The (all guards move) Eternal Domination number for $3 \times n$ Grids

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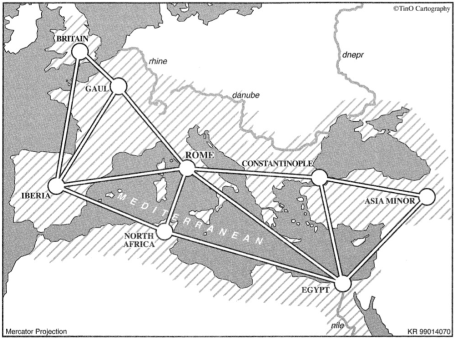
The Eternal Dominating Set Problem

• Deployed 4 powerful field armies (each comprised of 6 legions) over 8 regions

• An FA was considered capable of deploying to protect an adjacent region only if it moved from a region where there was at least one other FA to help launch it.



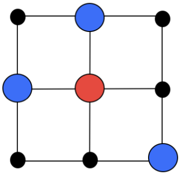
[ReVelle & Rosing]

• Consider a region to be *secure* if it has an FA stationed at it and *securable* if an FA can reach it in one step.

• Constantine's strategy is known in domination theory as **Roman domination**.

The Eternal Dominating Set Problem

- guards initially form a dominating set on G
- at each step, a vertex is attacked
- in a “move” for the guards, each guard may remain where it is or move to a neighbouring vertex



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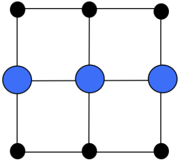


if the guards "move" so that a guard is located at the attacked vertex and the set of guards again forms a dominating set, then the guards have <u>*defended against the attack*</u>

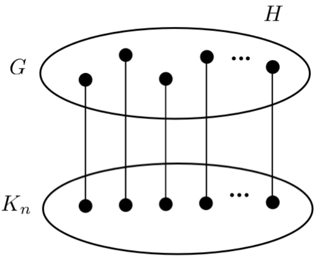
We wish to find the minimum number of guards to defend against <u>*any possible*</u> sequence of attacks on G.

$\gamma_{all}^{\infty}(G)$

The Eternal Dominating Set Problem

- special case of the (cops-first) GUARDING PROBLEM
  - given a board $[G; R, C]$, compute the minimum number of cops that can guard the cop-region C.

    $C \subsetneq V(G)$ and $R = V(G) \setminus C$; the cops move first and are only allowed to move within the cop-region C.



If the cop-region of H is $V(G)$ then G has an eternal dominating set of size k if and only if k cops can guard $V(G)$.

$\Rightarrow$ PSPACE-hard
[Fomin, Golovach, Lokshtanov 2009]

- γ_{all}^{∞} known for some small classes of graphs and trees
- $\gamma(G) \leq \gamma_{all}^{\infty}(G) \leq \alpha(G)$

Open Problem

Determine the classes of graphs G with $\gamma_{all}^{\infty}(G) = \gamma(G)$.

- If G has n vertices, $\gamma_{all}^{\infty}(G) + \gamma_{all}^{\infty}(\overline{G}) \leq n + 1$
- If G connected,

$$\gamma_{all}^{\infty}(G) \leq \left\lceil \frac{|V(G)|}{2} \right\rceil$$

$$\gamma_{all}^{\infty}(G) \leq 2\gamma(G) \qquad \text{[sharp for all values of } \gamma\text{]}$$

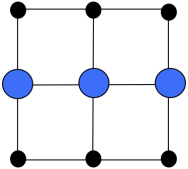
$$\gamma_{all}^{\infty}(G) \leq 2\tau(G) \qquad \text{[vertex cover number]}$$

$$\delta(G) \geq 2,\ \gamma_{all}^{\infty}(G) \leq \tau(G)$$

$$\delta(G) \geq 2,\ G \text{ girth } 7 \text{ or } \geq 9,\ \gamma_{all}^{\infty}(G) \leq \tau(G) - 1$$

[survey by Mynhardt, Klostermeyer]

$$\gamma(P_3 \,\square\, P_3) = 3 = \gamma^{\infty}_{all}(P_3 \,\square\, P_3)$$



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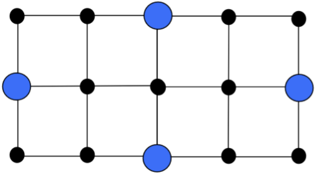

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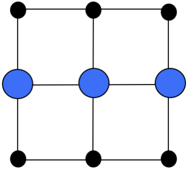

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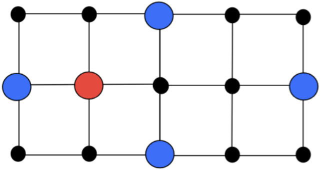

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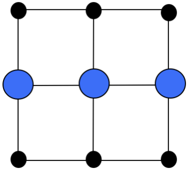

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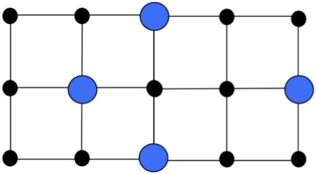

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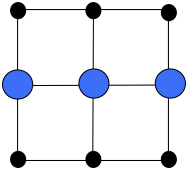

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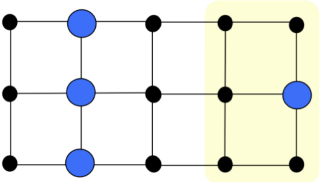


After determining that $\gamma_{all}^{\infty}(P_3 \square P_n) = n$ for $2 \leq n \leq 8$,

Goldwasser, Klostermeyer, Mynhardt [GKM 2012] found the surprising result that

$$\gamma_{all}^{\infty}(P_3 \square P_9) = 8$$

which yields the upper bound

Theorem 8 [GKM 2012]

For $n \geq 9$,

$$\gamma_{all}^{\infty}(P_3 \square P_n) \leq \left\lceil \frac{8n}{9} \right\rceil.$$

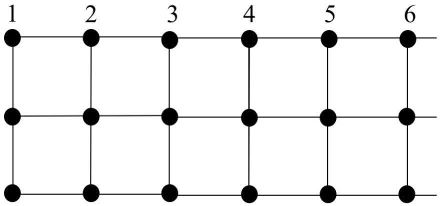
Conjecture 2 [GKM 2012]

For $n > 9$,

$$\gamma_{all}^{\infty}(P_3 \square P_n) = 1 + \left\lceil \frac{4n}{5} \right\rceil.$$

Lemma 10 [Goldwasser, Klostermeyer, Mynhardt 2012]

For $n > 5$, $P_3 \,\square\, P_n$ cannot be defended if at any step, there are only four guards in the first six columns.



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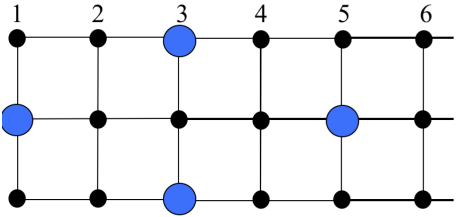

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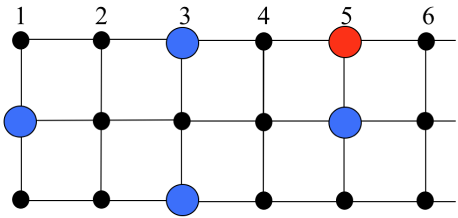

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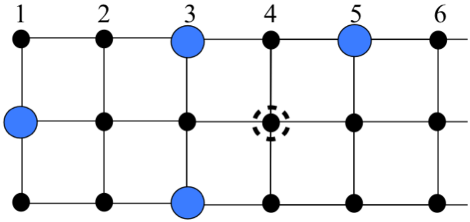

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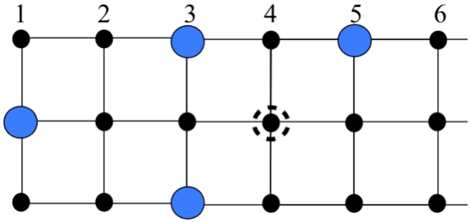

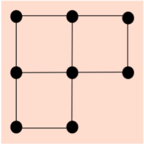

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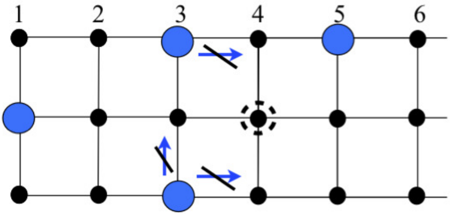

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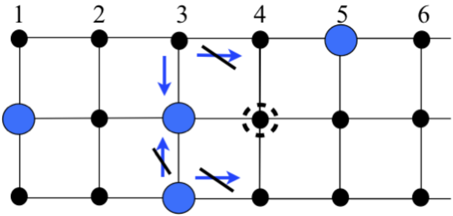


Theorem 6 [FMvB]

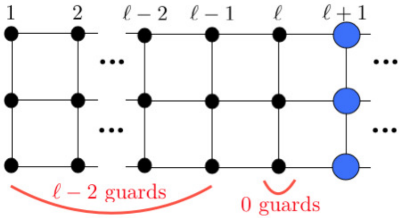
For $n \geq 15$, $\gamma_{all}^{\infty}(P_3 \,\square\, P_n) \geq 1 + \left\lceil \frac{4n}{5} \right\rceil$.

Corollary 4 [FMvB]

In any eternal dominating set of $P_3 \,\square\, P_n$, for any $\ell \geq 2$, the first ℓ columns contain at least $\lceil \frac{4\ell - 3}{5} \rceil$ guards.

Claim: Let $\mathcal{E}$ be an eternal dominating family of $P_3 \square P_n$ with fewer than $1 + \lceil \frac{4n}{5} \rceil$ guards. In every set of $\mathcal{E}$, there are at least $\ell - 1$ guards in the first ℓ columns, for any $\ell \geq 6$.

Proof: Let $\ell \geq 6$ be the smallest counterexample: in every set in $\mathcal{E}$, there are at least $\ell - 2$ guards in the first $\ell - 1$ columns, but there is a set $D \in \mathcal{E}$ in which there are $\ell - 2$ guards in the first ℓ columns.



Claim: Let $\mathcal{E}$ be an eternal dominating family of $P_3 \square P_n$ with fewer than $1 + \lceil \frac{4n}{5} \rceil$ guards. In every set of $\mathcal{E}$, there are at least $\ell - 1$ guards in the first ℓ columns, for any $\ell \geq 6$.

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- D has $\ell + 1$ guards in the first $\ell + 1$ columns.

Using Corollary 4, $|D| \geq \ell + 1 + \left\lceil \frac{4(n - (\ell + 1)) - 3}{5} \right\rceil$

By Lemma 2 [GKM], $\ell \geq 7 \Rightarrow |D| \geq 1 + \left\lceil \frac{4n}{5} \right\rceil$.

□

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By the Claim, for any $\ell \geq 6$, there are at least $\ell - 1$ guards in the first ℓ columns of every dominating set of $\mathcal{E}$.

This contradicts the assumption that the dominating sets of $\mathcal{E}$ use fewer than $1 + \lceil \frac{4n}{5} \rceil$ guards and the result follows.

$\square$

We actually do a little better:

Theorems 14 and 16 [FMvB]

For $n \geq 11$, $\quad 1 + \left\lceil \frac{4n+1}{5} \right\rceil \leq \gamma_{all}^{\infty}(P_3 \,\square\, P_n) \leq \left\lceil \frac{6n+2}{7} \right\rceil$.

And better still:

[DM 2014+]

For $n \geq 11$, $\quad 1 + \left\lceil \frac{4n+1}{5} \right\rceil \ \leq \ \gamma_{all}^{\infty}(P_3 \,\square\, P_n) \ \leq \ 2 + \left\lceil \frac{4n}{5} \right\rceil$.

Questions:

- What about $\gamma_{all}^{\infty}(P_n \square P_n)$ for $n \geq 5$?
- Or $\gamma_{all}^{\infty}(P_m \square P_n)$ for $m, n \geq 5$?

Thanks!

Mount Allison University

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