lasts T time periods and is successful if the walk intercepts the Attacker during the attack.

Pure Strategies

Attacker: (i, τ)
Patroller: w

Mixed Strategies:

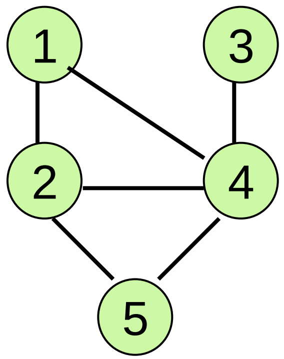
Playing (i, τ) with probability p(i, τ)
Playing w with probability p(w)

We assume: $T \geq m$

Patrolling Game on a Graph

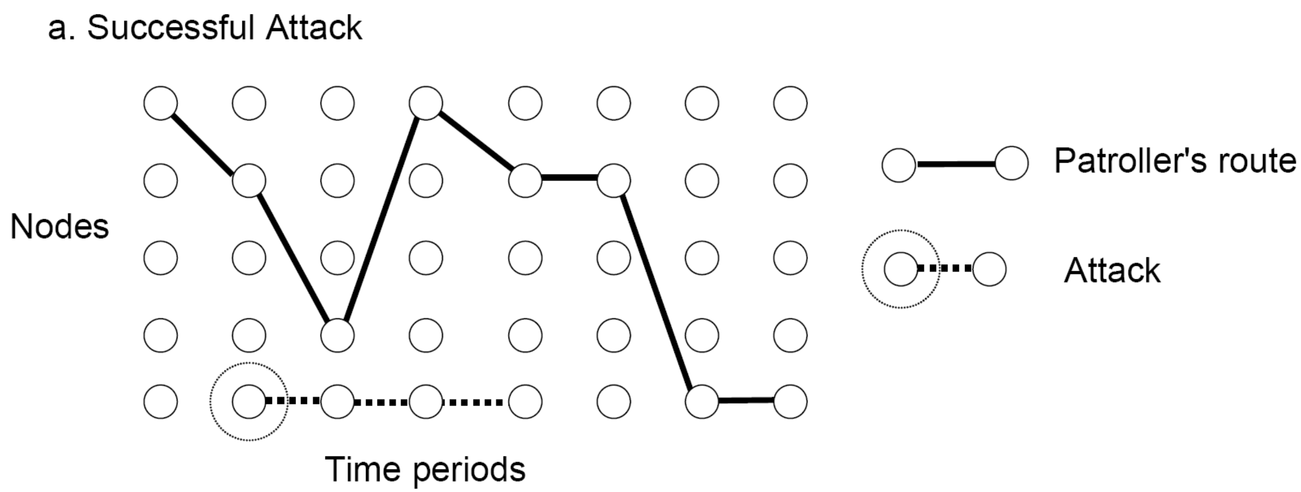
Space-time Network:

n=5, T=8, m=4



patroller picks: w = 1-2-4-1-2-2-5-5

attacker picks: (i, τ) =(5,2)

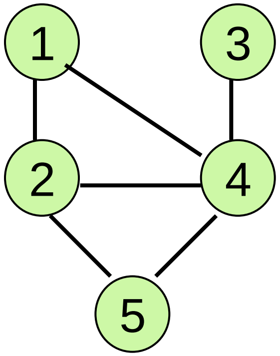


Since the patroller's walk does not intercept the attacker the attack is successful.

Patrolling Game on a Graph

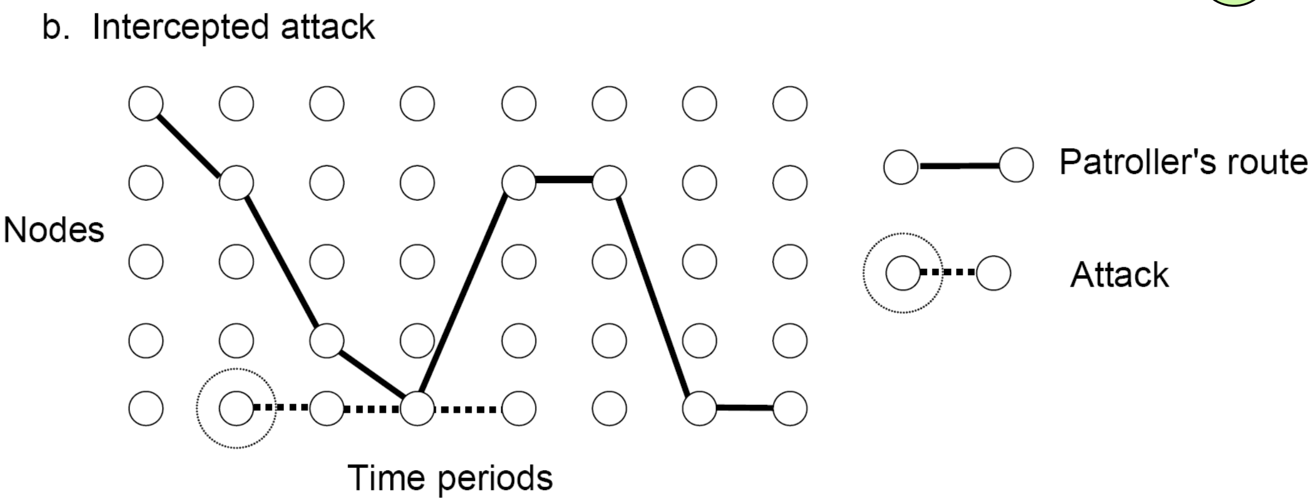
Space-time Network:

n=5, T=8, m=4



patroller picks: w = 1-2-4-5-2-2-5-5

attacker picks: (i, τ) =(5,2)



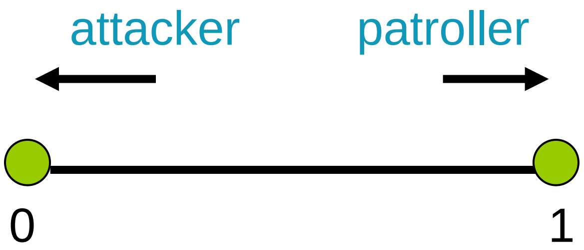
Since the patroller's walk intercepts the attacker the attack is not successful.

Patrolling Game on a Graph

The game is a zero-sum game with the following payoff:

$$\text{Payoff to the patroller} = \begin{cases} 1 & \text{if } (i, \tau) \text{ is intercepted by } w \\ 0 & \text{otherwise} \end{cases}$$

Value of the game = probability that the attack is intercepted



We denote the value of the game V or V(Q, T, m).

Types of Games

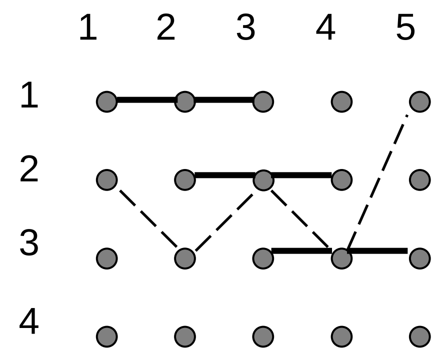
- Patrolling a Gallery:

  T = fixed shift

  (e.g. one working day)

  We call this the one-off game and denote it G^o with value V^o.

one-off game:



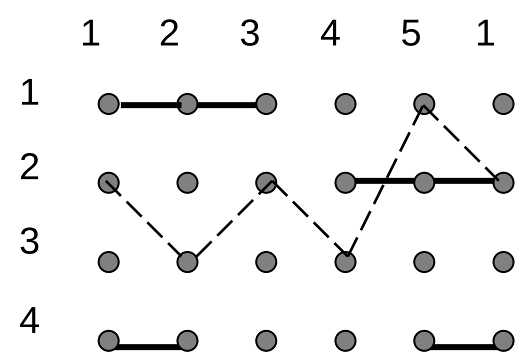
attacker can only start attack at times 1,2,3.

- Patrolling an Airport :

  continuous patrolling

  We call this the periodic game and we let T be the period. We denote it with G^p, V^p.

periodic game:



patroller must return to starting node.

$$V^p(Q, T, m) \le V^o(Q, T, m)$$

the one-off game has more patroller strategies and less attacker strategies

Generic Strategies

We have:

$$\frac{1}{n} \leq V \leq \frac{m}{n}$$

The patroller can guarantee the lower bound by:

- picking a node equiprobably and
- waiting there

The attacker can guarantee the upper bound by:

- fixing an attack time interval and
- attacking at a node equiprobably during that interval;
- out of these n pure attacker strategies, the patroller can intercept at most m of them, in a time interval of length m

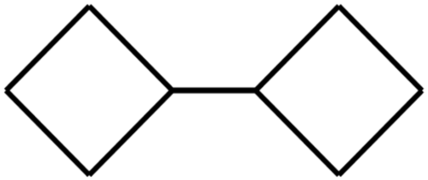
Uniform Attacker Strategy

The attacker attacks equiprobably over all time intervals and over all nodes.

Generic Strategies

Attacker's Diametrical Strategy

d(i,j) = minimum number of edges between nodes i and j
d = diameter of Q = maximum d(i,j) for all pairs i, j.



The attacker picks random attack time τ and attacks equiprobably nodes i and j that have distance d.

We have: $V \leq \max\left[m/2d, 1/2\right]$

The diametrical strategy guarantees the above upper bound:

- If m is large as compared to d, the best the patroller can do against the diametrical strategy is to go back and forth across the graph diameter (m/2d)

- If m is small as compared to d, the best the patroller can do against the diametrical strategy is to stay at the diametrical nodes and win half the time (1/2).

Independent/Covering strategies

Independent strategies

Independent set: set of nodes where no simultaneous attacks at any two nodes of the set can be covered by the same patrol during any fixed time interval (of length m).

Independence number I : maximum cardinality of an independent set.

Independent attack strategy: attack equiprobably nodes in a maximum independence set.

Covering strategies

Intercepting Patrol: a patrol w that intercepts every attack on a node that it contains.

Covering set of Q: a set of intercepting patrols such that every node of Q is contained in at least one of the patrols.

Covering number J: minimum cardinality of any covering set.

Covering patrol strategy: choose equiprobably from a minimum set of covering patrols.

Independence/Covering Strategies

Independent and Covering strategies

$$\frac{1}{\mathcal{J}} \leq V \leq \frac{1}{\mathcal{I}}$$

Upper bound: independent attack strategy
Lower bound: covering patrol strategy

When $\mathrm{I} = \mathrm{J}$ we can determine the value of the game:

Independence/Covering strategies

Example: The discrete line

m=3, L7 (n=7)

Maximum Independence set
= {1,4,7}
I = 3 ⇨ $V \leq 1/3$

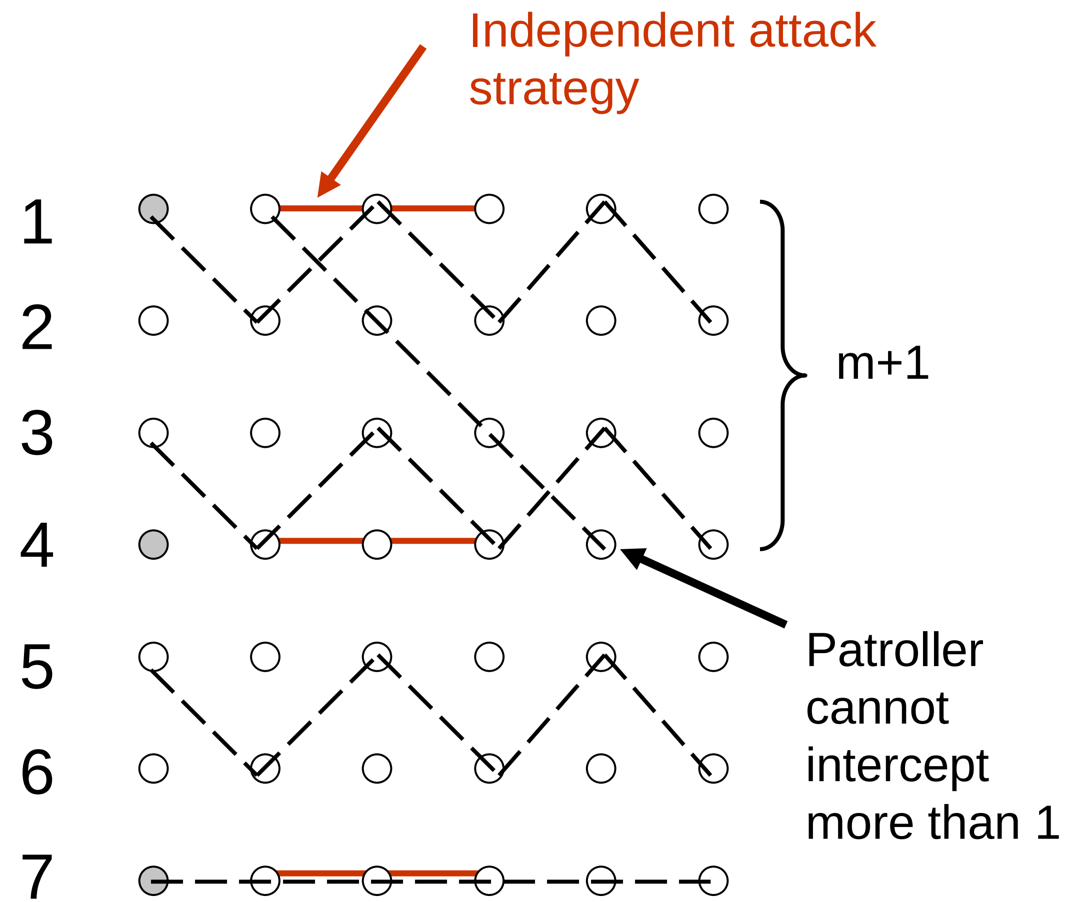
Minimum covering set of walks:

J = 4 ⇨ $V \geq 1/4$

$1/4 \leq V \leq 1/3$

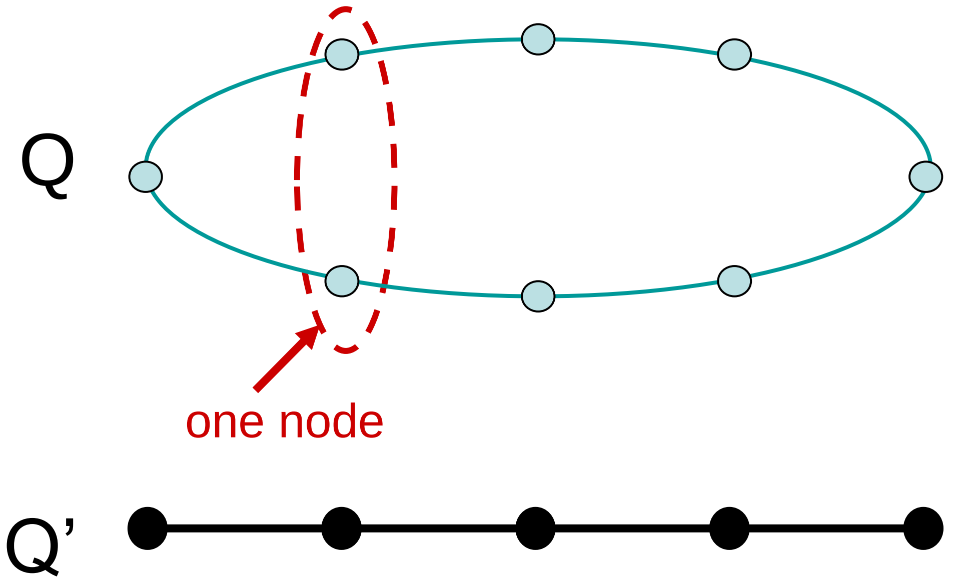
patroller
can do better

optimal



Earlier Results

Node Identification



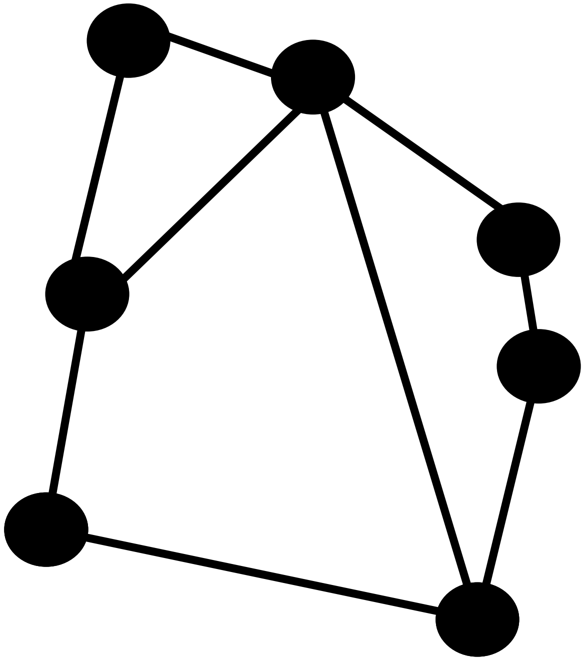
If Q' is obtained from Q by node identification, then

$$V(Q') \geq V(Q)$$

since any patrol on Q that intercepts an attack, has a corresponding patrol on Q' that intercepts the same attack

Earlier Results

Hamiltonian Graph



Any graph with a Hamiltonian cycle:

- Value (of V^o) is $\dfrac{m}{n}$
- Patroller - Random Hamiltonian patrol: pick a node at random and follow the Hamiltonian cycle in a fixed direction

  For any attack interval, the nodes visited by the patroller form an m-arc of the Hamiltonian cycle, which contains attack node i with probability m/n.

- Attacker - uniform attacking strategy, attack equiprobably over time and nodes

The discrete line - results

Case A: If $n \leq m + 1$*:*

$$V^o = \frac{m}{2(n-1)}$$

n small compared to m

Case B: If $n = m + 2$ *and* n, m *even,*

$$V^o = \frac{1}{2}$$

n similar compared to m

Case C: If $n = m + 2$ *and at least one of* n, m *odd, or* $n \geq m + 3$*:*

$$V^o = \frac{m}{n+m-1}$$

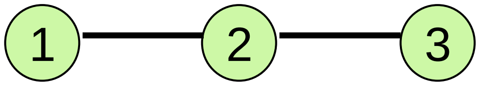
n large compared to m

We concentrate on the one-off game. The value for the periodic game is the same when either T goes to infinity, or when T is the appropriate multiple.

The discrete line – Case A

n small compared to m

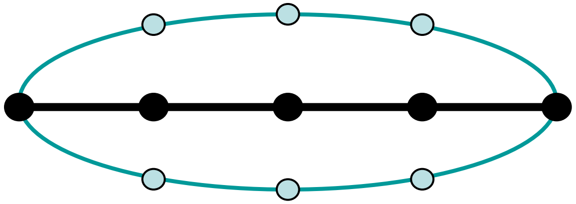
$$If\ n \leq m+1\ \ V^o = \frac{m}{2(n-1)}$$



- d = diameter = n-1
  The diametrical attacker strategy guarantees the upper bound for the attacker

- We use node identification, to show that
  the upper bound is achieved:

$$V(L_n) \geq V(C_{2(n-1)}) = \frac{m}{2(n-1)}$$

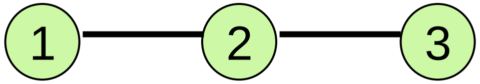


The Hamiltonian patrol on the cycle graph is equivalent to walking up and down the line graph (oscillation strategy).

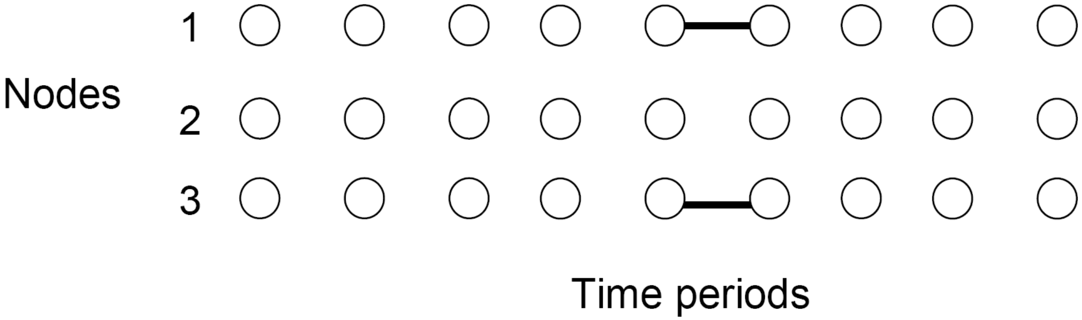
The discrete line – Case A

n small compared to m:

Consider L_3 the line graph with n=3. Let m=2.

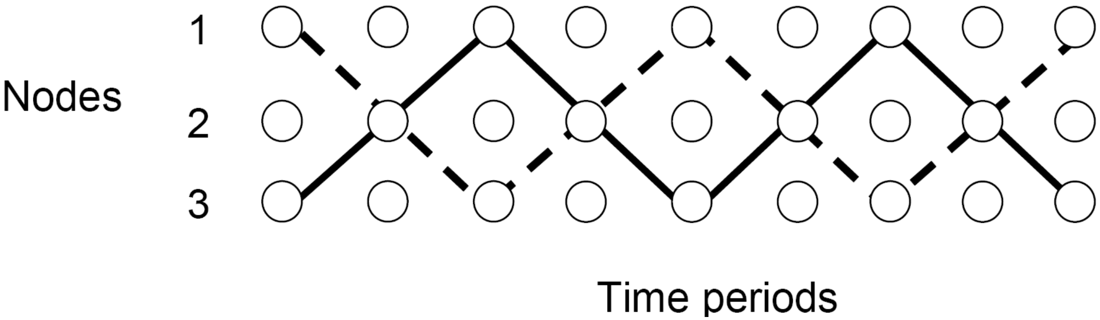


Attacker can guarantee ½ by attacking at the endpoints equiprobably: no walk can intercept both.



$$V \leq \frac{1}{2}$$

Patroller can guarantee ½ by playing equiprobably the following oscillations: every attack is intercepted by at least one oscillation.



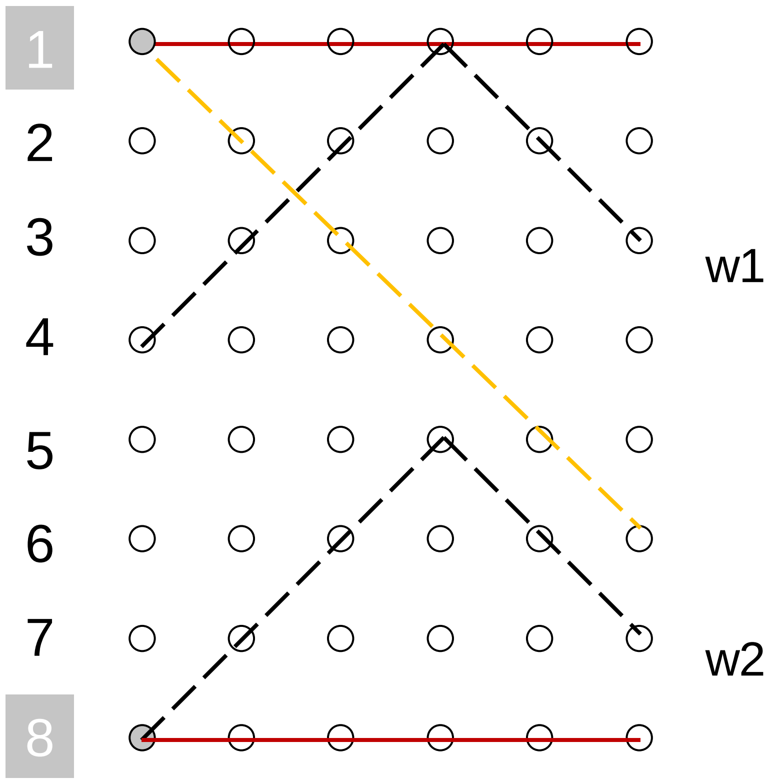
$$V \geq \frac{1}{2}$$

The discrete line – Case B

n similar compared to m: n=m+2 and both even

V= 1/2

Example: n=8, m=6



Patrols:

w1 oscillate between 1 and n/2

w2 oscillate between n/2+1 and n

w1, w2 are intercepting patrols

{w1,w2} is a covering set

J ≤ 2 and thus V ≥1/2

Attacks:

nodes {1,n} are an independent set

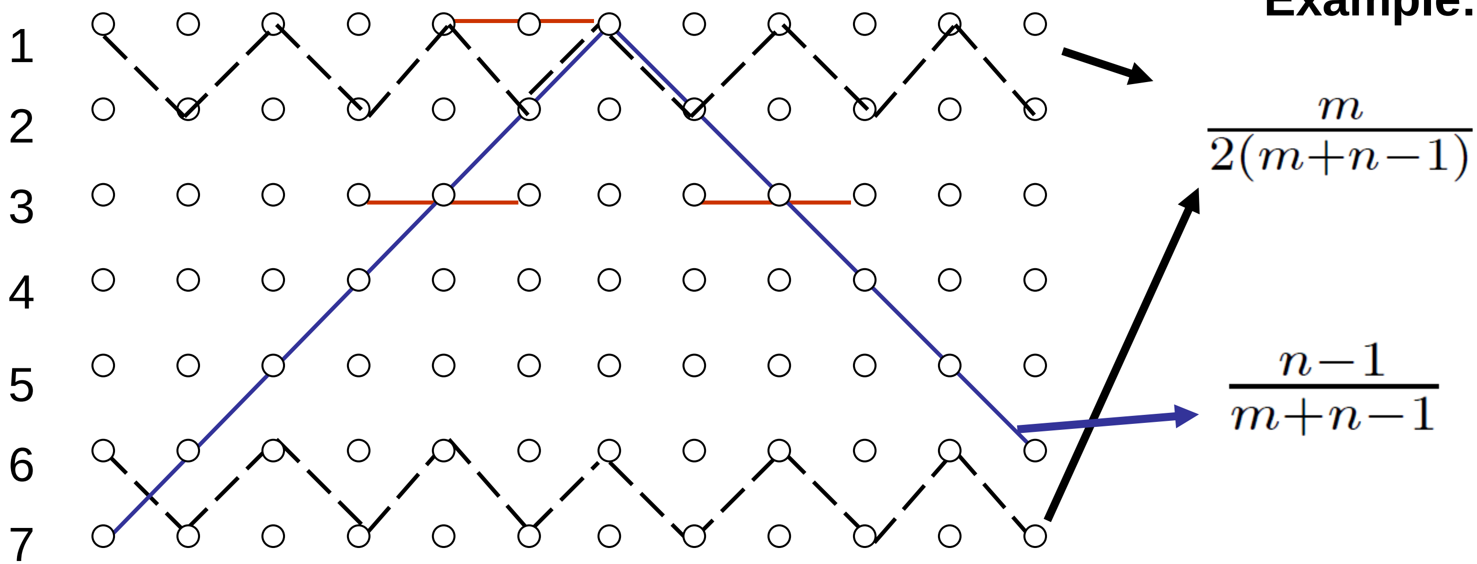
I ≥ 2 and thus V ≤ 1/2

The discrete line – Case C

n large compared to m

Patroller Strategy – Lower bound

Example: m=3, L7



Pr(interception at end node) = $\frac{n-1}{m+n-1}\,\frac{m}{2(n-1)} + \frac{m}{2(m+n-1)} = \frac{m}{n+m-1}$

Pr(interception at nodes 3-5) = $\frac{n-1}{m+n-1}\,\frac{2m}{2(n-1)} = \frac{m}{n+m-1}$

Pr(interception at nodes 2 and 6) ≥ Pr(interception at end node)

$$\boxed{V \geq \frac{m}{n+m-1}}$$

V ≥ 1/3

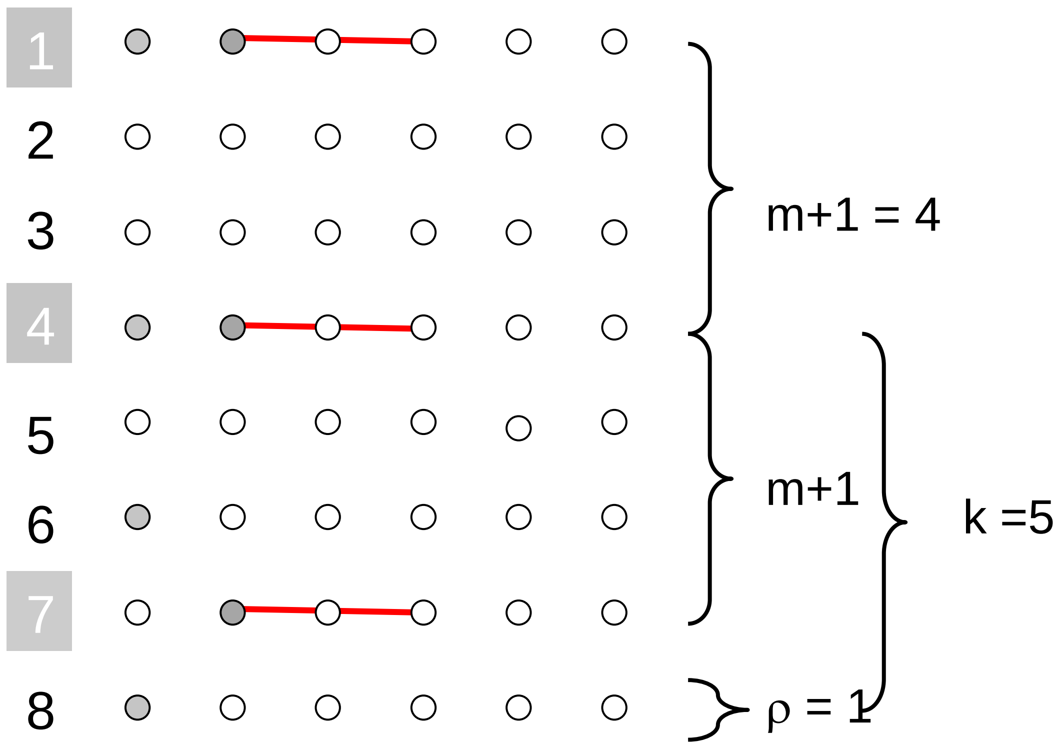
The discrete line – Case C

Attacker Strategies – Upper bound

Let q be the quotient and ρ be the remainder when $n-1$ is divided by m:

$$n - 1 = qm + \rho$$

$$k = m + 1 + \rho.$$



Cases for attacker strategies:

1. $\rho = 0$.
2. $\rho > 0$ and k odd.
3. $\rho > 0$ and k even, m odd
4. $\rho > 0$ and k even , m even and $k > m+2$.
5. $\rho > 0$ and k even , m even and $k = m+2$.

The discrete line – Case C1

n large compared to m:

$$\textit{If } \rho = 0, \ \textit{we have } V^o \leq \frac{m}{n+m-1}.$$

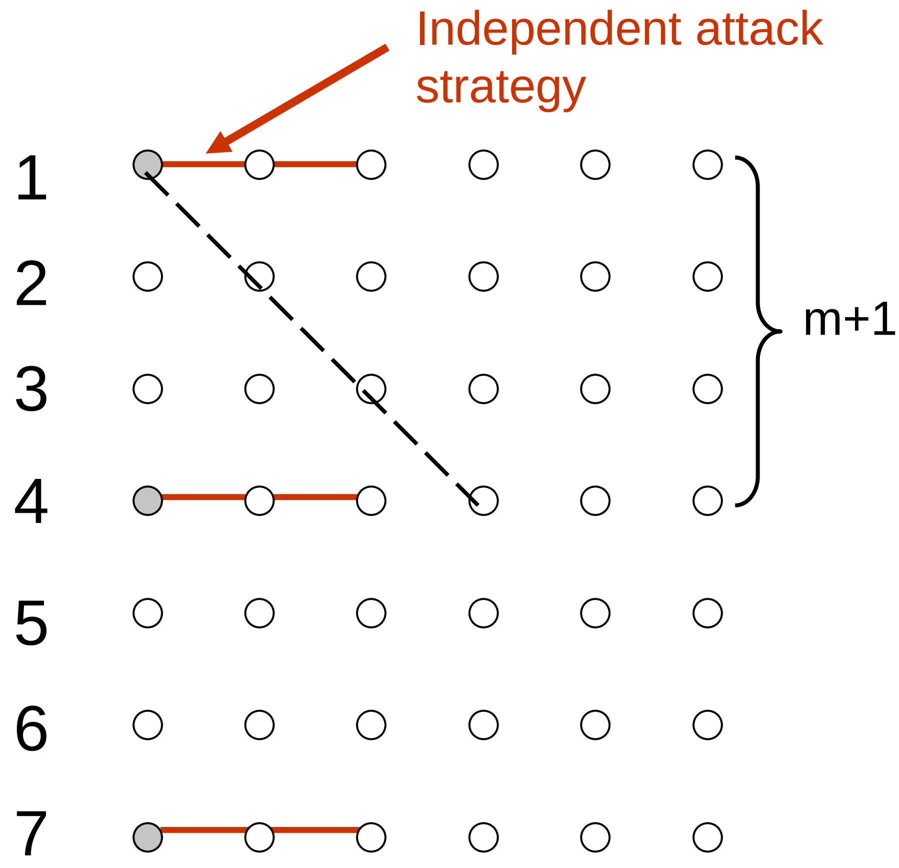
Attacker plays Independent strategy:

Attack at equiprobably at nodes {1, m+1, 2m+1,…,qm+1=n}. Patroller can intercept at most 1 out of q+1 attacks, where q = (n-1)/m :

$$\frac{m}{n+m-1} = \frac{1}{\frac{n-1}{m}+1} = \frac{1}{q+1}$$

Example with ρ = 0:
n = 7, m=3

Maximal Independence set
= {1,4,7}
I = 3 ⇨ V ≤ 1/3



The discrete line – Case C2

n large compared to m

$$If\ \rho > 0\ \ and\ k\ \ odd,\ we\ have\ V^o \leq \frac{m}{n+m-1}$$

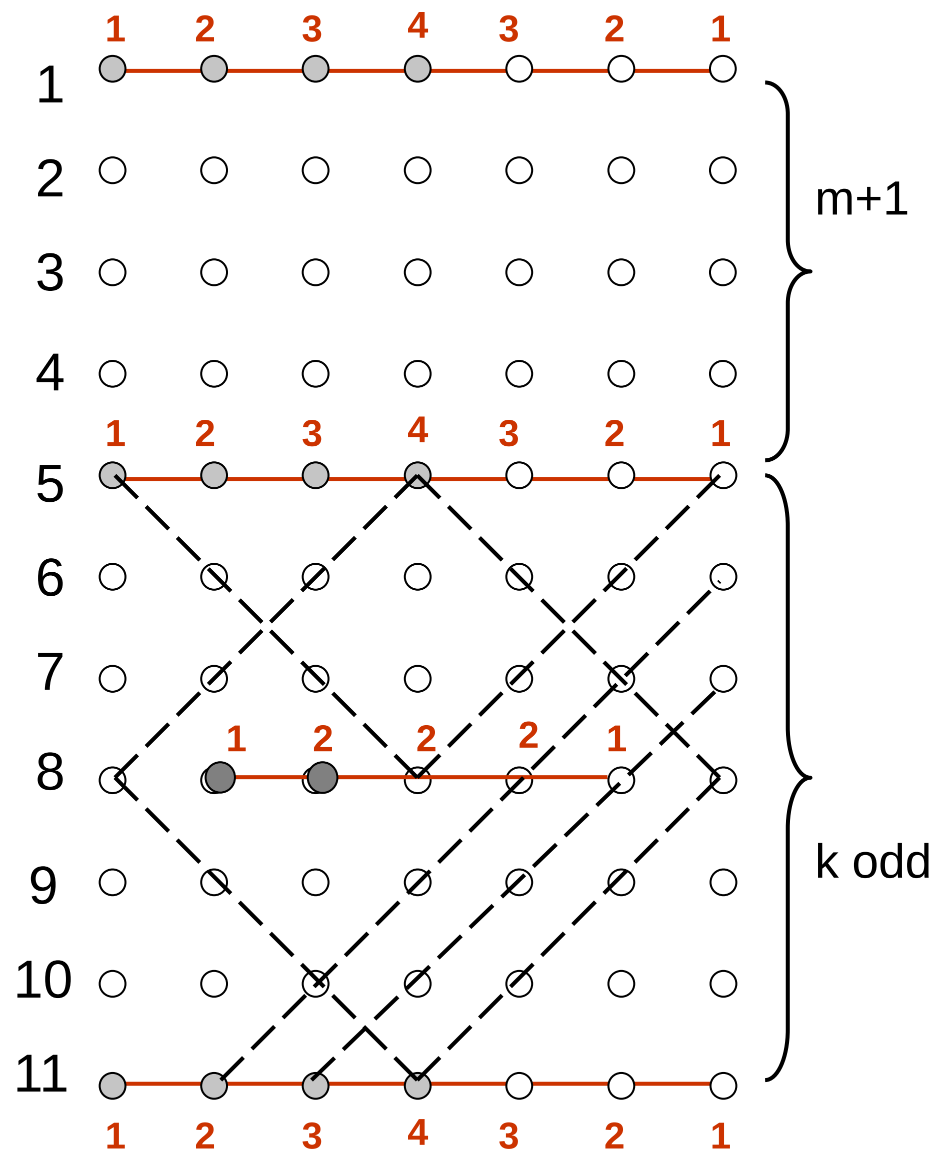
Example with ρ > 0 and k odd:
m=4, L11 (n=11)

Can we place n+m-1 attacks such that only m are intercepted by a single patrol?

Divide n-1 by m:
quotient q, remainder d.

- attack at nodes {1,m+1, …,(q-1)m+1,n} m times with attacks shifted by 1 time step

- attack at node in the middle of the odd interval



The discrete line – Case C3

n large compared to m

If $\rho > 0$, k *is even and* m *is odd*,

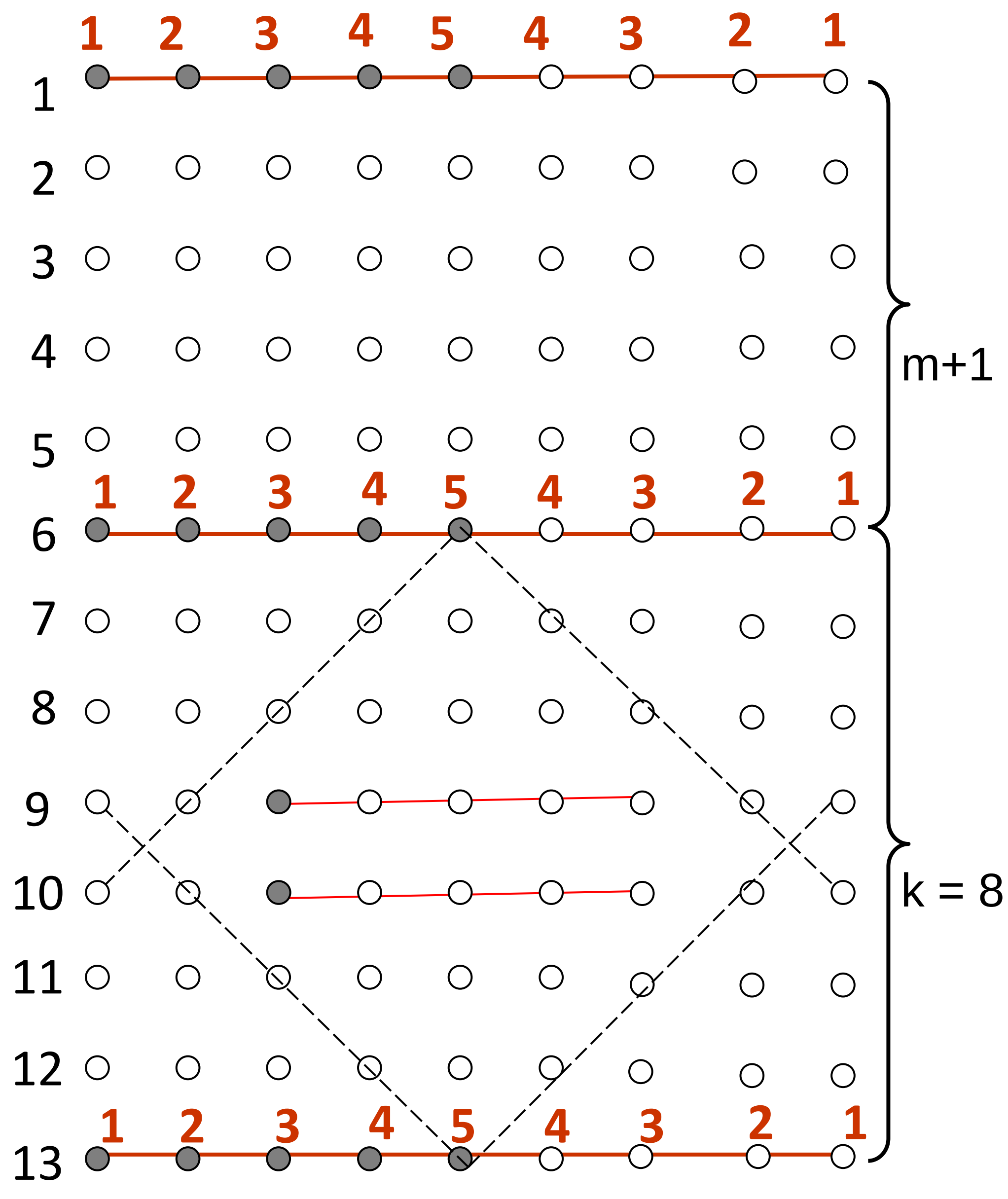
$$V^o \leq \frac{m}{n+m-1}$$

Example with **ρ > 0** **and** **k even**, **m odd**: n = 13, m = 5
Thus, q = 2 and ρ = 2, k = 8.

Can we place n+m-1 attacks such that only m are intercepted by a single patrol?

External attacks: at nodes {1,6,13} at time periods {1,2,3,4,5}

Internal attacks: nodes {9,10} at time period 3.



The discrete line – Case C4

n large compared to m

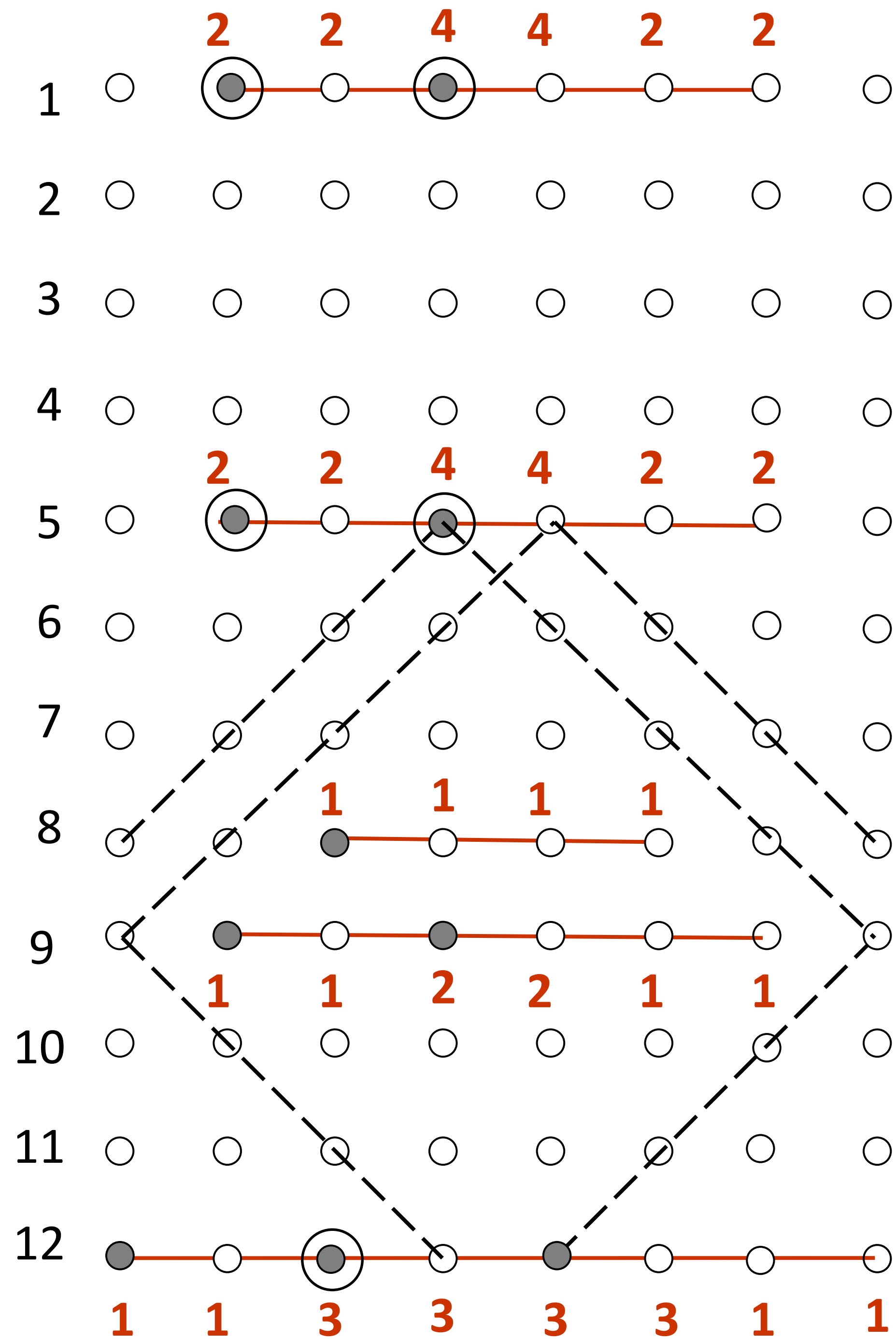
Example with ρ > 0 and k even, m even, k > m+2:
n = 12, m = 4, k = 8.

One attack

Two attacks

k

k attacks are intercepted
If the patroller passes
from a node labeled k



The discrete line – Case C5

n large compared to m

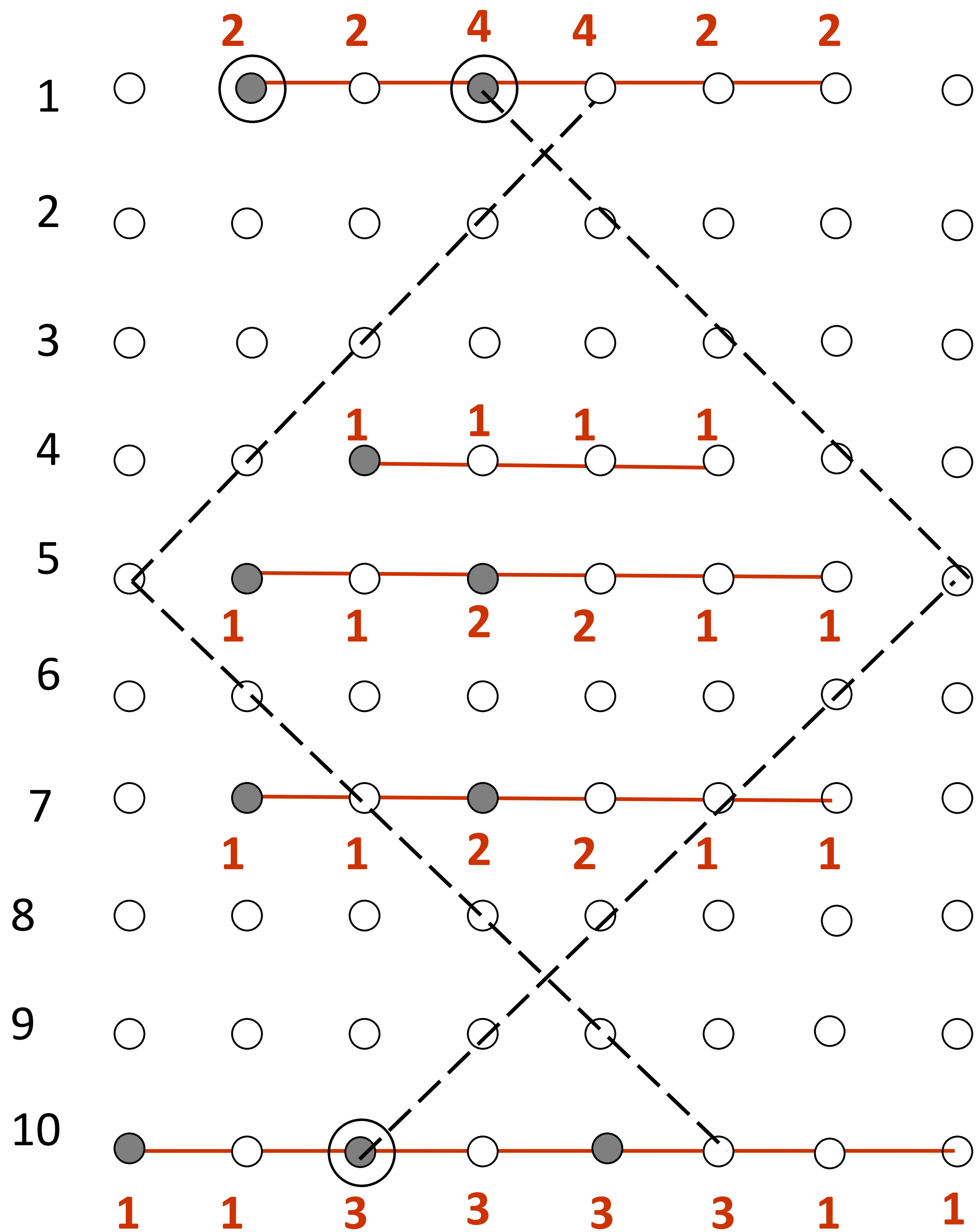
Example with ρ > 0 and k even, m even, k = m+2:
n = 10, m = 4, k = 6.

One attack

Two attacks

k

k attacks are intercepted
If the patroller passes
from a node labeled k



The continuous line

The game is played on the unit interval [0,1] over a time horizon T.

Patroller: patrols at unit speed, picks a walk $w: t \rightarrow [0,1]$

Attacker: picks a point x in [0,1] and a time τ, and stays there for time r. Thus the attack interval is $[\tau, \tau + r]$.

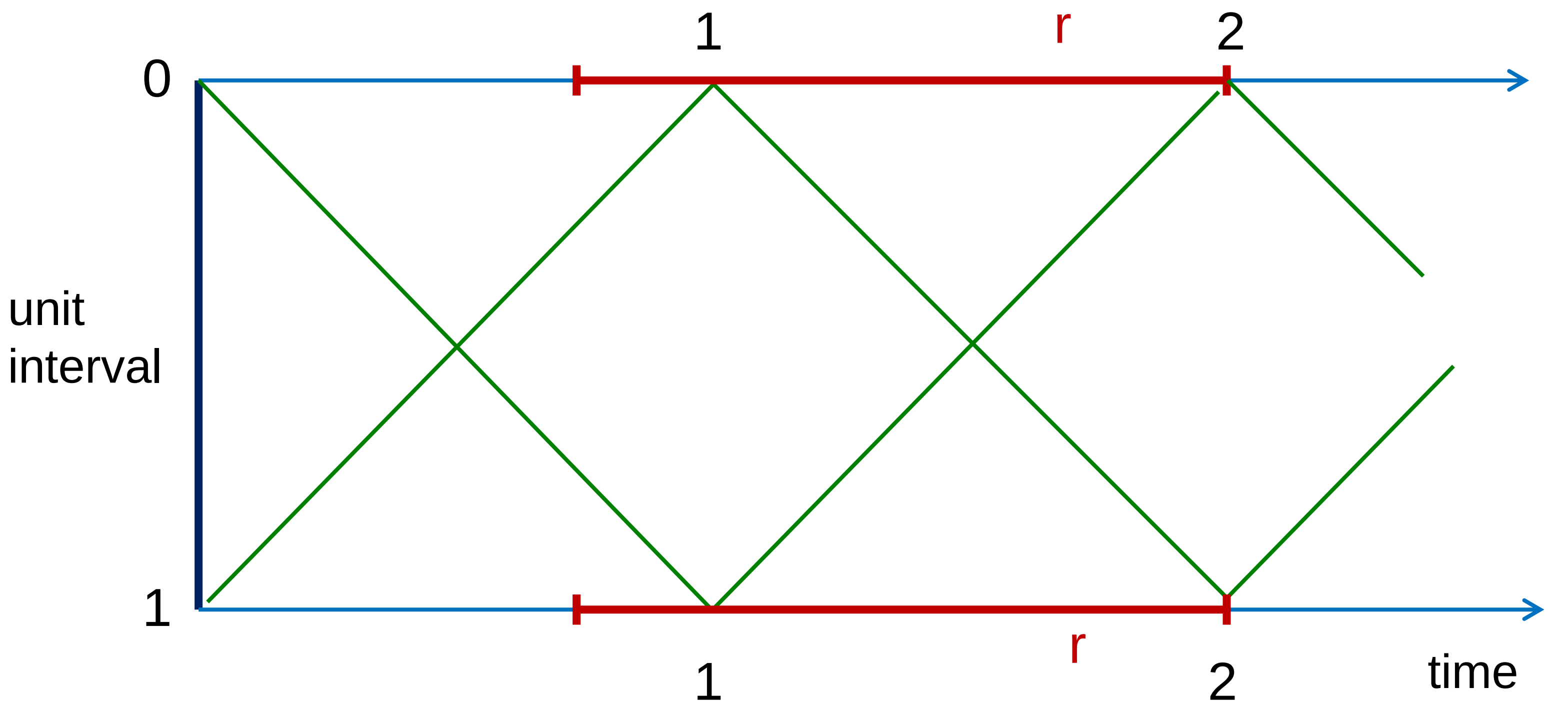
The attack is intercepted if $w(t) = x$ for some t in $[\tau, \tau + r]$.

Value of the game is 1 is the attack is intercepted, otherwise it is 0.

We assume $0 \leq r \leq 2$, otherwise the patroller can always intercept the attacker by going up and down the unit interval.

The continuous line

$$If\ r \geq 1,\ then\ V = \frac{r}{2}.$$



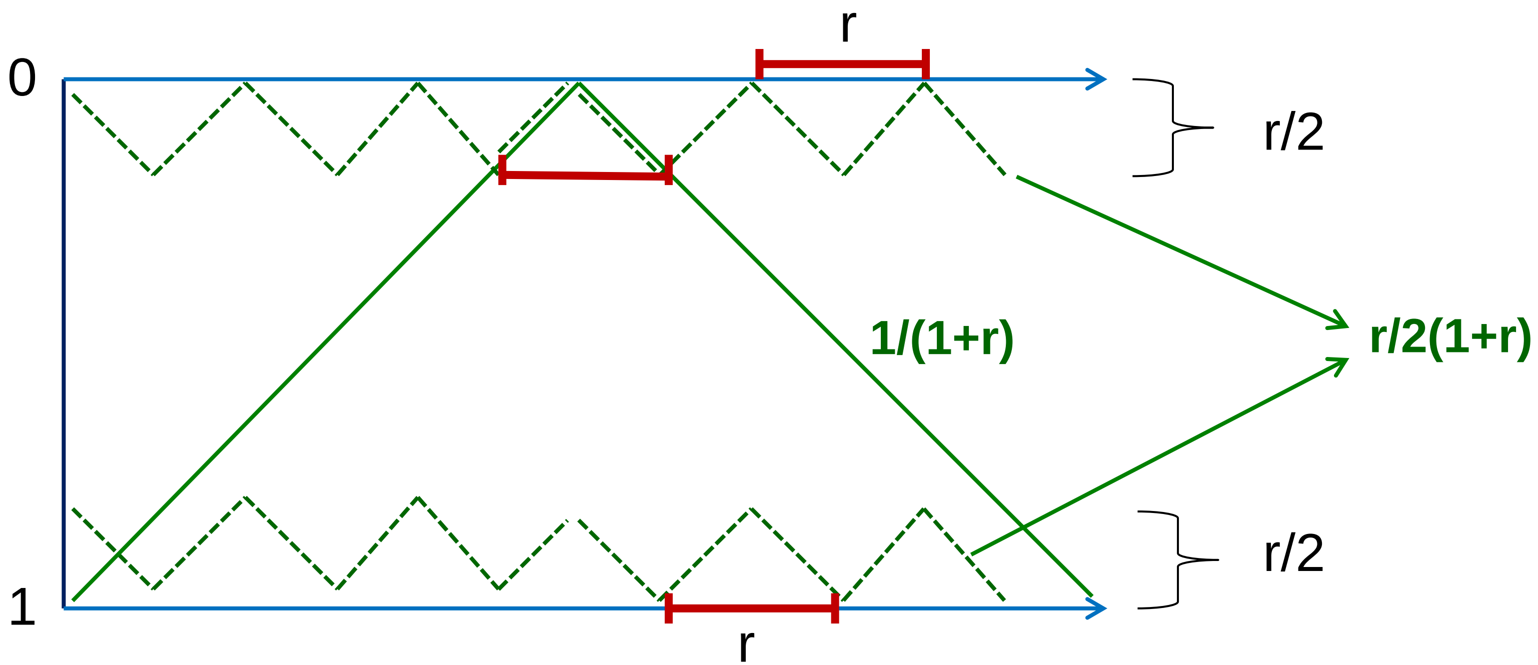
the patroller: picks equiprobably between two oscillations on [0,1] in opposite directions

the attacker: attacks equiprobably between the two endpoints

The continuous line

$$If\ r \le 1,\ then\ V = \frac{r}{1+r}$$

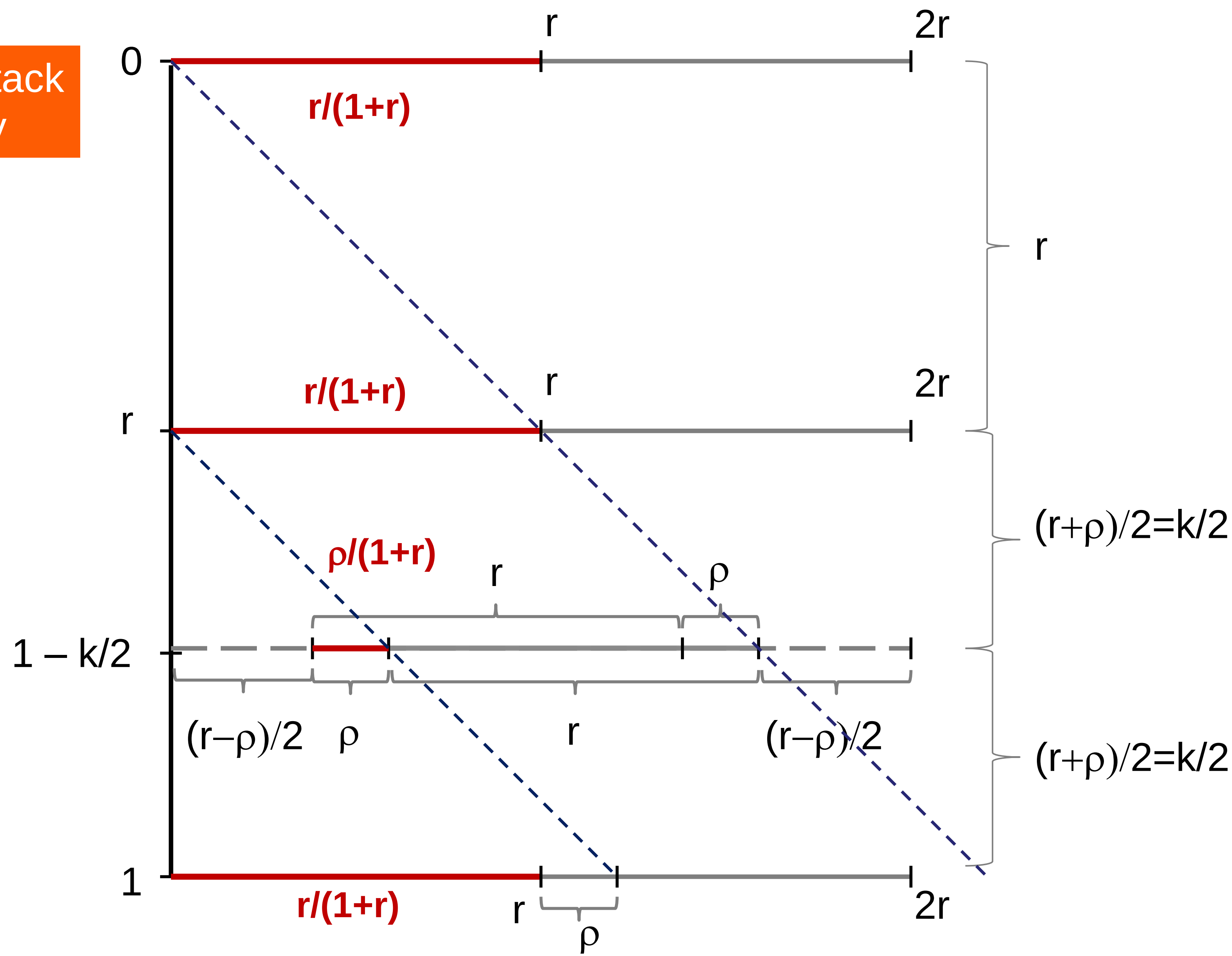
the patroller strategy



The continuous line

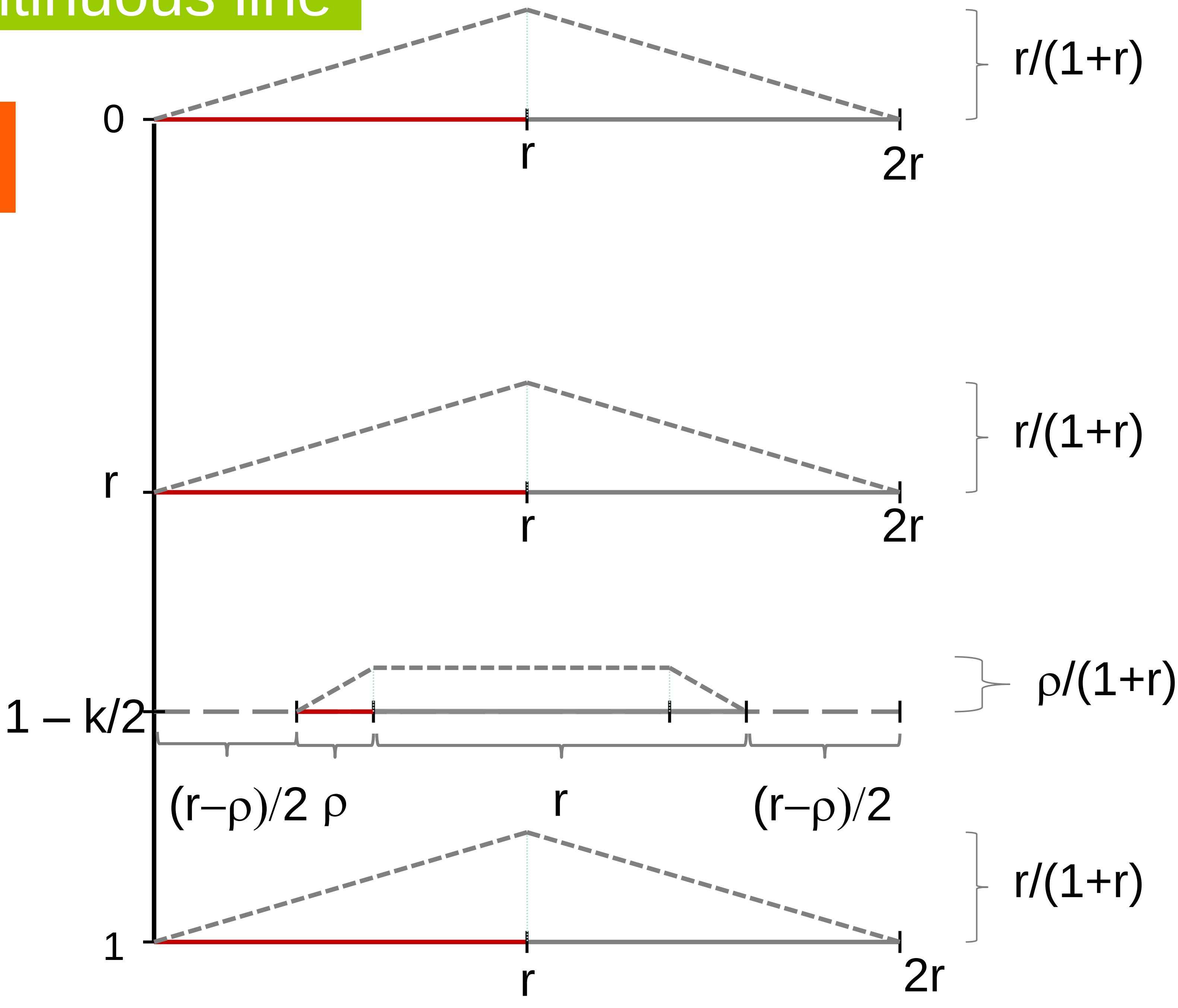
$$If\ r \leq 1,\ then\ V = \frac{r}{1+r}$$

the r-attack strategy



The continuous line

the r-attack strategy



The End

Thank you.

Assumptions

We make some simplifying assumptions:

- The attacker will attack during the time interval:

  By patrolling as if an attack will take place, the patroller deters the attack on this network and gives an incentive to the attacker to attack another network.

- The nodes have equal values:

  Nodes with different values can be easily modelled in the mathematical programming formulations of the game. All games that can be solved computationally, can also be solved using different valued nodes.

- The nodes on the network are equidistant:

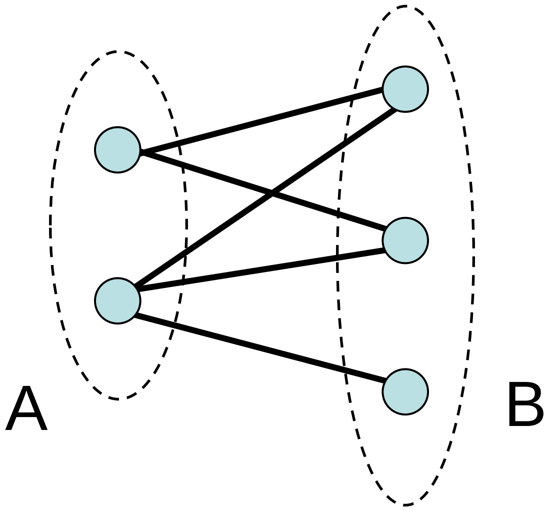
  This can also be modelled in the mathematical programming formulations.

Applications

- Security guards patrolling a museum or art gallery.
- Antiterrorist officers patrolling an airport or shopping mall.
- Patrolling a virtual network for malware.
- Police forces patrolling a city containing a number of potential targets for theft, such as jewellery stores.
- Soldiers patrolling a military territory.
- Air marshals patrolling an airline network.
- Inspectors patrolling a container yard or cargo warehouse.

Solutions for Special Graphs

Bipartite Graphs $K_{a,b}$



- No odd cycles

$$a = |A|, b = |B|, a \leq b$$

We assume: $m \leq 2b$

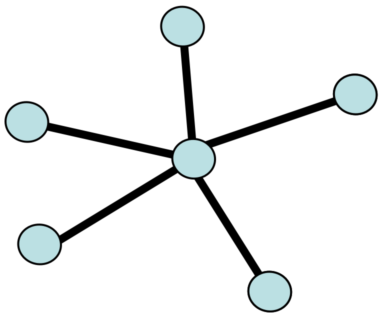
1. $V^o \leq m/(2b)$, *with equality if* Q *is complete bipartite*
2. $V^p \leq m/(2b)$, *with equality if* Q *is complete bipartite and* T *is a multiple of* $2b$.
   if Q *is complete bipartite then* $V^p \to m/(2b)$ *as* $T \to \infty$.

Attacker can guarantee $m/2b$, if he fixes the attack interval and attacks equiprobably on each node of the larger set B.

When Q is complete bipartite and a=b, there exists a Hamiltonian cycle and the value is achieved.

Solutions for Special Graphs

Bipartite Graphs: The Star Graph



S_n : star graph with n nodes

$C_{2(n-1)}$: cycle graph with 2(n-1) nodes

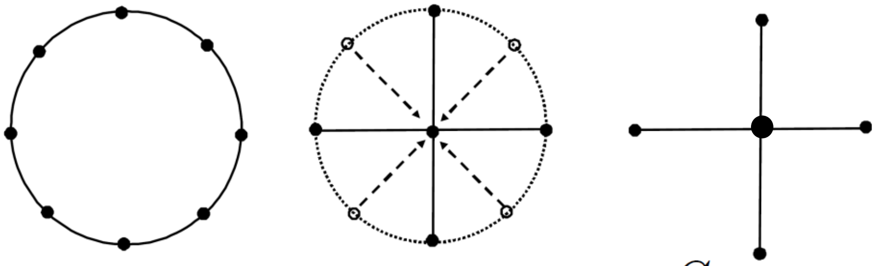
a = 1, b = n-1
T is a multiple of 2(n-1)

By node identification:

$$V(S_n) \geq V(C_{2(n-1)}) = \frac{m}{2(n-1)}$$

Since S_n is bipartite:

$$V(S_n) \leq \frac{m}{2b} = \frac{m}{2(n-1)}$$



C_8 S_5

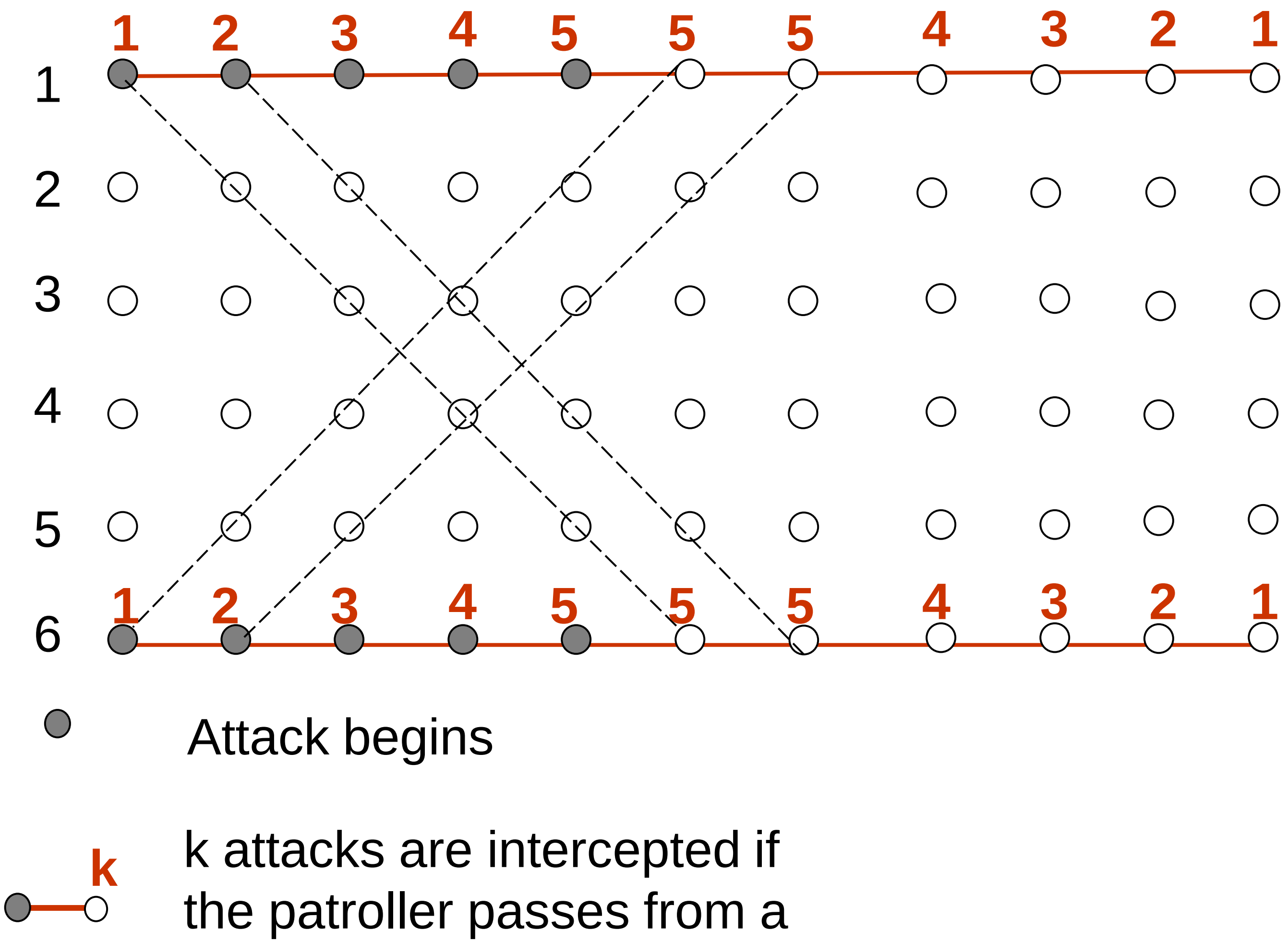
Thus, $V(S_n) = \dfrac{m}{2(n-1)}$

- attack leaf nodes equiprobably
- patrols leaf nodes every second period

The discrete line – Case A

n=6, m = 7

Time-dependent attacker strategies



k attacks are intercepted if the patroller passes from a node labeled k

The discrete line – Case C2

n large compared to m

If $\rho > 0$ and k odd, we have $V^o \leq \frac{m}{n+m-1}$

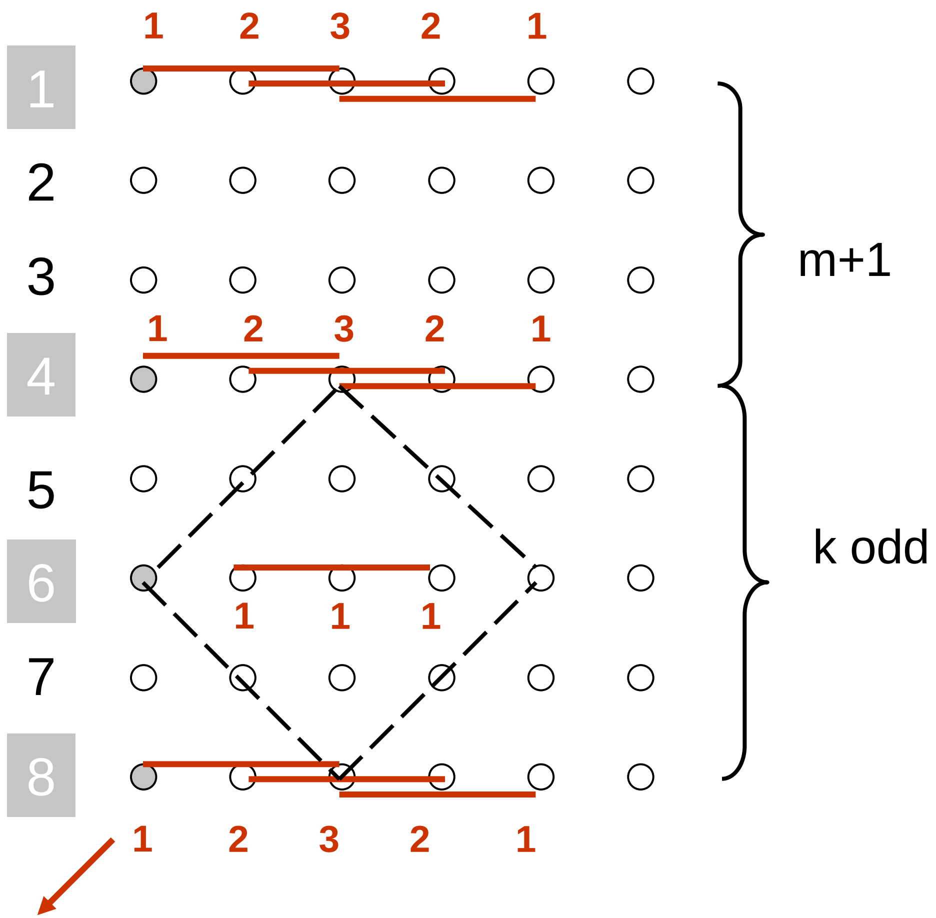
Example with ρ > 0 and k odd:
m=3, L8

Patroller cannot intercept more than 3 out of 3(3)+1 = 10 attacks.

Attacker can guarantee:
Value ≤ 3/10

Patroller can guarantee:

Value ≥ $\frac{m}{n+m-1}$ = 3/10



Number of attacks = n + m - 1