On the Acquaintance Time of Random Graphs

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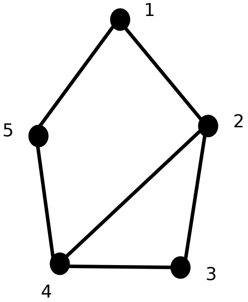
joint work with Bill W. Kinnersley and Paweł Prałat

Introduction to the acquaintance time

Example

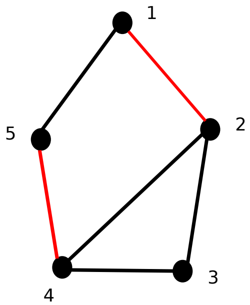
- Acquainted pairs at Round 0: 1-2, 2-3, 3-4, 4-5, 1-5, 2-4



Introduction to the acquaintance time

Example

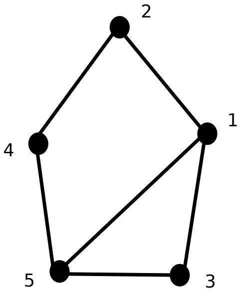
- Choose matching and switch agents



Introduction to the acquaintance time

Example

- Acquainted pairs after Round 1: 1-2, 2-3, 3-4, 4-5, 1-5, 2-4, 1-3, 3-5



Introduction to the acquaintance time

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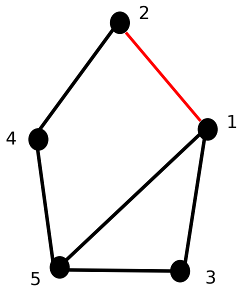
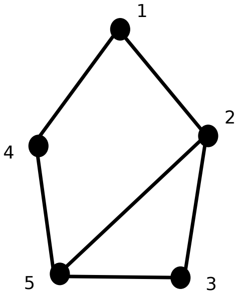


Introduction to the acquaintance time

Example

- Acquainted pairs after Round 2: 1-2, 2-3, 3-4, 4-5, 1-5, 2-4, 1-3, 3-5, 1-4, 2-5



Definition

Let $G = (V, E)$ connected.
$AC(G)$ = min number of rounds to get all pairs of agents acquainted

- Introduced by Benjamini, Shinkar, Tsur (2012)

- If $G_1 \subseteq G_2$, then $AC(G_2) \leq AC(G_1)$

Definition and basic properties

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Simple lower and upper bounds

- $AC(G) \geq \frac{\binom{|V|}{2}}{|E|} - 1$

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- $AC(G) = \Omega(Diam(G))$
- $AC(G) = O(n^2)$

Some more results

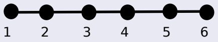
- Better upper bound was known (Benjamini, Shinkar, Tsur 2012)
- Gave slight improvement
- But new upper bound: $AC(G) = O(n^{3/2})$ (Angel, Shinkar 2014)
- Upper bound tight (explicit constructions for graphs with $AC(G) = f(n)$ with any $f(n) \in O(n^{3/2})$ in Benjamini, Shinkar, Tsur 2012)

Some more results

Proposition

If G has a Hamiltonian path, $AC(G) = O(n)$

- suppose that the path is $v_1, \ldots, v_n$, with $e_i = v_i v_{i+1}$
- Idea: On odd-numbered rounds, swap agents on odd edges, on even-numbered rounds, swap agents on even edges



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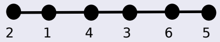

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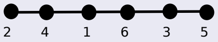

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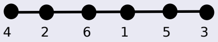

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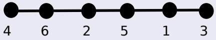

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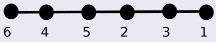

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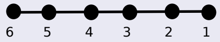

Random graphs

Definition

- Erdős-Rényi model $\mathcal{G}(n, p)$: sequence of probability spaces
- For each n and any G with $V(G) = \{1, \ldots, n\}$,
  $Pr(G \in \mathcal{G}(n, p)) = p^{|E|}(1-p)^{\binom{|V|}{2}-|E|}$
- For graph property P, we say that P holds a.a.s. if $Pr(G \ni P) \to 1$

Facts about random graphs

- $p = \log n / n$ threshold for connectivity
- $p = (\log n + \log\log n)/n$ threshold for Hamiltonicity

Our results

Conjecture (Benjamini, Shinkar, Tsur 2012)

- *For* $G \in \mathcal{G}(n,p)$, $p \gg \log n/n$, $AC(G) = O(\log n/p)$

Theorem (Kinnersley, M., Prałat 2013)

- *For* $G \in \mathcal{G}(n,p)$, $p \gg \log n/n$, $AC(G) = O(\log n/p)$

Our results

Definition

- Variant of the game: agents may use helicopter to go to any other vertex
- Formally, one round: for a permutation π, agent at v goes to $\pi(v)$
- Let $\overline{AC}(G)$ = min number of rounds until all agents get acquainted in this variant

Observation

$\overline{AC}(G) \leq AC(G)$

Theorem (Kinnersley, M., Prałat 2013)

- *For* $G \in \mathcal{G}(n,p)$, $n^{-1/2} \ll p \leq 1 - \varepsilon$, $\overline{AC}(G) = \Omega(\log n/p)$

Corollary

For $G \in \mathcal{G}(n,p)$, $n^{-1/2} \ll p \leq 1 - \varepsilon$, $AC(G) = \Theta(\log n/p)$, $\overline{AC}(G) = \Theta(\log n/p)$

Idea of proof of upper bound $O(\log n/p)$

- Use two-round exposure: $\mathcal{G}(n,p)$=$\mathcal{G}(n,p_1)$ + $\mathcal{G}(n,p_2)$ with $p = p_1 + p_2 - p_1 p_2$
- Choose $p_1 = (1+\varepsilon)\log n/n$, small but enough to have Hamiltonian path
- Break path into subpaths of length $\Theta(\log n/p)$
- For each subpath, agents perform $O(\log n/p)$ rounds as in proof before (pairs of the same subpath are acquainted)
- For the remaining pairs, a.a.s. in $\mathcal{G}(n,p_2)$ at least once an edge is found

Idea of proof of lower bound $\Omega(\log n/p)$

- Consider a random permutation π
- Show that after $k = \varepsilon \log n/p$ rounds the probability that all pairs are acquainted is $o(\left(\frac{1}{n!}\right)^k)$
- idea: expose edges one after the other, and consider pairs of vertices visited by one pair of agents
- If edge discovered, discard all at most k pairs of agents that correspond to this edge
- In this way investigate at least $\frac{\binom{|V|}{2}}{k}$ pairs of agents
- Show that Pr(one pair acquainted) is small (if pair of vertices was exposed but the pair of agents was not discarded, even harder to be acquainted)
- Then Pr(all pairs acquainted) really small

Publicity

If you like random graphs, then ...

- Workshop on Random graphs in Nice, May 14 and May 15, 2014