Exclusive Graph Searching in Various Graph Classes

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Graph Searching

In Graph Searching a team of searchers has to capture an invisible fugitive who is moving arbitrarily fast in a graph. The fugitive is considered captured when he is traveling on an edge whose endpoints are guarded by searchers.

The Graph Searching Problem

Design an algorithm that leads a minimum number of searchers to capture a fugitive in a graph within finite time.

Another definition for Graph Searching

Another definition for Graph Searching:

A graph G is given whose edges and nodes are *contaminated*. Design an algorithm that leads a minimum number of searchers to clear the graph within a finite time.

- A node of G is *cleared* when it is occupied by a searcher.
- An edge e of G is *cleared* if either searchers occupy simultaneously both its ends, or a searcher slides along e.

Recontamination

- An unoccupied clear node u is *recontaminated* as soon as there is a path free of searchers from u to a contaminated node.
- Similarly, an edge is *recontaminated* if at least one of its endpoints is recontaminated.

Graph Searching variants

A number of variants have been defined with respect to the searchers' strategies allowed.

Mixed Graph Searching [Bienstock and Seymour, 1991]

Each step consists of either:

- sliding a searcher along an edge, or
- placing a searcher at some node of the graph, or
- removing a searcher from a node of the graph.

Node Graph Searching [Kirousis and Papadimitriou, 1986], [Bienstock, 1991]

Each step consists of either:

- placing a searcher at some node of the graph, or
- removing a searcher from a node of the graph.

Classical Graph Searching

Edge Graph Searching [Parsons, 1978], [Petrov, 1982]

Same strategy as in mixed search allowed, but an edge can be cleared only by sliding a searcher along it.

Related Work

The minimum number of searchers capable of clearing a graph G is called:

- $s(G)$: search number of G if a mixed strategy is used,
- $ns(G)$: node search number of G if a node strategy is used,
- $es(G)$: edge search number of G if an edge strategy is used.

[Kirousis and Papadimitriou, 1986], [Bienstock and Seymour, 1991]

For any graph G, the following hold:

- $ns(G) - 1 \leq es(G) \leq ns(G) + 1$
- $s(G) \leq ns(G) \leq s(G) + 1$

Properties and the pathwidth problem

A strategy is called *monotone* if recontamination never occurs.

[Bienstock and Seymour, 1991]

For any graph G, there is a monotone mixed (resp., node, resp., edge) strategy that clears G using $s(G)$ (resp., $ns(G)$, resp., $es(G)$) searchers.

[Bienstock, 1991]

For any graph G, $ns(G) = pw(G) + 1$.

Complexity

[Monien and Sudborough, 1988]

The problem of computing the edge search number is NP-complete in the class of cubic planar graphs.

[Kirousis and Papadimitriou, 1986], [Monien and Sudborough, 1988]

The problem of computing the (mixed) search number is NP-complete in the class of cubic planar graphs where the set of vertices with degree exactly three induces an independent set.

Complexity

The pathwidth problem and several variants of graph searching have been studied in many graph classes. No classes of graphs where the computational complexity of these problems differ are known.

- The problems can be solved in polynomial time in *cographs* and *split graphs* but,
- they are NP-complete in a subclass of *starlike graphs* (where every peripheral clique has at least two nodes).

Exclusive Graph Searching

[Blin, Burman, Nisse, 2013]

- The searchers are initially placed at distinct nodes (i.e., at most one searcher at a node)
- Then at each step, a searcher may only slide (not jump) along an edge and only at a node which is not occupied.

The clearing and recontamination of nodes and edges are defined similarly as for mixed search.

- The exclusive search number of a graph G is denoted by $xs(G)$.
- It is easy to see that $s(G) \leq xs(G)$, for any graph G.

Exclusive Graph Searching

[Blin, Burman, Nisse, 2013]

- The exclusive search number and the search number of a graph may differ exponentially.
- The exclusive graph searching (in contrast with mixed, node or edge searching) does not satisfy the monotonicity property, even in tree topologies.
- The exclusive search number of an arbitrary tree can be computed in polynomial time.
- $ns(G) - 1 \leq xs(G) \leq (\Delta - 1)ns(G)$ in any graph G with maximum degree Δ.
- $ns(G) - 1 \leq xs(G) \leq ns(G)$ in any graph G with maximum degree 3.

Our Results

NP-hardness of Exclusive Graph Searching

It is NP-hard to compute the Exclusive Search Number of cubic planar graphs.

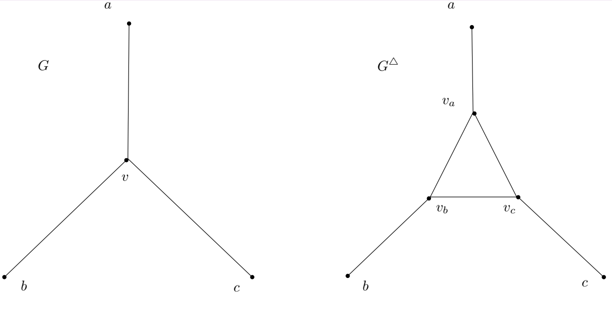
Exclusive Graph Searching vs Pathwidth

The complexities of *monotone* exclusive search and Pathwidth problems cannot be compared:

- Monotone exclusive search is NP-complete in split graphs (where Pathwidth can be computed in polynomial time).
- Monotone exclusive search is in P in a subclass of starlike graphs (where Pathwidth is NP-hard).
- Exclusive Search satisfies monotonicity and is in P in the class of cographs (where Pathwidth is also in P).

NP-hardness of Exclusive Graph Searching

We reduce the problem of computing the mixed search number of cubic planar graphs with no two adjacent nodes with degree three (called graph family $\mathcal{C}$) to our problem. The construction part of the reduction consists into replacing any node of degree three by a triangle.



(a) (b)

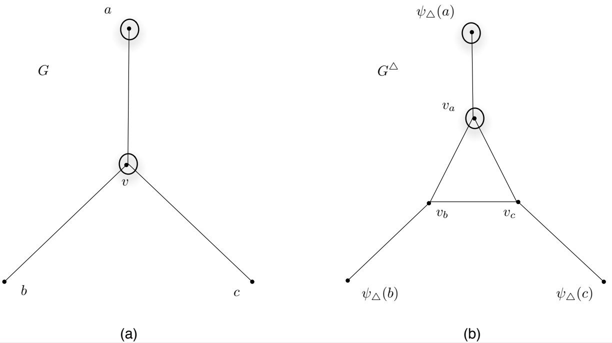
NP-hardness of Exclusive Graph Searching

Exclusive search differs from mixed search because searchers can only slide and therefore, because of the exclusivity property, the searchers have to avoid to meet other searchers at the same node. Intuitively, the triangles allow the searchers to bypass each other.

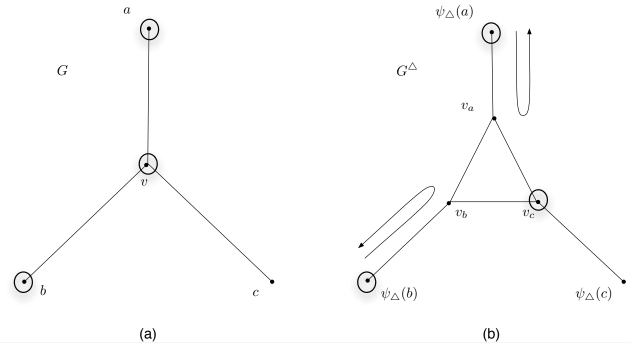
For any $G \in \mathcal{C}$, $xs(G^{\triangle}) \leq s(G)$.

Transform a monotone mixed strategy for G to an exclusive strategy (not monotone) for $G^{\triangle}$.

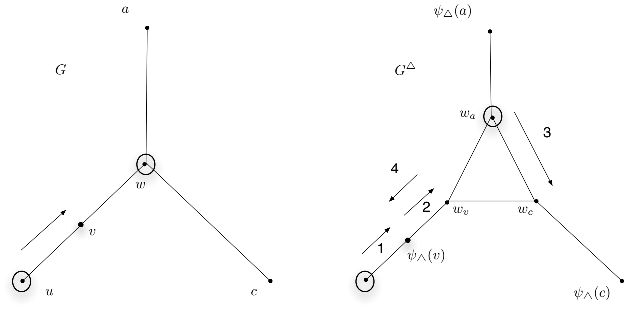
NP-hardness of Exclusive Graph Searching



NP-hardness of Exclusive Graph Searching

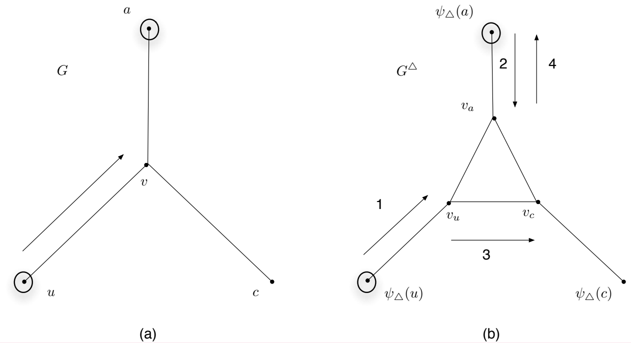


NP-hardness of Exclusive Graph Searching

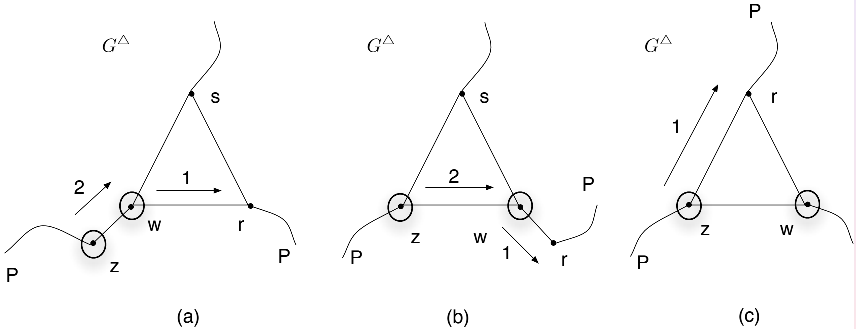


(a) (b)

NP-hardness of Exclusive Graph Searching



NP-hardness of Exclusive Graph Searching



NP-hardness of Exclusive Graph Searching

For any graph $G \in \mathcal{C}$, $s(G) \leq xs(G^{\triangle})$.

Transform an exclusive strategy for $G^{\triangle}$ to a mixed strategy for G.

Exclusive Graph searching in Starlike Graphs

Lemma

Exclusive graph searching is not monotone in starlike graphs.

Theorem

Let G be a starlike graph with cliques $(C_0, \cdots, C_r)$ such that $|C_i \setminus C_0| > 1$ for any $0 < i \leq r$, that is each non central clique has at least two peripherical nodes.

1. Either there is $(u, v) \in E(C_0)$ with $\{0 < i \leq r \mid u \in C_i\} \cap \{0 < i \leq r \mid v \in C_i\} = \emptyset$ and $\mathit{monotone\text{-}xs}(G) = |V(G)| - r - 1$,
2. or $\mathit{monotone\text{-}xs}(G) = |V(G)| - r$.

Exclusive Graph searching in Starlike Graphs

Corollary

The monotone exclusive search number can be computed in polynomial time in the class of star-like graphs where each peripheral clique has at least 2 peripheral nodes.

Note that the pathwidth is NP-hard in this class of graphs.

Exclusive Graph searching in Split Graphs

Theorem

Monotone exclusive graph searching is NP-complete in split graphs.

Note that the pathwidth is in P for this class of graphs.

Maximum Augmenting Cover (MAC)

input: A family $\mathcal{F} = \{S_1, \cdots, S_r\}$ of subsets of a set A (i.e., $S_i \subseteq A$, for all $i \leq r$) and $k \in \mathbb{N}$.

question: Is there an augmenting sequence of at least k sets $\mathcal{S} = (S'_1, \cdots, S'_k)$ of $\mathcal{F}$ (i.e., $S'_i \in \mathcal{F}$, for all $i \leq k$)?

We prove that MAC is NP-hard and we reduce MAC to our problem.

Exclusive Graph searching in Cographs

Theorem

The exclusive search number of cographs can be computed in polynomial time.

Note that the pathwidth is also in P for this class of graphs.

Discussion - Open Questions

- We can compute in polynomial time a 3 approximation factor of the exclusive search number for split graphs.
- We study Exclusive Graph Searching in distributed settings.
- Can Exclusive Graph Searching provide a polynomial-time approximation of Pathwidth in some graph class (or the other way around)?