On the variants of treewidth and minor-closedness property

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GRASTA 2014

Joint work with Hans Bodlaender, Vincent Kreuzen, Stefan Kratsch and Seongmin Ok

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- Motivations of our research.
- Notions.
  - (i) Intersection models of graphs.
  - (ii) Variants of treewidth.
- Basic properties.
  - (i) Algorithms.
  - (ii) Characterizing small width ($k = 1, 2$) in terms of cycle models and minor obstructions.
  - (iii) Non-minor-closedness of these parameters for $k \geqslant 3$.
- Discussion

Cops and Robbers

Treewidth

Cops move by helicopters, robbers cannot move the vertices occupied by cops.

Pathwidth

Cops move by helicopters, robbers cannot move the vertices occupied by cops, + cops do not see where the robber is located.

Question

Can we describe new parameters, which we will define later, in terms of a graph searching or a cops and robbers game?

Courcelle's Theorem

Every monadic second-order logic representable graph properties can be decided in linear time on bounded treewidth.

- It needs to construct complicated automata to represent it.
- One escape for this complexity is to use a relatively new parameter "cliquewidth".
- In 2012, Courcelle asked whether we can obtain a similar result by restricting the conditions of tree-decompositions.

Theorem (Courcelle, 2012)

Bounded special treewidth has much simpler representation than bounded treewidth.

A graph G has treewidth at most k if and only if there exists a **chordal graph** H such that G is a subgraph of H with maximum clique size at most $k+1$.

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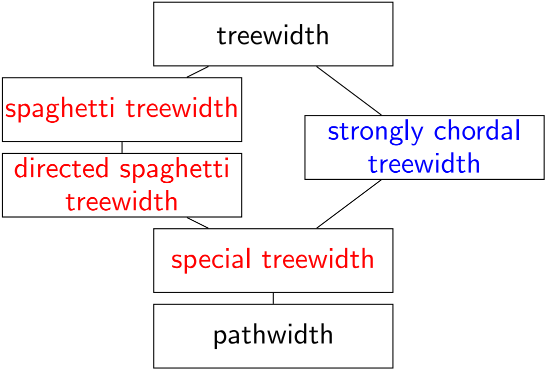

A graph G has special treewidth at most k if and only if there exists a **rooted directed path graph** H such that G is a subgraph of H with maximum clique size at most $k+1$.

- red $\rightarrow$ variations of the intersection model.
- blue $\rightarrow$ one more condition on even cycles

- G is called an **undirected path graph**
  $\Leftrightarrow$ G has an intersection model of paths on a tree.
- G is called a **directed path graph**
  $\Leftrightarrow$ G has an intersection model of directed paths on a directed tree(the underlying graph is a tree).
  (i) Forbidden induced subgraph characterizations / fast recognition algorithms for both classes are known.
- G is called a **rooted directed path graph**
  $\Leftrightarrow$ G has an intersection model of directed paths on a rooted directed tree.
  (i) Forbidden induced subgraph characterization is open.
  (ii) Dietz (1984, Ph.D. thesis) provided a recognition algorithm in time $\mathcal{O}(n+m)$. (not published)

- G is called **strongly chordal** if and only if it is chordal and every even cycle C of length at least 6 has an odd chord which divides C into two odd paths of length at least 3.
- A graph G is called a sun if $V(G)$ has two partition $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ such that A induces an independent set and $a_i b_j \in E(G)$ iff $i = j, j-1 \pmod{k}$.

Theorem (Farber, 1983)

A graph is strongly chordal if and only if it is chordal and it has no induced subgraph isomorphic to a sun.

Treewidth

A graph G has treewidth at most k if and only if there exists a **chordal graph** H such that G is a subgraph of H with maximum clique size at most $k + 1$.

- Pathwidth $\mathrm{pw}(G)$: Interval graphs
- Special treewidth $\mathrm{spctw}(G)$: Rooted directed path graphs (Courcelle, 2012)
- Spaghetti treewidth $\mathrm{spghtw}(G)$: Undirected path graphs
- Directed spaghetti treewidth $\mathrm{dspghtw}(G)$: Directed path graphs
- Strongly chordal treewidth $\mathrm{sctw}(G)$: Strongly chordal graphs

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Algorithms to compute the parameters

Theorem (Bodlaender, Kratsch, and Kreuzen 13)

For fixed k, there exists a linear time algorithm that decides whether the **special treewidth** (or **spaghetti treewidth**) of a given graph is at most k, which runs in time $\mathcal{O}(2^{\mathcal{O}(k^3)})$.

There exists an $\mathcal{O}(3^n)$-time algorithm to compute exact value of the **special treewidth**.

Open

1. Fixed parameter tractability for strongly chordal treewidth.
2. Non-trivial exact algorithms for new parameters.

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Graph classes of bounded width

Are the graphs having special treewidth $\leqslant k$ minor-closed?

Theorem (Courcelle 12)

All trees have special treewidth at most 1.

Observation

Let G be a connected graph. Then TFAE:
G is a tree $\Leftrightarrow \mathrm{tw}(G) \leqslant 1 \Leftrightarrow \mathrm{spghtw}(G) \leqslant 1 \Leftrightarrow \mathrm{sctw}(G) \leqslant 1$
$\Leftrightarrow \mathrm{dspghtw}(G) \leqslant 1 \Leftrightarrow \mathrm{spctw}(G) \leqslant 1$

Theorem (Courcelle 12)

For $k \geqslant 5$, the graphs of $\mathrm{spctw}(G) \leqslant k$ are not minor-closed.

Graph classes of bounded width

Are the graphs having special treewidth $\leqslant k$ minor-closed?

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Main result

Theorem

For each new parameter
(special, (directed) spaghetti, strongly chordal treewidth),
the graphs of width at most 2 are minor-closed.

- We generate new subclasses of graphs of treewidth at most 2.

Theorem

For each integer $k \geqslant 3$ and for each new parameter
(special, (directed) spaghetti, strongly chordal treewidth),
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For each integer $k \geqslant 3$ and for each new parameter
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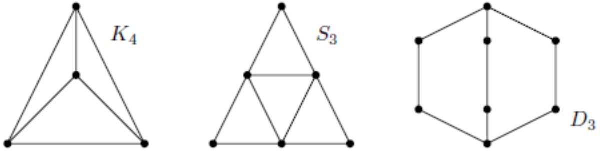
the graphs of width at most k are not minor-closed.

Graph classes	Minor obstructions for 2-connected graphs	Minor obstructions for general graphs
$\mathrm{tw} \leqslant 2$	K_4	K_4
$\mathrm{spghtw} \leqslant 2$	?	?
$\mathrm{sctw} \leqslant 2$	?	?
$\mathrm{dspghtw} \leqslant 2$	?	?
$\mathrm{spctw} \leqslant 2$	?	?
$\mathrm{pw} \leqslant 2$	K_4, D_3, S_3 [Barát et el, 12]	110 graphs [Kinnersley, Langston 94]



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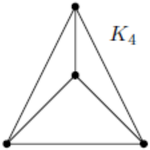
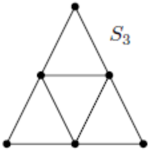
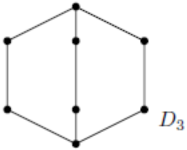
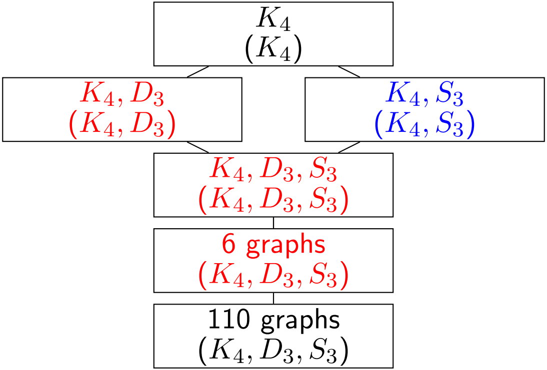
$\mathrm{spghtw} \leqslant 2$	K_4, D_3	K_4, D_3
$\mathrm{sctw} \leqslant 2$	K_4, S_3	K_4, S_3
$\mathrm{dspghtw} \leqslant 2$	K_4, D_3, S_3	K_4, D_3, S_3
$\mathrm{spctw} \leqslant 2$	K_4, D_3, S_3	6 graphs
$\mathrm{pw} \leqslant 2$	K_4, D_3, S_3 [Barát et el, 12]	110 graphs [Kinnersley, Langston 94]







The obstructions (2-connected obstructions)

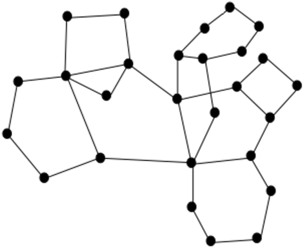


- For each class, we fully describe it as a cycle model.

The graph $\widetilde{G}$ is **the cell completion** of a 2-connected graph G if $\widetilde{G}$ is obtained from G by adding an edge vw for all pairs of nonadjacent vertices $v, w \in V(G)$ such that $G[V(G)\backslash v\backslash w]$ has at least three connected components.

Theorem (Bodlaender, Kloks 1993)

Let G be a 2-connected graph.
G has treewidth at most 2 if and only if $\widetilde{G}$ is **a tree of cycles**.
(They are recursively defined by attaching a cycle on an edge)

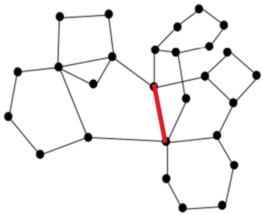


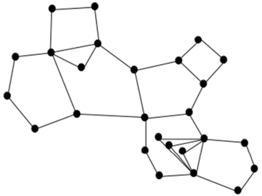
Theorem

Let G be a 2-connected graph. TFAE.

(i) G has **spaghetti treewidth** at most 2

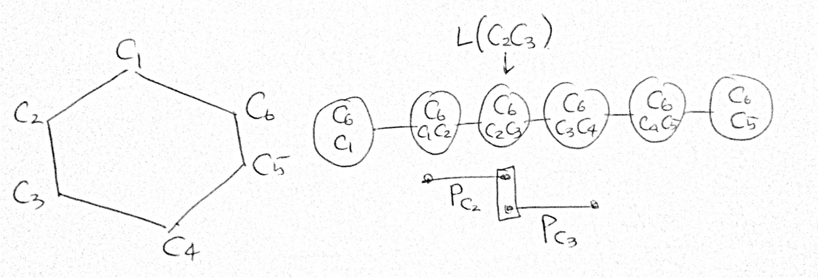
(ii) $\widetilde{G}$ is a tree of cycles and for every edge separator u, v in $\widetilde{G}$, uv is not contained in 3 non-trivial induced cycles.





(i) $\Rightarrow$ (ii). Straightforward.

(iii) $\Rightarrow$ (i) We prove, by induction on the number of induced cycles, that there exists a spaghetti tree-decomposition satisfying that for each edge uv which is not an edge separator, there exists a bag P_u and P_v ends in the same bag $L(uv)$.

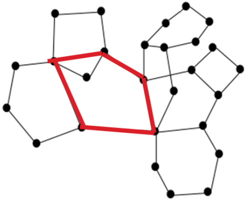


Theorem

Let G be a 2-connected graph. TFAE.

(i) G has **strongly chordal treewidth** at most 2

(ii) $\widetilde{G}$ is a tree of cycles and for each induced cycle C, it contains no 3 edge separators.



(i) $\Rightarrow$ (ii) We showed that if G is a strongly chordal graph of maximum clique size 3 and $e \in E(G)$, then G/e is strongly chordal. (It is not true when $\geqslant 4$)

(ii) $\Rightarrow$ (i) Suppose G is a tree of cycles such that for each induced cycle C, it contains at most 2 edge separators.

We can triangulate each chordless cycle so that no sun appear. By Farber's characterization, the triangulated graph is strongly chordal.
Therefore, G has strongly chordal treewidth two.

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Corollary

Let G be a 2-connected graph. TFAE.

(i) G has **pathwidth** at most 2 (Bodlaender and Fluiter 1996)
(ii) G has **special treewidth** at most 2
(iii) G has **directed path treewidth** at most 2
(iv) G has no minor isomorphic to K_4, S_3 and D_3
(v) $\tilde{G}$ is a tree of cycles and for each induced cycle C, it contains no 3 edge separators, and for every edge separator u, v in $\tilde{G}$, uv is not contained in 3 non-trivial induced cycles.

Theorem

A graph is special treewidth at most 2 if and only if (2-connected graphs of pathwidth at most two, or edges) are attached in a sense of a rooted tree.

Conclusion

We showed that

Let k be an integer.
The graphs of $\mathrm{spghtw}(G) \leqslant k$ are minor-closed iff $k \leqslant 2$.
The graphs of $\mathrm{sctw}(G) \leqslant k$ are minor-closed iff $k \leqslant 2$.
The graphs of $\mathrm{dspghtw}(G) \leqslant k$ are minor-closed iff $k \leqslant 2$.

- Can we do better than $O(3^n)$ for computing special treewidth exactly?
- Want to find non-trivial exact algorithms for other parameters.
- Can we describe those parameters in terms of cops and robbers game?

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